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The Two-Fund Theorem

- Minimum variance set is defined by the equations (Markowitz problem):

$$\sum_{j=1}^{n} \sigma_{ij} w_j + \lambda \bar{r}_i + \mu = 0 \quad (i = 1, .., n)$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}, \quad \sum_{i=1}^{n} w_i = 1.$$

- If we have two sets of solutions $(w_1^{(1)}, .., w_n^{(1)})$ and $(w_1^{(2)}, .., w_n^{(2)})$ for two fixed value of $\bar{r}$, say $\bar{r}^{(1)}$ and $\bar{r}^{(2)}$, the portfolio with the weights

$$(\alpha w_1^{(1)} + (1-\alpha) w_1^{(2)}, .., \alpha w_n^{(1)} + (1-\alpha) w_n^{(2)})$$

will be the minimum variance set for $\bar{r} = \alpha \bar{r}^{(1)} + (1-\alpha) \bar{r}^{(2)}$!

- You can check that this is true!

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- If there are two portfolios (funds) with different $\bar{r}$'s, the <u>Two-Fund Theorem</u> is valid:

TWO-FUND THEOR.

Two minimum variance funds can be established so that any minimum variance portfolio can be constructed in terms of mean and variance, as a combination of these two. In other words, all investors seeking minimum variance portfolios need only invest in combination of these two funds.

- According to this theorem, two mutual funds (investment company that accept capital from individuals and reinvests that capital in a diversity of individual stocks; each individual is entitled to his/her proportionate share of

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the fund's portfolio value) could provide a complete investment services for everyone!

- Of course, this is based on assumption that everyone cares only about mean and variance; that everyone is estimating this quantities using the same method etc.

- If we have a risk-free asset (lending or borrowing at the fixed interest rate) with the expected rate of return r_f ($\sigma_f = 0$ by definition), and form a portfolio of this asset and another, risky ($\bar{r}_r, \sigma_r$), asset, we will have

$$\bar{r} = \alpha r_f + (1-\alpha)\bar{r}_r$$

$$\sigma^2 = \alpha^2 \cancel{\sigma_f^2}^{\,0} + 2\,\cancel{\sigma_{rf}}^{\,0}\,\alpha(1-\alpha) + (1-\alpha)^2\sigma_r^2$$

$$\Rightarrow \sigma = (1-\alpha)\,\sigma_r$$

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- <u>The one-fund theorem</u>: There is a single fund F of risky assets such that any minimum variance portfolio can be constructed as a combination of the fund F and the risk-free asset.

Market Equilibrium

- Suppose that every one is mean-variance optimizer; ~~and~~ that every one agrees on the probabilistic structure of assets (same $\bar{r}_i$, same σ_i^2, σ_{ij}); that there is a unique risk-free rate of borrowing and lending available to all
- From the one-fund theorem we know that every one will purchase a single fund of risky assets and form a

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portfolio using the risk-free asset

- Since ~~all~~ everyone uses the same $\bar{r}_i$'s, σ_i^2's and σ_{ij}'s, they will all use the same risky fund!
- Of course, the mix of these two assets will likely vary across individuals according to their individual tastes of risk
- Hence, the <u>one fund</u> in the theorem is really the only fund that is used!!
- This fund is <u>Market Portfolio</u>, i.e. the totality of all available assets
- The weight of an asset in the market portfolio is equal to the proportion of that asset's total capital value to the total market

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Capital value (capitalization weights)

- Example: If we have only three ~~compan~~ companies on the stock market, with the details given in a table

Company	Shares	Price	Capital
Jazz, Inc.	10,000	\$6.00	\$60,000.00
Classical, Inc.	30,000	\$4.00	\$120,000.00
Rock, Inc.	40,000	\$5.50	\$220,000.00

since the total capital is \$400,000.00, the capitalization weights will be

$$w_J = \frac{\$60,000}{\$400,000} = \frac{3}{20}, \quad w_C = \frac{\$120,000}{\$400,000} = \frac{3}{10}, \text{ and}$$

$$w_R = \frac{\$220,000}{\$400,000} = \frac{11}{20}.$$

You can check that $w_J + w_C + w_R = 1$.

