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- So far, we have solved system of n linear equations in n unknowns of the form

$$A x = \alpha,$$

where $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \in \mathbb{R}^{nn}$, $\underline{\underline{\det A \neq 0}}$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ are unknowns, and $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

- The solution was given by $x = A^{-1}\alpha$, and the condition $\det A \neq 0$ was very important, since it guarantees that A^{-1} exists.

- We will now mention two other methods for solving such systems (with $\det A \neq 0$)

- Cramer's rule: The solution is given by

$$x_1 = \frac{D_1}{D}, \dots, \quad x_n = \frac{D_n}{D},$$

where $D = \det A$ and D_k is the determinant obtained from D by replacing the k-th column by the column α

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– Example:

$$x+y+z=1$$
$$2x+y+z=2$$
$$x+2y+z=3$$

$$A=\begin{bmatrix}1&1&1\\2&1&1\\1&2&1\end{bmatrix} \qquad \alpha=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$D=\det A=\begin{vmatrix}1&1&1\\2&1&1\\1&2&1\end{vmatrix}=\begin{vmatrix}1&1\\2&1\end{vmatrix}-2\begin{vmatrix}1&1\\2&1\end{vmatrix}+\underbrace{\begin{vmatrix}1&1\\1&1\end{vmatrix}}_{0}=$$

$$=-1-2\cdot(-1)=1$$

$$D_1=\begin{array}{c}\begin{vmatrix}1&1&1\\2&1&1\\3&2&1\end{vmatrix}\\ \begin{array}{lll}\uparrow&&\\ \alpha&&\end{array}\end{array}=\begin{vmatrix}1&1\\2&1\end{vmatrix}-2\begin{vmatrix}1&1\\2&1\end{vmatrix}+3\underbrace{\begin{vmatrix}1&1\\1&1\end{vmatrix}}_{0}=$$

$$=-1-2\cdot(-1)=1 \Rightarrow x=\frac{D_1}{D}=1$$

$$D_2=\begin{array}{c}\begin{vmatrix}1&1&1\\2&2&1\\1&3&1\end{vmatrix}\\ \begin{array}{ccc}&\uparrow&\\ &\alpha&\end{array}\end{array}=\begin{vmatrix}2&1\\3&1\end{vmatrix}-2\begin{vmatrix}1&1\\3&1\end{vmatrix}+\begin{vmatrix}1&1\\2&1\end{vmatrix}=$$

$$=-1-2\cdot(-2)-1=2 \Rightarrow y=\frac{D_2}{D}=2$$

$$D_3=\begin{array}{c}\begin{vmatrix}1&1&1\\2&1&2\\1&2&3\end{vmatrix}\\ \begin{array}{rrr}&&\uparrow\\ &&\alpha\end{array}\end{array}=\begin{vmatrix}1&2\\2&3\end{vmatrix}-2\begin{vmatrix}1&1\\2&3\end{vmatrix}+\begin{vmatrix}1&1\\1&2\end{vmatrix}=$$

$$=-1-2\cdot(1)+1=-2 \Rightarrow z=\frac{D_3}{D}=-2$$

– we can check if the solution is OK

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- Second method is Gauss' Method of Elimination

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned}$$

- We will eliminate x_1 from equations 2-n using equation 1 by multiplying it by $-\frac{a_{21}}{a_{11}}$ and adding it to the second equation; multiplying it by $-\frac{a_{31}}{a_{11}}$ and adding it to the third equation etc.

- Example:

$$\begin{aligned} x+y+z&=1 \;/\cdot\left(-\tfrac{2}{1}\right) \\ 2x+y+z&=2 \\ x+2y+z&=3 \end{aligned}$$

$$\begin{aligned} -\cancel{2x}-2y-2z&=-2 \\ \cancel{2x}+y+z&=2 \quad + \\ \hline -y-z&=0 \end{aligned}$$

- Now the third equation

$$\begin{aligned} x+y+z&=1 \;/\cdot\left(-\tfrac{1}{1}\right) \\ x+2y+z&=3 \end{aligned}$$

$$\begin{aligned} -x-y-z&=-1 \\ x+2y+z&=3 \quad + \\ \hline y&=2 \end{aligned}$$

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- After this process is finished, we will consider the system of $n-1$ equations (obtained equations $2-n$ after the addition) in $n-1$ unknowns $(x_2, .., x_n)$; now x_2 will be eliminated in the same manner

- Example:

$$\begin{aligned} -y - z &= 0 \;/\cdot\left(-\tfrac{1}{-1}\right) \\ y &= 2 \end{aligned} \qquad \begin{aligned} -y - z &= 0 \\ y &= 2 \quad + \\ \hline -z &= 2 \end{aligned}$$

- After the process is repeated $n-1$ times, we have only one equation in one unknown; we can solve it easy, and then use the obtained value to solve first equation above for another unknown etc.

