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Constrained Optimization

- Again we want to find $\max_{x\in S} f(x)$ or $\min_{x\in S} f(x)$ for some function $f: \mathbb{R}^n \to \mathbb{R}$
- Here we have some constraints:

$$g_1(x)=0, \dots, g_M(x)=0 \qquad \text{(equalities)}$$

$$h_1(x)\geq 0, \dots, h_p(x)\geq 0 \qquad \text{(inequalities)}$$

Equality-Constrained Optimization: Lagrange Method.

Theorem: If x^* is a local optimum, and

$$\operatorname{rank}\begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \vdots & & \vdots \\ \frac{\partial g_M}{\partial x_1}(x^*) & \cdots & \frac{\partial g_M}{\partial x_n}(x^*) \end{bmatrix} = M,$$

then there exists a vector $\lambda^* = \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_M^* \end{bmatrix} \in \mathbb{R}^m$ such that

$$\frac{\partial f}{\partial x_i}(x^*) + \sum_{j=1}^{M} \lambda_j^* \frac{\partial g_j}{\partial x_i}(x^*) = 0. \qquad (i=1,\dots,n)$$

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- This is equivalent to finding max or min of the Lagrangian:

$$L = f + \sum_{j=1}^{m} \lambda_j g_j : \frac{\partial L}{\partial x_i} = 0 \ \underline{\text{and}} \ \overbrace{\frac{\partial L}{\partial \lambda_j} = 0}^{\text{constraint}}$$

- Additional conditions (maximality or minimality) are the same as in unconstrained case for f, i.e. we need to investigate definiteness of Hessian matrix for L

- Be aware that $\text{rank} \left[\frac{\partial g_j}{\partial x_i}(x^*) \right]$ __must__ be equal to number of constraints M! If not, this procedure will not work, although max or min can exist.

- Example: $f(x,y) = x^2 - y^2$, $g(x,y) = 1 - x^2 - y^2$

$$L = f(x,y) + \lambda g(x,y) = x^2 - y^2 + \lambda (1 - x^2 - y^2)$$

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$$\frac{\partial L}{\partial x} = 2x - 2\lambda x = 0 \Rightarrow x^*(1-\lambda^*) = 0$$

$$\frac{\partial L}{\partial y} = -2y - 2y\lambda = 0 \Rightarrow y^*(1+\lambda^*) = 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 = 0 \Rightarrow x^{*2} + y^{*2} = 1$$

If $x^* = 0$, from the third equation follows $y^* = \pm 1$, and from the second $\lambda^* = -1$, so we have two solutions $(x^*, y^*, \lambda^*) = (0, \pm 1, -1)$

If $x^* \neq 0$, then $\lambda^* = 1$ from the first, and $y^* = 0$ from the second equation. From the third equation we have $x^* = \pm 1$, so we have two additional solutions: $(\pm 1, 0, 1)$.

$\text{rank}\left[\frac{\partial g_j}{\partial x_i}\right] = \text{rank}\left[-2x^* \quad -2y^*\right]$, so for the first solution we have

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$$\text{rank}\left[\frac{\partial f_j}{\partial x_i}\right] = \text{rank}\begin{bmatrix} 0 & \mp 2 \end{bmatrix} = 1 = M,$$

so all conditions for the theorem are satisfied.

For the second pair is

$$\text{rank}\left[\frac{\partial f_j}{\partial x_i}\right] = \text{rank}\begin{bmatrix} \mp 2 & 0 \end{bmatrix} = 1 = M,$$

so everything is OK here also.

– Hessian matrix is

$$H = \begin{bmatrix} 2-2\lambda & 0 \\ 0 & -2-2\lambda \end{bmatrix}$$

For $(0, \pm 1, -1)$ is $H = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0$

For $(\pm 1, 0, 1)$ is $H = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \geqslant 0$

All solutions are minimums, although not strict ones.

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Inequality-Constrained Optimization: Kuhn-Tucker Method

Theorem: If x^* is a local maximum, and $\text{rank}\left[\frac{\partial h_{Ej}}{\partial x_i}\right] = M_E$, where h_E are constraints with the equality sign valid for x^*, and M_E is the number of those constraints; then there exists a vector $\lambda^* = \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_N^* \end{bmatrix} \in \mathbb{R}^N$ such that

$$\frac{\partial f}{\partial x_i}(x^*) + \sum_{j=1}^{N} \lambda_j^* \frac{\partial h_j}{\partial x_i}(x^*) = 0 \qquad (i = 1, .., n)$$

$$h_i(x^*) \geq 0$$

$$\lambda_i^* \geq 0$$

$$\lambda_i^* h_i(x^*) = 0.$$

– Note that you need to transform the problem of optimization into the

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maximization problem with all conditions of the form $h_i(x^*) \geqslant 0$, which can easily be done.