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- When everyone follows the mean-variance methodology, we know that the minimum variance fund of risky assets is market portfolio! No need to solve the problem!
- Using risk-free asset, we can have desired portfolio!
- How does this happen? The return of an asset depends on both its initial price and its final price. All investors solve the mean-variance portfolio problems using their common estimates, and place orders in the market to acquire their portfolios. If the orders do not match with the available assets the prices must change (assets under heavy demand will ~~rise~~ increase and vice versa)

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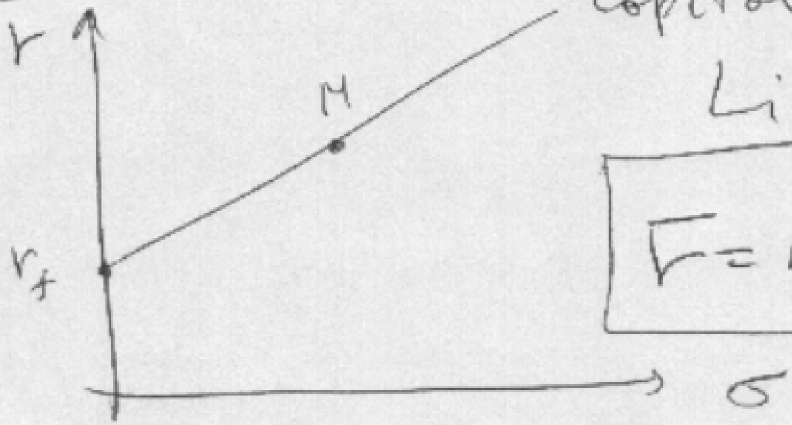
- These price changes affect the estimates of asset returns, and investors will recalculate their optimal portfolios.
- This process continues until demand matches supply – until there is equilibrium



$$\boxed{\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma}$$

$$K = \frac{\bar{r}_M - r_f}{\sigma_M} = \text{price of risk}$$

(how much the $\bar{r}$ must increase if the standard deviation increases by one unit)

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The Capital Asset Pricing Model (CAPM)

- Theorem: The expected rate of return of any asset i in the market portfolio is

$$\bar{r}_i - r_f = \beta_i (\bar{r}_M - r_f),$$

where $\beta_i = \dfrac{\sigma_{iM}}{\sigma_M^2}$.

- Example: $r_f = 8\%$, $\bar{r}_M = 12\%$, $\sigma_M = 15\%$

If $\sigma_{iM} = 0.045$, then $\beta = \dfrac{0.045}{0.15^2} = 2$,

and $\bar{r} = r_f + \beta(\bar{r}_M - r_f) = 0.08 + 2 \cdot (0.12 - 0.08) =$
$= 0.16 = 16\%$.

- Values of β and σ (volatility) can be found for companies!

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- Beta of a portfolio: $\beta = \sum_{i=1}^{n} w_i \beta_i$
- Example: CAPM applied to performance evaluation

The ABC mutual fund has the 10-year record given in table below, together with Standard & Poor's 500 stock average (Market Portfolio) and Treasury bill rates

Year	ABC r [%]	S&P 500 r [%]	T-bill r [%]
1	14	12	7
2	10	7	7.5
3	19	20	7.7
4	−8	−2	7.5
5	23	12	8.5
6	28	23	8
7	20	17	7.3
8	14	20	7
9	−9	−5	7.5
10	19	16	8

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Is ABC fund a good fund that we would recommend? Can it serve as the one-fund for a mean-variance investor?

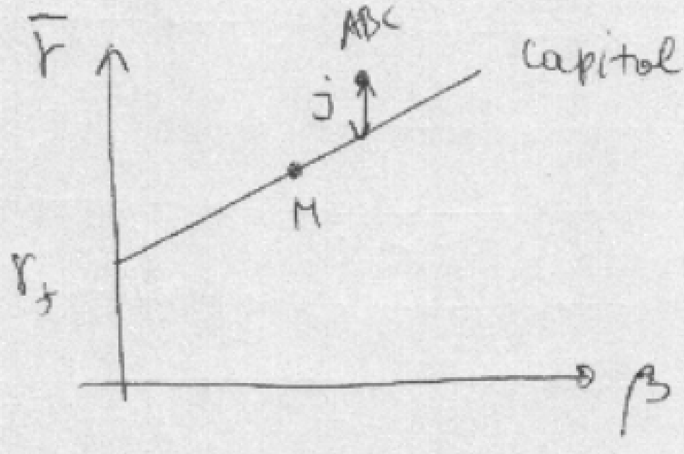
Means: $\bar{r}_{ABC} = 13\%$, $\bar{r}_M = 12\%$, $r_f = 7.6\%$

