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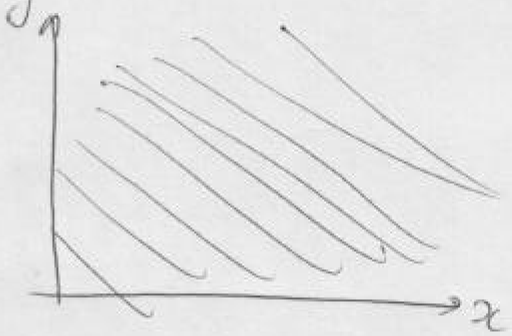
– Gauss' Integral

$$\int_0^\infty e^{-x^2}\,dx = I$$

It can't be calculated in a standard way, but we will do some tricks

$$I^2 = \int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy$$



Area of integration is first quadrant

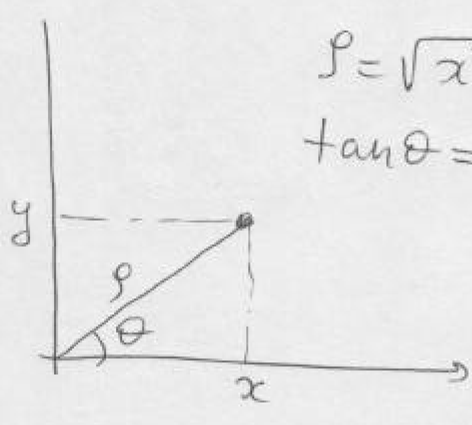
We will use polar coordinates:

$x = \rho \cos\theta$

$y = \rho \sin x$

$\rho = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$



$dx\,dy = \rho\, d\theta\, d\varphi$

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http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
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$$I^2 = \int_0^\infty d\rho \int_0^{\pi/2} d\theta \ \rho \cdot e^{-\rho^2} = \int_0^\infty \rho e^{-\rho^2} d\rho \underbrace{\int_0^{\pi/2} d\theta}_{\theta \big|_0^{\pi/2} = \frac{\pi}{2}} \Rightarrow$$

from $dxdy$ (Jacobian)

$$I^2 = \frac{\pi}{2} \cdot \int_0^\infty e^{-\rho^2} d\left(\frac{1}{2}\rho^2\right) = \frac{\pi}{4}\int_0^\infty e^{-\rho^2} d\rho^2 =$$

$$= \frac{\pi}{4}\left(-e^{-\rho^2}\right)\Big|_0^\infty = \frac{\pi}{4} \Rightarrow I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

If we are calculating $\int_{-\infty}^{\infty} e^{-x^2} dx$, we will get $\sqrt{\pi}$.

– Other forms of Gauss' Integral

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx, \quad ax^2 = y^2 \Rightarrow x = \frac{y}{\sqrt{a}}$$

$$dx = \frac{dy}{\sqrt{a}}$$

$$= \int_{-\infty}^{\infty} e^{-y^2} \frac{dy}{\sqrt{a}} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}},$$

Obviously, $a > 0$.

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$$I(a,b,c)=\int_{-\infty}^{\infty} e^{-ax^2-bx-c}\,dx \quad , \quad a>0$$

$$I(a,b,c)=\int_{-\infty}^{\infty} e^{-a\left(x^2+\frac{b}{a}x\right)-c}\,dx =$$

$$= e^{-c}\int_{-\infty}^{\infty} e^{-a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]+a\cdot\left(\frac{b}{2a}\right)^2}\,dx =$$

$$= e^{\frac{b^2}{4a}-c}\int_{-\infty}^{\infty} e^{-a\left(x+\frac{b}{2a}\right)^2}\,dx$$

$$x+\frac{b}{2a}=y, \quad dx=dy$$

$$I(a,b,c)=e^{\frac{b^2}{4a}-c}\underbrace{\int_{-\infty}^{\infty} e^{-ay^2}\,dy}_{I(a)}=e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}$$