- We can again construct Lagrangian

$$L = f + \sum_{j=1}^{N} \lambda_j h_j ,$$

and the equations for x^* are

$$\frac{\partial L}{\partial x_i} = 0 , \quad \frac{\partial L}{\partial \lambda_j} \geqslant 0 , \quad \lambda_j^* \geqslant 0$$

$$\lambda_j^* h_j^*(x^*) = 0 .$$

- Be aware of the problems with rank!

- Example: $f(x,y) = x+y$, $h(x,y) = xy - 1$

$$L = f + \lambda h = x + y + \lambda(xy - 1)$$

$$\frac{\partial L}{\partial x} = 1 + \lambda y = 0 \Rightarrow \lambda^* y^* = -1 \Rightarrow \lambda^* \neq 0,\ y^* \neq 0$$

$$\frac{\partial L}{\partial y} = 1 + \lambda x = 0 \Rightarrow \lambda^* x^* = -1 \Rightarrow \lambda^* \neq 0,\ x^* \neq 0$$

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$$y^* = x^* = -\frac{1}{\lambda^*}$$

$$\frac{\partial L}{\partial \lambda} = xy - 1 \geq 0 \quad , \quad \lambda^* \geq 0 \quad , \quad \lambda^*(x^* y^* - 1) = 0$$

$$x^* y^* - 1 = \frac{1}{\lambda^{*2}} - 1 \geq 0 \Rightarrow \lambda^{*2} \leq 1 .$$

with $\lambda^* \geq 0 \Rightarrow 0 \leq \lambda^* \leq 1$

Since $\lambda^* \neq 0$, we have $x^* y^* = 1 \Rightarrow \lambda^* = 1$,

$x^* = y^* = -1$, so the solution is $(-1, -1, 1)$.

$$\text{rank}\left[\frac{\partial h_{E_i}}{\partial x_i}(x^*)\right] = \left[x^* \;\; y^*\right] = \left[-1 \;\; -1\right] = 1 ,$$

So the conditions are OK.

- For homework you can check if this is a maximum or not (it cannot be a minimum!)

- For the definition and introduction to the Markowitz problem, you can consult lecture notes from Jan 9, 2004., or Ch. 6 of „Investment Science" by D. Luenberger

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Homework problems for constrained optimization:

1) $f(x,y) = x^2 - y^2$, $g(x,y) = 1 - x - y$
2) $f(x,y) = \frac{x^3}{3} - \frac{3y^2}{2} + 2x$, $g(x,y) = x - y$

equality const.

3) $f(x,y) = x^2 - y$, $h(x,y) = 1 - x^2 - y^2$
4) $f(x,y) = 2x^3 - 3x^2$, $h(x,y) = (3 - x)^3$
5) $f(x,y) = x^2 - x$, $h(x,y) = x$

inequality constrained

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Economic Example: Markowitz Problem

If we have n assets with expected rates of return $\bar{r}_1, \dots, \bar{r}_n$ and the covariances σ_{ij}, $(i,j=1,\dots,n)$, a portfolio is defined by a set of n weights w_i, $i=1,\dots,n$, so that $\sum_{i=1}^{n} w_i = 1$. The problem is to

$$\text{minimize } \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij},$$

subject to constraints:

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}, \quad \sum_{i=1}^{n} w_i = 1.$$

- Lagrangian is

$$L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} + \lambda \left(\sum_{i=1}^{n} w_i r_i - \bar{r} \right) +$$

$$+ \mu \left(\sum_{i=1}^{n} w_i - 1 \right)$$

– To simplify things, we will work with $n=2$, so that

$$L = \frac{1}{2}\left(w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2^2 \sigma_2^2\right) +$$

$$+ \lambda\left(w_1 \bar{r}_1 + w_2 \bar{r}_2 - \bar{r}\right) + \mu\left(w_1 + w_2 - 1\right)$$

– The equations are

$$\frac{\partial L}{\partial w_1} = w_1 \sigma_1^2 + \frac{1}{2} w_2 \sigma_{12} + \frac{1}{2} w_2 \sigma_{21} + \lambda \bar{r}_1 + \mu = 0$$

$$\frac{\partial L}{\partial w_2} = \frac{1}{2} w_1 \sigma_{12} + \frac{1}{2} w_1 \sigma_{21} + w_2 \sigma_2^2 + \lambda \bar{r}_2 + \mu = 0$$

$$\frac{\partial L}{\partial \lambda} = w_1 \bar{r}_1 + w_2 \bar{r}_2 - \bar{r} = 0$$

$$\frac{\partial L}{\partial \mu} = w_1 + w_2 - 1 = 0$$

– Since $\sigma_{12} = \sigma_{21}$, we have a set of 4 equations in 4 unknowns w_1, w_2, λ, and μ:

$$\begin{aligned}
w_1 \sigma_1^2 + w_2 \sigma_{12} + \lambda \bar{r}_1 + \mu &= 0 \\
w_1 \sigma_{12} + w_2 \sigma_2^2 + \lambda \bar{r}_2 + \mu &= 0 \\
w_1 \bar{r}_1 + w_2 \bar{r}_2 &= \bar{r} \\
w_1 + w_2 &= 1
\end{aligned}$$