- Example

$$\begin{aligned} x + y + z &= 1 \\ -y - z &= 0 \\ -z &= 2 \Rightarrow z = -2 \end{aligned}$$

$$y = -z = 2$$

$$x = 1 - y - z = 1$$

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– Example:

$$2x + 3y + z = 2 \;/\cdot\left(-\tfrac{3}{2}\right) \qquad -3x - \tfrac{9}{2}y - \tfrac{3}{2}z = -3$$
$$3x + 2y + 2z = 1 \qquad\qquad\qquad\qquad 3x + 2y + 2z = 1 \quad +$$
$$x + y - z = 2 \qquad\qquad\qquad\qquad -\tfrac{5}{2}y + \tfrac{1}{2}z = -2$$

$$2x + 3y + z = 2 \;/\cdot\left(-\tfrac{1}{2}\right) \qquad -x - \tfrac{3}{2}y - \tfrac{1}{2}z = -1$$
$$x + y - z = 2$$
$$-\tfrac{1}{2}y - \tfrac{3}{2}z = 1$$

The system is now

$$2x + 3y + z = 2$$
$$-\tfrac{5}{2}y + \tfrac{1}{2}z = -2$$
$$-\tfrac{1}{2}y - \tfrac{3}{2}z = 1$$

We will now eliminate y from the third equation by using the second equation

$$-\tfrac{5}{2}y + \tfrac{1}{2}z = -2 \;/\cdot\left(-\frac{-\frac{1}{2}}{-\frac{5}{2}}\right) = -\tfrac{1}{5} \qquad \tfrac{1}{2}y - \tfrac{1}{10}z = \tfrac{2}{5} \quad +$$
$$-\tfrac{1}{2}y - \tfrac{3}{2}z = 1$$
$$-\tfrac{8}{5}z = \tfrac{7}{5}$$

$$-\tfrac{1}{10} - \tfrac{3}{2} = -\tfrac{1}{10} - \tfrac{3\cdot 5}{2\cdot 5} = -\tfrac{16}{10} = -\tfrac{8}{5}$$

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The system becomes

$$2x + 3y + z = 2$$
$$-\frac{5}{2}y + \frac{1}{2}z = -2$$
$$-\frac{8}{5}z = \frac{7}{5} \Rightarrow z = -\frac{7}{8}$$

$y = -\frac{2}{5}\left(-2 - \frac{1}{2}z\right)$

$x = \frac{1}{2}(2 - z - 3y)$

$$y = -\frac{2}{5}\left(-2 - \frac{1}{2}\cdot\left(-\frac{7}{8}\right)\right) = -\frac{2}{5}\left(-2 + \frac{7}{16}\right) = -\frac{2}{5}\left(\frac{-32+7}{16}\right) =$$
$$= \frac{2}{5}\cdot\frac{25}{16} = \frac{5}{8}$$

$$x = \frac{1}{2}\left(2 + \frac{7}{8} - 3\cdot\frac{5}{8}\right) = \frac{1}{2}\ \frac{16+7-15}{8} = \frac{1}{2}\frac{8}{8} = \frac{1}{2}$$

The solution is $x = \frac{1}{2}$, $y = \frac{5}{8}$, $z = -\frac{7}{8}$

We can check if it is OK

- If we have more equations than unknowns, we will choose some of the equations (the same number as the number of unknowns), solve it, and then check if remaining equations are satisfied; if not, the system doesn't have the solution; if yes, we found the solution

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- If we have more unknowns than equations, we will choose some of the unknowns (the same number as the number of equations), and put all other unknowns on the right side of all equation; then we will solve the obtained system; effectively, we will express chosen unknowns in terms of other unknowns, from the right side of equations
- System of m equations in n unknowns

  1) $m > n$ } we just discussed this
  2) $m < n$ }

  3) $m = n \wedge \det A \neq 0$: Cramer, Gauss,
     $x = A^{-1}\alpha$

  4) $m = n \wedge \det A = 0$?