– We can even calculate $\int_{-\infty}^{\infty} x^n e^{-ax^2-bx-c}\,dx$,

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since

$$\frac{\partial I(a,b,c)}{\partial b} = \frac{\partial}{\partial b}\int_{-\infty}^{\infty} e^{-ax^2-bx-c}\,dx =$$

$$= \int_{-\infty}^{\infty} (-x)\, e^{-ax^2-bx-c}\,dx = -\int_{-\infty}^{\infty} x\, e^{-ax^2-bx-c}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} x^n e^{-ax^2-bx-c}\,dx = (-1)^n \frac{\partial^n I(a,b,c)}{\partial b^n} =$$

$$= (-1)^n \frac{\partial^n}{\partial b^n}\left(e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}\right)$$

– Example:

$$\int_{-\infty}^{\infty} x\, e^{-ax^2-bx-c}\,dx = -\frac{\partial}{\partial b}\left(e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}}\right) =$$

$$= -e^{-c}\sqrt{\frac{\pi}{a}}\,\frac{\partial}{\partial b}\left(e^{\frac{b^2}{4a}}\right) = e^{-c}\sqrt{\frac{\pi}{a}}\cdot e^{\frac{b^2}{4a}}\cdot\frac{2b}{4a} =$$

$$= \frac{b}{2a}\, e^{\frac{b^2}{4a}-c}\sqrt{\frac{\pi}{a}} = \frac{b}{2a}\cdot I(a,b,c)$$

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tel: 011 32 44 953, 011 32 40 345

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– We can now formulate general solution of Linear First-Order Differential Equation

$$f' + g(x) f = h(x)$$

using the integrals:

$$f(x) = e^{-\int g(x)dx} \cdot \left[\int e^{\int g(x)dx} h(x)\, dx + C \right]$$

– Example:

$$f' + x f = x \quad , \; g(x) = x, \; h(x) = x$$

$$\int g(x) dx = \int x\, dx = \frac{x^2}{2}$$

$$\int e^{\int g(x)dx} h(x)\, dx = \int e^{x^2/2} x\, dx =$$

$$= \int e^{x^2/2} d(x^2/2) = e^{x^2/2} \Rightarrow$$

$$f(x) = e^{-\frac{x^2}{2}} \left[e^{\frac{x^2}{2}} + C \right] = Ce^{-\frac{x^2}{2}} + 1$$

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check: $f' = Ce^{-\frac{x^2}{2}} \cdot (-x) = -Cxe^{-\frac{x^2}{2}}$

$$f' + xf = -Cxe^{-\frac{x^2}{2}} + Cxe^{-\frac{x^2}{2}} + x = x \quad \text{O.K.}$$

— Further remarks:

1) Linear n^{th}-order Differential Equation is Homogeneous, if it has a form

$$f^{(n)} + g_1(x) f^{(n-1)} + \cdots + g_n(x) f = 0 \qquad (1)$$

2) General solution of a Linear n^{th}-order Differential Equation

$$f^{(n)} + g_1(x) f^{(n-1)} + \cdots + g_n(x) f = h(x) \qquad (2)$$

has a form

$$f(x) = f_H(x) + f_p(x),$$

where $f_H(x)$ is a general solution of Homogeneous equation (1), a $f_p(x)$ is <u>any</u> particular solution of (2).

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— Example.

$$f' + 2f = x$$

Homogeneous equation: $f' + 2f = 0$

General solution: $f_H(x) = C e^{-2x}$

Particular solution? Let us try polynomial:

If $f = A_1 x^2 + A_2 x + A_3$, f' is $f' = 2A_1 x + A_2$,

so

$$f' + 2f = 2A_1 x + A_2 + 2A_1 x^2 + 2A_2 x + 2A_3$$

If we want this to be equal to x, we must fulfil

$$2A_1 = 0, \quad 2A_1 + 2A_2 = 1, \quad A_2 + 2A_3 = 0 \quad \Rightarrow$$

$$A_1 = 0, \quad A_2 = \frac{1}{2}, \quad A_3 = -\frac{1}{2} A_2 = -\frac{1}{4},$$

So, the particular solution is $f_P(x) = \frac{1}{2} x - \frac{1}{4}$,

and the general solution is

$$f(x) = C e^{-2x} + \frac{1}{2} x - \frac{1}{4}.$$

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– Homework $\underline{VIII}$: Solve the diff. equations:

1) $f' + \frac{1}{x} f = 2\sin x$ 2) $f' + e^{\sin x} f = \cos x$
$f(1) = 1$

3) $f'' - 2f' + f = x^2 + 1$ 4) $f'' - 2f' + 2f = x + 2$

– Solution of HW $\underline{V}$, assigned on December 29, 2003.