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- And now we have a system of linear equations, which can be easily solved
- General set of equations is

$$\sum_{j=1}^{n} \sigma_{ij} w_j + \lambda \bar{r}_i + \mu = 0 \quad (i = 1, \dots, n)$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

- we have system of $n+2$ ~~unknowns~~ equations in $n+2$ unknowns (w_i, λ, μ).
- Home work Problem: Solve the Markowitz problem for a set of three uncorrelated assets, with $\sigma_1^2 = 1$, $\sigma_2^2 = 1$, $\sigma_3^2 = 1$, and $\bar{r}_1 = 1$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$.

QM, Dec 23, 2004. (11)

- Solutions of Homework problems assigned on Dec 22, 2004.

1) $f(x) = x^2 + x^3$, $f'(x) = 2x + 3x^2 = x(2+3x)$

$f'(x)=0 \Leftrightarrow x(2+3x)=0 \Leftrightarrow x=0 \vee x=-\frac{2}{3}$

If $x^*=0$, then $f''(x) = 2+6x$ is

at x^* equal to $f''(0) = 2 > 0$, so

$\boxed{x^*=0 \text{ is minimum}}$

If $x^* = -\frac{2}{3}$, then $f''(x^*) = f''\left(-\frac{2}{3}\right) = 2+6\cdot\left(-\frac{2}{3}\right) =$

$= 2-4 = -2 < 0$, so $\boxed{x^* = -\frac{2}{3} \text{ is maximum}}$

2) $f(x) = e^x \sin x$, $f'(x) = e^x \sin x + e^x \cos x$

$f''(x) = \cancel{e^x \sin x} + e^x \cos x + e^x \cos x - \cancel{e^x \sin x} =$

$= 2e^x \cos x$

$f'(x) = 0 \Leftrightarrow e^x(\sin x + \cos x) = 0$

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- Since $e^x > 0$, the condition for x^* is
$\sin x + \cos x = 0$
- Using mathematical identities
$\cos x = \sin(\frac{\pi}{2} - x) = -\sin(x - \frac{\pi}{2})$ and
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$, we can transform this equation into

$$\sin x + \cos x = \sin(x - \tfrac{\pi}{4} + \tfrac{\pi}{4}) - \sin(x - \tfrac{\pi}{4} - \tfrac{\pi}{4}) =$$

$$= \cancel{\sin(x - \tfrac{\pi}{4})\cos\tfrac{\pi}{4}} + \cos(x - \tfrac{\pi}{4})\sin\tfrac{\pi}{4} -$$

$$- \cancel{\sin(x - \tfrac{\pi}{4})\cos\tfrac{\pi}{4}} + \cos(x - \tfrac{\pi}{4})\sin\tfrac{\pi}{4} =$$

$$= 2\sin\tfrac{\pi}{4}\cos(x - \tfrac{\pi}{4}) = 0 \implies$$

$$x - \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}, \quad \text{so}$$

we have $x_k^* = \frac{3\pi}{4} + k\pi$

$$f''(x_k^*) = 2e^{x_k^*}\cos x_k^* = 2e^{\frac{3\pi}{4} + k\pi}\cos(\tfrac{3\pi}{4} + k\pi)$$

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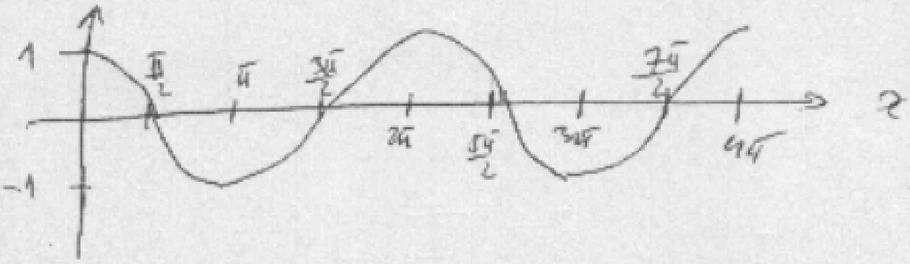
From the graph of cos function



we can see that

$\cos(\frac{3\pi}{4}) < 0$, $\cos(\frac{3\pi}{4}+\pi) > 0$,

$\cos(\frac{3\pi}{4}+2\pi) < 0$, ... , so

$$\cos(\frac{3\pi}{4}+k\pi) \begin{cases} < 0 & \text{for } k \text{ even} \\ > 0 & \text{for } k \text{ odd, and} \end{cases}$$

$$x_k^* = \frac{3\pi}{4} + k\pi \begin{cases} \text{is maximum for } k \text{ even} \\ \text{is minimum for } k \text{ odd} \end{cases}$$

