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FACULTY OF ECONOMICS

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South European Center for Contemporary Finance

QS, Dec 29, 2003 (11)

- Homework V: Find general and particular solutions to the given diff. equations:

1) $f' + 10f = 0$, $f(1) = -1$

2) $2ff' + 3f^2 = 0$, $f(0) = 1$

3) $f'' - 9f = 0$, $f(0) = 2$, $f'(0) = 3$

4) $f'' + 4f' + 5f = 0$, $f(0) = 2$, $f'(0) = 3$

5) $f'' + 4f' + 4f = 0$ $f(0) = 1$, $f'(0) = 2$

6)* $f' + 2f \ln f = 0$, $f(0) = e$

7)** $f''f - f'^2 + 2f'f = 0$ (only general solution)

*, ** Problems 6 and 7 require to tranform given diff. equation into the linear diff. equations with constant coefficient, by some transformation of dependent variable f!

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– Solutions of Homework $\underline{III}$, assigned on December 27, 2003.

1) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n^2-2n+2}{n^3} = \lim \left(\frac{1}{n} - \frac{2}{n^2} + \frac{2}{n^3}\right) = 0$

2) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\sin n}{n}$; $a_n = \frac{\sin n}{n}$

$b_n \leq a_n \leq c_n$, $b_n = \frac{-1}{n}$, $c_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \left(-\frac{1}{n}\right) = 0 = \lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \lim\limits_{n\to\infty} b_n \leq \lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} c_n$

$0 \leq \lim\limits_{n\to\infty} a_n \leq 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

3) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{e^n}{n} = \lim\limits_{n\to\infty} \frac{1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\cdots}{n} =$

$= \lim\limits_{n\to\infty} \left(\overset{\nearrow 0}{\cancel{\frac{1}{n}}} + 1 + \frac{n}{2!} + \frac{n^2}{3!} + \cdots\right)$ divergent, since it is not bounded

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4) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-1}{n}+\frac{n^2+2}{n^2}\right) = \lim_{n\to\infty}\left(1-\cancel{\frac{1}{n}}^{0}+1+\cancel{\frac{2}{n^2}}^{0}\right) =$

$= 2$

– Solutions of Homework $\underline{IV}$, assigned on December 27, 2003

1) $f(x) = \frac{\sin x}{x}$, $\frac{df}{dx} = \frac{(\cos x)\cdot x - \sin x\cdot 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$

2) $f(x) = \ln(\tan\sqrt{x})$, $f'(x) = \frac{1}{\tan\sqrt{x}}\cdot\frac{1}{\cos^2\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} =$

$$= \frac{\cos\sqrt{x}}{\sin\sqrt{x}}\cdot\frac{1}{\cos^2\sqrt{x}}\,\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sin\sqrt{x}\cos\sqrt{x}} = \frac{1}{\sqrt{x}\sin 2\sqrt{x}}$$

since $2\sin\alpha\cos\alpha = \sin 2\alpha$.

$$f''(x) = \frac{d}{dx}\left(x^{-1/2}\cdot(\sin 2\sqrt{x})^{-1}\right) = -\frac{1}{2}x^{-3/2}\cdot(\sin 2\sqrt{x})^{-1} +$$

$$+ x^{-1/2}\cdot(-1)(\sin 2\sqrt{x})^{-2}\cdot\cos 2\sqrt{x}\cdot\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x}} =$$

$$= \frac{-1}{2\sqrt{x^3}\sin 2\sqrt{x}} - \frac{\cos 2\sqrt{x}}{x\sin^2 2\sqrt{x}}$$

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3) $f(x) = a^{\sin x}\sqrt{\sinh(\alpha x)}$

$$f'(x) = a^{\sin x}\ln a \cdot \cos x \cdot \sqrt{\sinh(\alpha x)} + a^{\sin x}\frac{\cosh(\alpha x)\cdot \alpha}{2\sqrt{\sinh(\alpha x)}} =$$

$$= \frac{a^{\sin x}}{2\sqrt{\sinh(\alpha x)}}\left[2\ln a \cos x \sinh(\alpha x) + \alpha\cosh(\alpha x)\right]$$

$$f''(x) = \frac{a^{\sin x}\ln a \cdot \cos x \cdot 2\sqrt{\sinh(\alpha x)} - \frac{\alpha\cosh(\alpha x)}{\sqrt{\sinh(\alpha x)}}a^{\sin x}}{4\sinh(\alpha x)} \cdot$$

$$\cdot\left[2\ln a \cos x \sinh \alpha x + \alpha\cosh(\alpha x)\right] +$$

$$+ \frac{a^{\sin x}}{2\sqrt{\sinh(\alpha x)}}\Big[2\ln a\,(-\sin x)\sinh(\alpha x) + 2\ln \alpha \cos x \cdot \cosh(\alpha x)\cdot\alpha +$$
$$+ \alpha\sinh(\alpha x)\cdot\alpha\Big]$$