1) $f(x) = Ce^{-10x}$, $f(1) = Ce^{-10} = -1 \Rightarrow C = -e^{10}$

$f_p(x) = -e^{10} \cdot e^{-10x} = -e^{10(1-x)}$

2) $2ff' + 3f^2 = 0 \,/ : f \Rightarrow 2f' + 3f = 0 \Rightarrow$

$f' + \frac{3}{2} f = 0 \Rightarrow f(x) = Ce^{-\frac{3}{2}x}$

$f(0) = Ce^{-\frac{3}{2}\cdot 0} = C = 1 \Rightarrow f_p(x) = e^{-\frac{3}{2}x}$

3) $f'' - 9f = 0$ $\lambda^2 - 9 = 0 \Rightarrow \lambda_1 = 3,\ \lambda_2 = -3$

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$f(x) = C_1 e^{3x} + C_2 e^{-3x}$, $f'(x) = 3C_1 e^{3x} - 3C_2 e^{-3x}$

$f(0) = C_1 + C_2 = 2$ $\qquad C_1 + C_2 = 2$

$f'(0) = 3C_1 - 3C_2 = 3 \Rightarrow \qquad C_1 - C_2 = 1$

$C_1 = \frac{3}{2}, \; C_2 = \frac{1}{2}$

$f_p(x) = \frac{3}{2} e^{3x} + \frac{1}{2} e^{-3x}$

4) $f'' + 4f' + 5f = 0 \Rightarrow \lambda^2 + 4\lambda + 5 = 0$

$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$\lambda_1 = -2 + i, \; \lambda_2 = -2 - i$ $\qquad p = -2, \; q = 1$

$f(x) = e^{-2x}(A \sin x + B \cos x)$, $f'(x) = -2e^{-2x}(A \sin x + B \cos x) + e^{-2x}(A \cos x - B \sin x)$

$f(0) = B = 2,$

$f'(0) = -2B + A = 3 \Rightarrow A = 3 + 2B = 7$

$f_p(x) = e^{-2x}(7 \sin x + 2 \cos x)$

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5) $f'' + 4f' + 4f = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = -2$$

$$\Rightarrow f(x) = Ae^{-2x} + Bx\,e^{-2x}$$

$$f'(x) = -2Ae^{-2x} + Be^{-2x} - 2Bx\,e^{-2x}$$

$f(0) = A = 1$

$f'(0) = -2A + B = 2 \Rightarrow B = 2 + 2A = 4$

$$f_p(x) = e^{-2x} + 4x\,e^{-2x}$$

6)* $f' + 2f\ln f = 0 \;/ : f \Rightarrow \frac{f'}{f} + 2\ln f = 0$

If we introduce function $g(x) = \ln f(x)$, then

$g'(x) = \frac{f'(x)}{f}$, so the equation becomes

$g' + 2g = 0 \Rightarrow g(x) = Ce^{-2x} \Rightarrow$

$$f(x) = e^{g(x)} = e^{Ce^{-2x}}$$

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$$f(0) = e^{c} = e \Rightarrow c = 1$$

$$f_p(x) = e^{e^{-2x}}$$

7)** $f''f - f'^2 + 2f'f = 0$

If we find the first derivative of the previous equation, we will get

$$\frac{d}{dx}\left(\frac{f'}{f} + 2\ln f\right) = 0, \text{ or}$$

$$\frac{f'' \cdot f - f'^2}{f^2} + \frac{2f'}{f} = 0 \quad / \cdot f^2$$

$f''f - f'^2 + 2f'f = 0$, which is equation 7**

So, these two equations have the same solution.

General solution is

$$f(x) = e^{Ce^{-2x}}$$