3) $f(x,y) = x^4y^2 + x^2y^4 - x^2y^2$

$$\frac{\partial f}{\partial x} = 4x^3y^2 + 2xy^4 - 2xy^2 = 0$$

$$\frac{\partial f}{\partial y} = 2x^4y + 4x^2y^3 - 2x^2y = 0$$

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If $x=0$, then both eq. are satisfied.
If $y=0$, again both eq. are satisfied.

So, we have solutions $(0,y)$, $(x,0)$, where x, y are free parameters.

If $x\neq 0$ and $y \neq 0$ we can divide first eq. by xy^2 and second eq. by x^2y:

$$4x^2+2y^2=2 \quad /:2 \qquad 2x^2+y^2=1$$
$$2x^2+4y^2=2 \quad /:2 \qquad x^2+2y^2=1$$

The unique solution of this system of linear equation in x^2, y^2 is $x^2=y^2=\frac{1}{3}$, so we have for solutions (two pairs)

$$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right) \text{ and } \left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right).$$

Hessian matrix is

$$H = \begin{bmatrix} 12x^2y^2+2y^4-2y^2 & 8x^3y+8xy^3-4xy \\ 8x^3y+8xy^3-4xy & 2x^4+12x^2y^2-2x^2 \end{bmatrix}$$

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- For the solution $(0,y)$, we have

$$H=\begin{bmatrix} 2y^4-2y^2 & 0 \\ 0 & 0 \end{bmatrix}$$

This matrix is positive semidefinite for $y^2>1$, and negative semidefinite for $0<y^2<1$.

- For the solution $(x,0)$, we have

$$H=\begin{bmatrix} 0 & 0 \\ 0 & 2x^4-2x^2 \end{bmatrix},$$

and the answer is the same as in the previous case.

To be sure if these points are minima or maxima, we need further investigation.

- For $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ we have

$$H=\begin{bmatrix} \frac{12}{9}+\frac{2}{9}-\frac{2}{3} & \frac{8}{9}+\frac{8}{9}-\frac{4}{3} \\ \frac{8}{9}+\frac{8}{9}-\frac{4}{3} & \frac{2}{9}+\frac{12}{9}-\frac{2}{3} \end{bmatrix}=\begin{bmatrix} \frac{8}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{8}{9} \end{bmatrix}$$

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$H_{11} > 0, \quad \det H = \left(\frac{8}{9}\right)^2 - \left(\frac{4}{9}\right)^2 > 0 \Rightarrow$

H is positive definite $\Rightarrow \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ is minimum.

- For $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right)$ we have

$$H = \begin{bmatrix} \frac{12}{9} + \frac{2}{9} - \frac{2}{3} & -\frac{8}{9} - \frac{8}{9} + \frac{4}{3} \\ -\frac{8}{9} - \frac{8}{9} + \frac{4}{3} & \frac{2}{9} + \frac{12}{9} - \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{8}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{8}{9} \end{bmatrix}$$

$H_{11} > 0, \quad \det H = \left(\frac{8}{9}\right)^2 - \left(-\frac{4}{9}\right)^2 > 0$, so

H is positive definite $\Rightarrow \left(\pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right)$ is minimum.

4) $f(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz$

$\frac{\partial f}{\partial x} = 4x + 2y + 2z = 0$

$\frac{\partial f}{\partial y} = 4y + 2x + 2z = 0$

$\frac{\partial f}{\partial z} = 4z + 2x + 2y = 0$

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The determinant of this system is

$$\det\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} = 4\begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} =$$

$$= 4\cdot(16-4) - 2\cdot(8-4) + 2(4-8) =$$

$$= 48 - 8 - 8 = 32 \neq 0$$

Since the system is homogeneous, it has a unique solution $x^* = y^* = z^* = 0$.

Hessian matrix is

$$H = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix},$$ the same as the matrix of the above solved system.

$$H_{11} > 0, \quad \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 16-4 = 12 > 0,$$

$\det H = 32 > 0 \Rightarrow H$ is positive definite, and $x^* = y^* = z^* = 0$ is minimum of f.