$\sigma_{ABC} = 12.4\%$, $\sigma_M = 9.4\%$, $\sigma_f = 0.5\% \approx 0$

$\sigma_{ABC,M} = 0.0107 \Rightarrow \beta_{ABC} = \frac{\sigma_{ABC,M}}{\sigma_M^2} = 1.20375$

We will introduce Jensen index J:

$$\bar{r}_{AB} - r_f = J + \beta(\bar{r}_M - r_f)$$



If $J > 0$, the fund is excellent!

In our case, $J = 0.00104 > 0$

But we don't know if ABC is efficient!

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- Example: CAPM as a pricing formula

Suppose that an asset is purchased at price P and later sold at (unknown) price Q. Rate of return is $r = \frac{Q-P}{P}$. CAPM gives us

$$\frac{\overline{Q}-P}{P} - r_f = \beta(\overline{r}_M - r_f) \Rightarrow$$

$$P = \frac{\overline{Q}}{1+r_f+\beta(\overline{r}_M - r_f)}$$

So, the appropriate price of an asset with payoff $\overline{Q}$ is given by this formula.

- In the deterministic case, the price would be $\frac{\overline{Q}}{1+r_f}$; $\beta(\overline{r}_M - r_f)$ is adjustment for risk, since usually $\beta(\overline{r}_M - r_f) > 0$, so $\frac{\overline{Q}}{1+r_f} > \frac{\overline{Q}}{1+r_f+\beta(\overline{r}_M - r_f)}$

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Linear Regression with one Variable

- CAPM has the form $\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$ for expected values
- For real data

$$r_i = r_f + \beta_i (r_M - r_f) + \xi_i,$$

where ξ_i are random variables
- Using real data we have to estimate parameters of this function
- The model is then $y = \alpha + \beta x + \xi$, y is dependent variable and x is independent variable, α is intercept and β is slope
- We have a real data $(x_1, y_1), \ldots, (x_n, y_n)$

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- For a chosen parameters α and β we have $y_1' = \alpha + \beta x_1, \dots, y_n' = \alpha + \beta x_n$, and, of course, $y_1 \neq y_1', \dots, y_n \neq y_n'$ (in general; some of them may be equal).
- We want these values as close as possible — the method is to minimize residual sum of squares (RSS)

$$RSS = \sum_{i=1}^{n} (y_i - y_i')^2 =$$

$$= \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

- The equations are $\frac{\partial RSS}{\partial \alpha} = 0$ and $\frac{\partial RSS}{\partial \beta} = 0$
- The solution is:

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$$\beta^* = \frac{\sum_i (x_i y_i) - \frac{\sum_i x_i \sum_j y_j}{n}}{\sum_i x_i^2 - \frac{\left(\sum_i x_i\right)^2}{n}} = \frac{\mathrm{cov}(x,y)}{\mathrm{var}(y)}$$

$$\alpha^* = \overline{y} - \beta^* \overline{x}.$$

- Excel has built-in functions for linear regression (Try Tools/Data Analysis/Regression)

Multiple Linear Regression

- The model: $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \varepsilon$
- Variable y depends on several independent variables $x_1, \ldots, x_n$
- Now we have a set of data consisting of N values for y, and N values for each independent variable

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- We will again minimize RSS
- Excel is suitable tool for doing multiple regressions
- Multiple Regression has been used to estimate multi-factor models of asset returns, including testing of arbitrage pricing theory, development of many macro-economic models, time-series analysis etc.
- Examples are given in LinearRegression.xls file
- Exam: Jan 11, 6-10 pm, Computer Classroom C
- Consultations: Jan 8, noon, SECCF office

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Overview

1) Linear Algebra
   a) Systems of Linear Equations
   b) Quadratic Forms

2) Mathematical Analysis
   a) Sequences, Limits
   b) Derivatives and Taylor's Theorem
   c) Integrals

3) Optimization
   a) Unconstrained Min/Maximization
   b) Constrained Optimization
   c) Markowitz Problem

4) Empirical Data
   a) Distributions, Measures of Dispersion
   b) Hypotheses Testing

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5) CAPM, Linear Regressions

The End

Marry Christmass and Happy New Year!!

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