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$f(x)$	$f'(x)$
x^a	$a x^{a-1}$
$\ln x$	$1/x$
e^x	e^x
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1/\cos^2 x$
$\cotan x$	$-1/\sin^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1/\sinh^2 x$
$\cotan x$	$-1/\cosh^2 x$

etc.

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- Rules for derivations: $f'(x) = \frac{df(x)}{dx}$

$$\left(f_1(x) \pm f_2(x)\right)' = f_1'(x) \pm f_2'(x), \quad \left(Cf(x)\right)' = Cf'(x)$$

$$\left(f_1(x) \cdot f_2(x)\right)' = f_1'(x) \cdot f_2(x) + f_1(x) f_2'(x)$$

$$\left(\frac{f_1(x)}{f_2(x)}\right)' = \frac{f_1'(x) \cdot f_2(x) - f_1(x) \cdot f_2'(x)}{f_2^2(x)}$$

$$\left(f_1(f_2(x))\right)' = f_1'(f_2(x)) \cdot f_2'(x)$$

- Examples:

$$(x^2)' = (x \cdot x)' = 1 \cdot x + x \cdot 1 = 2x$$

$$(x^2)' = 2 \cdot x^{2-1} = 2x$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

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$$\left(\ln(\sin x)\right)' = \frac{1}{\sin x} \cdot \cos x = \cot x$$

$$e^{\sin x} \cdot \tan(x^2) = e^{\sin x} \cdot \cos x \cdot \tan(x^2) +$$

$$+ e^{\sin x} \cdot \frac{1}{\cos^2(x^2)} \cdot 2x$$

$$\left(\sqrt{\cosh(x^3)}\right)' = \left(\left(\cosh(x^3)\right)^{1/2}\right)' = \frac{1}{2}\left(\cosh(x^3)\right)^{1/2-1} \cdot \sinh(x^3) \cdot 3 \cdot x^{3-1} =$$

$$= \frac{3}{2} \quad \frac{x^2 \sinh(x^3)}{\sqrt{\cosh(x^3)}}$$

— If we have a function that depends on several variables, e.g.

$f(x,y) = x+y$, or $f(x,y,z) = xyz$, then we have different possibilities

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to define first derivative

- We will here call them partial derivatives, and denote ~~them~~ them

$$\frac{\partial f(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y}$$

- When we are calculating partial derivative corresponding to one variable, we will assume that all other variables are constant:

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y) - f(x,y)}{\varepsilon}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\varepsilon \to 0} \frac{f(x, y+\varepsilon) - f(x,y)}{\varepsilon}$$

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- The same rule applies when we are calculating partial derivatives using the table:

$$\frac{\partial}{\partial x}\left(xyz\right) = \underbrace{yz}_{\text{constant}}\frac{\partial}{\partial x}(x) = yz(x)' = yz$$

$$\frac{\partial}{\partial y}\left(x^2 \cdot \sin y\right) = x^2 \frac{\partial}{\partial y}\sin y = x^2 \frac{d}{dy}\sin y = x^2 \cos y$$

- Second (n^{th}) derivative is first derivative of the first ($n-1^{th}$) derivative

$$f''(x) = \frac{d^2 f(x)}{dx^2}, \quad f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$$

$$f''(x) = \frac{df'(x)}{dx}, \quad f^{(n)}(x) = \frac{df^{(n-1)}(x)}{dx}$$

- The same applies to partial derivatives

$$\frac{\partial^2 f(x,y,z)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

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Homework IV: Find the derivatives:

1) $f(x) = \frac{\sin x}{x}$, $\frac{df}{dx} = ?$

2) $f(x) = \ln(\tan(\sqrt{x}))$, $\frac{df}{dx} = ?$, $\frac{d^2f}{dx^2} = ?$

3) $f(x) = a^{\sin x}\sqrt{\sinh(\alpha x)}$, $\frac{df}{dx} = ?$ $\frac{d^2f}{dx^2} = ?$

4) $f(x) = x^n$, $\frac{d^kf}{dx^k}$, $k \in \mathbb{N}$

5) $f(x,y,z) = x^2yz + \frac{1}{xyz}$, $\frac{\partial f}{\partial x} = ?$, $\frac{\partial f}{\partial y} = ?$, $\frac{\partial f}{\partial z} = ?$

6) $f(x,y) = e^{x^2+y^2}$, $\frac{\partial f}{\partial x} = ?$ $\frac{\partial f}{\partial y} = ?$, $\frac{\partial^2 f}{\partial x^2} = ?$ $\frac{\partial^2 f}{\partial x \partial y} = ?$ $\frac{\partial^2 f}{\partial y^2} = ?$

7) $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, $\frac{\partial^2 f}{\partial x \partial y} = ?$, $\frac{\partial^2 f}{\partial y \partial z} = ?$ $\frac{\partial^3 f}{\partial x \partial y \partial z} = ?$

8) $f(x,y) = \frac{\ln(x-y)}{\sqrt{x^2+y^2}}$ $\frac{\partial f}{\partial x} = ?$ $\frac{\partial^2 f}{\partial x^2} = ?$ $\frac{\partial f}{\partial y} = ?$ $\frac{\partial^2 f}{\partial y^2} = ?$ $\frac{\partial^2 f}{\partial x \partial y} = ?$

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– Taylor's Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) +$$

$$+ \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

– Definition of a series:

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n$$

We will use Taylor's theorem in a form

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0) + \sigma((x-x_0)^{n+1}),$$

with the meaning that if we approximate the function f with the first $n+1$ terms, the error will be proportional to the $(x-x_0)^{n+1}$. So, if $x-x_0$ is a small number, the error will be even smaller.

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- Of course, there are some limitations for this formula: it is valid only for some subset of the values of the domain of the function f
- There is a theorem that allows us to find all the values of x for which Taylor's theorem is valid for every function f, but we will not go into these details: we will assume that this information is known to us, because there are tables of Taylor's series for different functions, containing all the necessary things.
- If we choose $x_0 = 0$, i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

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or

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^{n+1}),$$

then we will call this type of series

Maclaurin series.

– Table of Series for elementary functions:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+\cdots, \quad \underline{|x|<1}$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots, \quad \forall x\in\mathbb{R}$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots, \quad \forall x\in\mathbb{R}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots, \quad \forall x\in\mathbb{R}$$

$$\ln(x+1)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots, \quad |x|<1$$

$$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots, \quad \forall x\in\mathbb{R}$$

$$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots, \quad \forall x\in\mathbb{R}$$

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- Solutions of Homework I, given on December 26, 2003.

1) $x=2,\ y=3$

2) $x=2,\ y=2,\ z=2$

3) $x=1,\ y=0,\ z=4$

4) $x=\frac{z}{3},\ y=\frac{2z}{3},\ z$ arbitrary

5) $x=-\frac{z}{4},\ y=0,$ z arbitrary

6) $w=z,\ x=0,\ y=2z,$ z arbitrary

7) no solution!

8) $x=1,\ y=2,\ z=3$

- Solutions of Homework II, given on December 26, 2003.

1) $A=\begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix},\ C=\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix},$ $Q>0$

2) $A=\begin{bmatrix} 0 & 4 \\ 0 & -5 \end{bmatrix},\ C=\begin{bmatrix} 0 & 2 \\ 2 & -5 \end{bmatrix},$ Q can be positive, $=0$, negative

3) $A=\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix},$ Q can be positive, $=0$, negative

4) $A=\begin{bmatrix} 5 & -2 \\ 0 & 1 \end{bmatrix},\ C=\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix},$ $Q>0$

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