- To solve this case, we will have to define a rank of a matrix
- A matrix mxn is said to be of rank r if it contains at least one rxr submatrix with nonvanishing determinant, while the

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determinant of any $(r+1)\times(r+1)$ submatrix is zero.

- Theorem: $r \le \min(m, n)$
- Theorem: $(m = n \wedge \det A \neq 0) \Rightarrow r = m = n$
- Theorem $(m = n \wedge \det A = 0) \Rightarrow \begin{matrix} r < m \\ r < n \end{matrix}$

- Example

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix} \qquad \det A = \begin{vmatrix} 3 & 0 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -2 & -1 \\ 2 & 4 \end{vmatrix} +$$

$$+2\begin{vmatrix} -2 & -1 \\ 3 & 0 \end{vmatrix} = 12 + 3\cdot(-6) + 2\cdot(3) =$$

$$= 0$$

case $m = n$ and $\det A = 0$

submatrix $\begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix}$ has a determinant

$\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} = 3 + 6 = 9 \neq 0$, so the rank of A is 2: $\operatorname{rank}(A) = 2$

Try other submatrices.

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- Now we are back at the case
  $m = n$ equations in $m = n$ unknowns, with
  $\det A = 0$
- Theorem: If $\operatorname{rank}(A) = r$, then we can rewrite the system so that

$$\det R = \det \begin{bmatrix} a_{11} & \dots & a_{1r} \\ \vdots & & \\ a_{r1} & \dots & a_{rr} \end{bmatrix} \neq 0;$$

the unknowns $x_{r+1}, \dots, x_n$ ($n-r$ unknowns) are arbitrary, and we can put them on the right hand side:

$$a_{11} x_1 + \dots + a_{1r} x_r = -a_{1r+1} x_{r+1} - \dots - a_{1n} x_n + \alpha_1$$

$$\vdots$$

$$a_{r1} x_1 + \dots + a_{rr} x_r = -a_{rr+1} x_{r+1} - \dots - a_{rn} x_n + \alpha_n;$$

now we can consider $x_1, \dots, x_r$ as new unknowns, and solve the above system for these variables. This is the final solution, $x_1, \dots, x_r$ solved and expressed in terms of $x_{r+1}, \dots, x_m$.

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– Example

1) $x+y+z=1$
$2x+y+z=1$
$x+y+z=1$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\det A = 0$, since 1^{st} and 3^{rd} rows are proportional

$\operatorname{rank}(A) = 2$, since $R = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ has $\det R = 1-2 = -1 \neq 0$

now we are solving

$\left.\begin{array}{l} x+y = 1-z \\ 2x+y = 1-z \end{array}\right]$ subtracting first from the second we obtain $x=0$

entering $x=0$ into first (or second) equati[on] we will get $y = 1-z$

<u>Solution: $x=0$, $y=1-z$</u>

<u>unknown z is arbitrary</u>

We can see that this solution is satisfying the third equation

$$x+y+z = 0 + 1 - z + z = 1$$

So, we can generate as many solutions as we want, varying the value of z

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– Homework I: Find the solution of the following systems of linear equations:

1)
$$\begin{aligned} 2x+y&=7\\ 3x-2y&=0 \end{aligned}$$

2)
$$\begin{aligned} 2x-y+3z&=8\\ -x+2y+z&=4\\ 3x+y-4z&=0 \end{aligned}$$

3)
$$\begin{aligned} -x+2y+z&=3\\ 2x+3y-z&=-2\\ x-y+2z&=9 \end{aligned}$$

4)
$$\begin{aligned} 2x-y&=0\\ x+y-z&=0 \end{aligned}$$

5)
$$\begin{aligned} -4x+3y-z&=0\\ 8x+y+2z&=0 \end{aligned}$$

6)
$$\begin{aligned} -2w+7x+y&=0\\ 3w-3x-y-z&=0\\ w+5x+2y-5z&=0 \end{aligned}$$

7)
$$\begin{aligned} -2x+2y-3z&=-2\\ -x+y-2z&=6\\ -x-y-z&=-1\\ 3x+y+z&=-1 \end{aligned}$$

8)
$$\begin{aligned} 3x+y+z&=8\\ -x+y-2z&=-5\\ 2x+2y+2z&=12\\ -2x+2y-3z&=-7 \end{aligned}$$

Try to use different methods!!

