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- Indefinite integral is operation which is inverse to the first derivative:

$$f'(x) = g(x) \iff \int g(x)\,dx = f(x) + C$$

constant ← (C)

We can always add a constant to the solution of indefinite integral, since its first derivative $\frac{d}{dx} C = 0$.

- The rules:

$$\int (af(x) + b g(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

- Differential of a function

$$df(x) = f'(x)\,dx$$

- We have a table of integrals, inverse to the table of first derivatives:

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$f(x)$	$\int f(x)dx$
x^a	$\frac{x^{a+1}}{a+1}$
$\frac{1}{x}$	$\ln x$
e^x	e^x
a^x	$\frac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\frac{1}{\cos^2 x}$	$\tan x$
$-1/\sin^2 x$	$\text{cotan} x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$1/\sinh^2 x$	$\tanh x$
$-1/\cosh^2 x$	$\text{cotanh}\, x$

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- Two most powerful methods for solving integrals:
  1) Change of variable $x = g(y)$
  2) Partial integration

- change of variable $x = g(y)$: $dx = g'(y)dy$

$$\int f(x)dx = \int f(g(y))g'(y)dy$$

We will choose the function g so that the integral is simpler after the change.

- Example:

1) $\int x \sin(x^2)dx$; $x^2 = y \Rightarrow x = \sqrt{y}$

$$dx = \frac{1}{2\sqrt{y}}dy \Rightarrow$$

$$\int x \sin(x^2)dx = \int \sqrt{y} \cdot \sin y \cdot \frac{1}{2\sqrt{y}}dy = \underbrace{\frac{1}{2}\int \sin y \, dy}_{\text{from the table}} =$$

$$= \frac{1}{2}(-\cos y) + C = -\frac{1}{2}\cos x^2 + C$$

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2) $\int e^{\sin x} \cos x \, dx$; $\sin x = y$

$d(\sin x) = \cos x \, dx = dy \Rightarrow$

$$\int e^{\sin x} \cos x \, dx = \int e^{y} dy = e^{y} + C = e^{\sin x} + C$$

– Partial integration is based on a formula

$$d(uv) = (uv)' dx = (u'v + uv') dx =$$

$$= u' dx \cdot v + uv' dx = du \cdot v + u \cdot dv,$$

and by integrating it we will get

$$\int d(uv) = \int (uv)' dx = uv + C =$$

$$= \int du \cdot v + \int u \, dv \Rightarrow$$

$\int u \, dv = uv - \int v \, du$ (C can be omitted, since we have integral on the right-hand side)

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– Examples:

1) $\int x e^x dx = \underbrace{\int x \, de^x}_{\int u\,dv} = \underset{uv}{x e^x} - \underbrace{\int e^x dx}_{\int v\,du} =$

$= x e^x - e^x + C$

check: $\frac{d}{dx}\left(x e^x - e^x + C \right) = 1 \cdot e^x + x \cdot e^x - e^x = x e^x$

2) $\int x^2 e^x dx = \int x^2 de^x = x^2 e^x - \int e^x dx^2 =$

$= x^2 e^x - \int e^x 2x\,dx = x^2 e^x - 2\int x e^x dx =$

$= x^2 e^x - 2\left(x e^x - e^x \right) + C = x^2 e^x - 2x e^x + 2e^x + C$

3) $\int x \sin x \, dx = \int x \, d(-\cos x) = -x\cos x + \int \cos x \, dx =$

$= -x\cos x + \sin x + C$

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4) $\int x \ln x \, dx = \int \ln x \, d\left(\frac{1}{2}x^2\right) =$

$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot d(\ln x) =$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}\, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx =$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\frac{x^2}{2} + C = \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C$

check: $\frac{d}{dx}\left(\frac{1}{2}x^2 \ln x - \frac{x^2}{4} + C\right) = \frac{1}{2}\cdot 2x \cdot \ln x + \frac{1}{2}x^2 \cdot \frac{1}{x} -$

$- \frac{1}{4}\cdot 2x = x \ln x + \frac{1}{2}x - \frac{x}{2} = x \ln x$.

— Homework VI: Find the integrals:

1) $\int 2x^2 \ln x \, dx$ 2) $\int \sin(x+1)\, dx$

3) $\int x \sin(x+1)\, dx$ 4) $\int \sqrt{y}\, \sinh(\alpha\sqrt{y})\, dy$

5) $\int \frac{\ln x}{x}\, dx$ 6) $\int \frac{x^2-3x+2}{x+1}\, dx$

7) $\int x^3 \cos x \, dx$ 8) $\int \ln(1+x^2)\, dx$, if we know $\int \frac{dx}{1+x^2} = \mathrm{Arctan}\, x$.