4) $f(x) = x^n$ $\quad \frac{df}{dx} = n x^{n-1}, \quad \frac{d^2 f}{dx^2} = n(n-1)x^{n-2}, \ldots$

$$\ldots \frac{d^n f}{dx^n} = n(n-1)\cdots 1\cdot x^0 = n!, \quad \frac{d^{n+1} f}{dx^{n+1}} = 0 \Rightarrow$$

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$$\frac{d^k}{dx^k}(x^n) = \begin{cases} \frac{n!}{(n-k)!}x^{n-k}, & k \le n \\ 0, & k > n \end{cases}$$

5) $f(x,y,z) = x^2yz + \frac{1}{xyz}$

$$\frac{\partial f}{\partial x} = 2xyz - \frac{1}{x^2yz}, \quad \frac{\partial f}{\partial y} = x^2z - \frac{1}{xy^2z},$$

$$\frac{\partial f}{\partial z} = x^2y - \frac{1}{xyz^2}$$

6) $f(x,y) = e^{x^2+y^2}$, $\frac{\partial f}{\partial x} = e^{x^2+y^2}\cdot 2x$, $\frac{\partial f}{\partial y} = e^{x^2+y^2}\cdot 2y$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = e^{x^2+y^2}\cdot 2x\cdot 2x + e^{x^2+y^2}\cdot 2 = 2e^{x^2+y^2}(2x^2+1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^{x^2+y^2}\cdot 2x\cdot 2y = 4xy\,e^{x^2+y^2} \quad \text{or}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = e^{x^2+y^2}\cdot 2y\cdot 2x = 4xy\,e^{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = e^{x^2+y^2}\cdot 2y\cdot 2y + e^{x^2+y^2}\cdot 2 = 2e^{x^2+y^2}(2y^2+1)$$

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7) $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$\frac{\partial f}{\partial x} = \frac{1}{y} - \frac{z}{x^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{y^2}$$

$$\frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = -\frac{1}{z^2}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial z}\right) = 0$$

8) $f(x,y) = \frac{\ln(x-y)}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial x} = \frac{\frac{\sqrt{x^2+y^2}}{x-y} - \ln(x-y)\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} =$

$$= \frac{1}{(x-y)\sqrt{x^2+y^2}} - \frac{x\ln(x-y)}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{-\sqrt{x^2+y^2} - \frac{(x-y)x}{\sqrt{x^2+y^2}}}{(x-y)^2(x^2+y^2)} -$$

$$- \frac{\left(\ln(x-y) + \frac{x}{x-y}\right)(x^2+y^2)^{3/2} - x\ln(x-y)\cdot 3x\sqrt{x^2+y^2}}{(x^2+y^2)^3}$$

$$\frac{\partial f}{\partial y} = \frac{-\frac{\sqrt{x^2+y^2}}{x-y} - \ln(x-y)\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{-1}{(x-y)\sqrt{x^2+y^2}} - \frac{y\ln(x-y)}{(x^2+y^2)^{3/2}}$$

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$$\frac{\partial^2 f}{\partial y^2} = \frac{-\sqrt{x^2+y^2} - \frac{(x-y)y}{\sqrt{x^2+y^2}}}{(x-y)^2(x^2+y^2)} - $$

$$- \frac{\left(\ln(x-y) - \frac{y}{x-y}\right)(x^2+y^2)^{3/2} - y\ln(x-y)\cdot 3y\sqrt{x^2+y^2}}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{(x-y)\sqrt{x^2+y^2}} - \frac{x\ln(x-y)}{(x^2+y^2)^{3/2}}\right) =$$

$$= \frac{\sqrt{x^2+y^2} - (x-y)\frac{y}{\sqrt{x^2+y^2}}}{(x-y)^2(x^2+y^2)} - \frac{\frac{-x}{x-y}(x^2+y^2)^{3/2} - x\ln(x-y)\cdot 3y\sqrt{x^2+y^2}}{(x^2+y^2)^3}$$

or

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-1}{(x-y)\sqrt{x^2+y^2}} - \frac{y\ln(x-y)}{(x^2+y^2)^{3/2}}\right) =$$

$$= \frac{\sqrt{x^2+y^2} + (x-y)\frac{x}{\sqrt{x^2+y^2}}}{(x-y)^2(x^2+y^2)} - \frac{\frac{y}{x-y}(x^2+y^2)^{3/2} - y\ln(x-y)\cdot 3x\sqrt{x^2+y^2}}{(x^2+y^2)^3}$$

you can check that the last two expressions are equal, and that they can simplify to

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{(x-y)^2\sqrt{x^2+y^2}} + \frac{1}{(x^2+y^2)^{3/2}} + \frac{3xy\ln(x-y)}{(x^2+y^2)^{5/2}}.$$