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- Bilinear form in $2n$ variables $x_1, .., x_n$ and $y_1, .., y_n$ is the expression of the form

$$B = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} y_j =$$

$$= x_1 a_{11} y_1 + x_1 a_{12} y_2 + \cdots + x_1 a_{1n} y_n +$$
$$+ x_2 a_{21} y_1 + \cdots \qquad + x_2 a_{2n} y_n +$$
$$+ \vdots \qquad \vdots$$
$$+ x_n a_{n1} y_1 + \quad \cdots \quad + x_n a_{nn} y_n$$

- Using matrix notation, we can write

$$B = x^T A y, \quad \text{where}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} \cdots & a_{1n} \\ \vdots & \\ a_{n1} \cdots & a_{nn} \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

- If $x = y$, then $B = Q$ is quadratic form in variables $x_1, .., x_n$:

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$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j =$$

$$= a_{11} x_1^2 + (a_{12} + a_{21}) x_1 x_2 + \dots + (a_{1n} + a_{n1}) x_1 x_n +$$
$$+ a_{22} x_2^2 + \dots\dots + (a_{2n} + a_{n2}) x_2 x_n +$$
$$+ \dots\dots\dots + $$
$$\ddots$$
$$+ a_{nn} x_n^2$$

– If we introduce matrix C with elements

$$C_{ij} = \frac{1}{2} (a_{ij} + a_{ji}), \quad C = \frac{1}{2}(A + A^T)$$

it is symmetric: $C^T = C$, and

$$Q = x^T C x$$

– Example: $Q = 2x_1^2 + x_1 x_2 - 3x_2^2$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}, \quad C = \frac{1}{2}\left(\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix} \right) =$$

$$= \frac{1}{2} \begin{bmatrix} 4 & 1 \\ 1 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 1/2 \\ 1/2 & -3 \end{bmatrix}$$ Lets check:

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$$Q = x^T C x = [x_1 \;\; x_2] \begin{bmatrix} 2 & 1/2 \\ 1/2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= \left[2x_1 + \tfrac{1}{2}x_2 \quad \tfrac{1}{2}x_1 - 3x_2\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + \tfrac{1}{2}x_1x_2 + \tfrac{1}{2}x_1x_2 - 3x_2^2 =$$

$$= 2x_1^2 + x_1x_2 - 3x_2^2 \quad \text{O.K.}$$

– Theorem: Quadratic form $Q = x^T C x$, where $C^T = C$ is symmetric $n \times n$ matrix and $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a real-valued column, is said to be positive definite, iff

$Q > 0$ for all values of $(x_1, \dots, x_n) \neq (0, \dots, 0)$

Q is negative definite if $Q < 0$ for all values of $(x_1, \dots, x_n) \neq (0, \dots, 0)$.

A necessary and sufficient condition for $Q > 0$ is that

$$\det[c_{11}] > 0, \; \det\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} > 0, \dots, \; \det C > 0.$$

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A necessary and sufficient condition for $Q<0$ is that

$$\det[c_{11}]<0,\ \det\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}>0,\ \det\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31}, & c_{32} & c_{33} \end{bmatrix}<0,$$

$$\ldots,\ \det C \begin{cases} <0, & n \text{ odd} \\ >0, & n \text{ even}. \end{cases}$$

– Example

$$Q = x_1^2 - 2x_1x_2 + x_2^2,\ A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$C = \frac{1}{2}(A+A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$n=2$

$$\det[c_{11}] = 1 > 0,\ \det C = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1-1=0$$

Q is nonnegative

$$Q = (x_1 - x_2)^2 \Rightarrow \begin{array}{ll} x_1 \neq x_2 & Q>0 \\ x_1 = x_2 & Q=0 \end{array}$$

but allways $Q \geq 0$

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- Homework II: Calculate matrices A and C and investigate definiteness of ~~a~~ quadratic forms:

1) $Q = 3x_1^2 - 2x_1x_2 + 4x_2^2$

2) $4x_1x_2 - 5x_2^2$

3) $4x_1x_3 + 2x_2x_3 + x_3^2$

4) $5x_1^2 - 2x_1x_2 + x_2^2$

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