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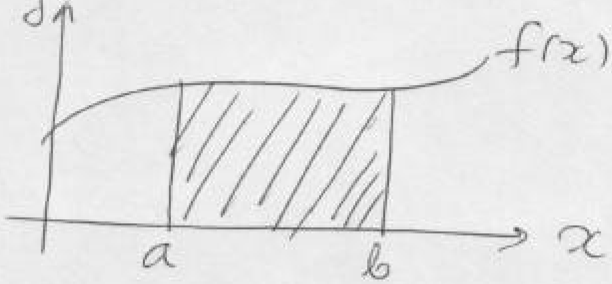
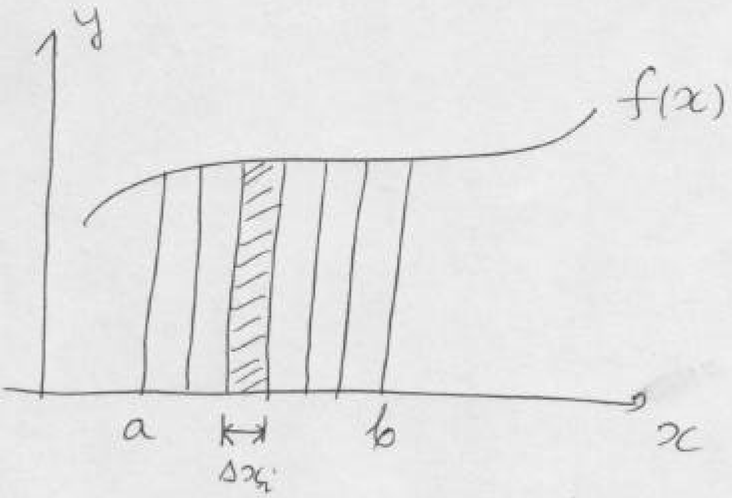
- Definite integrals are something completely different:

$$\int_a^b f(x)\,dx =$$



- We can try to calculate them numerically:



We can divide interval $[a, b]$ into n parts, each of the width Δx_i, $i = 1, \ldots, n$, and approximate the integral with the sum

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\cdot\Delta x_i ,$$ where x_i belongs to i-th subinterval

In fact, $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$, if all $x_i \to 0$.

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- But, we have alternative way, thanks to mathematicians

- If we know indefinite integral

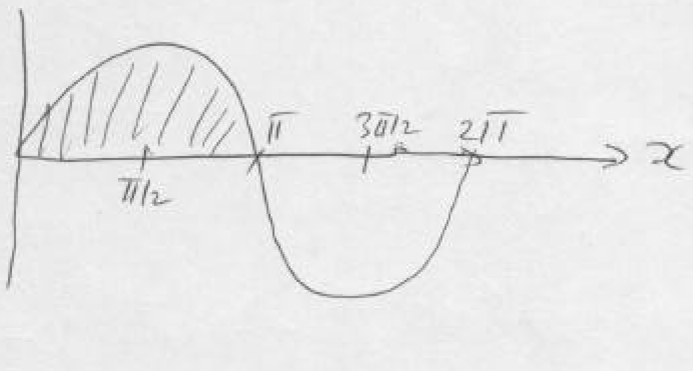
$$\int f(x)\,dx = g(x) + C,$$

then

$$\int_a^b f(x)\,dx = g(x)\Big|_a^b = g(b) - g(a)$$

- This is very important theorem.

- Example

1) $\int_0^{\pi} \sin x\,dx =$ 

$=$

$$= -\cos x\Big|_0^{\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = 2$$

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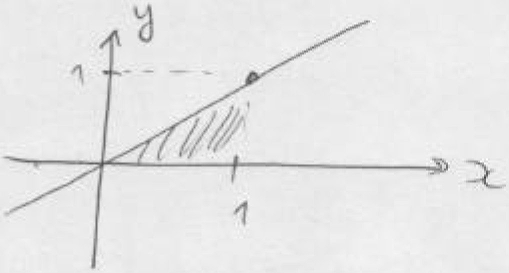
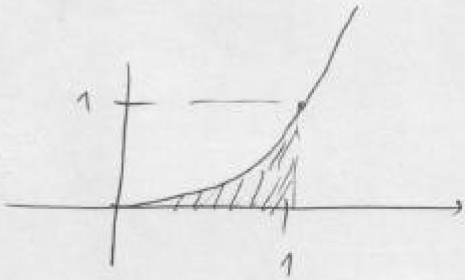
2) $\int_0^1 x\,dx =$



$= \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

We know that this integral is $\frac{1 \cdot 1}{2} = \frac{1}{2}$, since it is a well known formula for a triangle

3) $\int_0^1 x^2\,dx =$



$= \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

If we want to change independent variable, we also <u>must</u> change the points a and b!

4) $\int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e - 1 \approx 1.718281\ldots$

– Homework VII: Find the integrals:

1) $\int_{-1}^{1} x\sin x\,dx$

2) $\int_0^{\infty} e^{-x}\,dx$

3) $\int_{-1}^{1} x^2 e^x\,dx$

4) $\int_2^3 \ln^2 x\,dx$

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- There are also multiple integrals, both indefinite and definite.

$$\int \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$\int_{\Omega} \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$, where Ω is some area in n-dimensional space

- Change of independent variables:

$$\left.\begin{array}{l} x_1 \rightarrow y_1 = y_1(x_1, \ldots, x_n) \\ \vdots \\ x_n \rightarrow y_n = y_n(x_1, \ldots, x_n) \end{array}\right\}$$

$$F(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n)$$

$$\Omega \longrightarrow \Omega'$$

$$\int_{\Omega} \cdots \int f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \int_{\Omega'} \cdots \int F(y_1, \ldots, y_n)\, \dot{J}\, dy_1 \cdots y_n$$

$$\dot{J} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$ is Jacobian.

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