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- Let us now consider cash flow streams

$$(x_0, x_1, \dots, x_n)$$

compounded yearly, with the interest rate r.

- If the flow is negative, we cover it by taking out a loan.

- Future value of a stream (at the end of n periods) is

$$FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \dots + x_{n-1}(1+r) + x_n.$$

- Example: $(-2, 1, 1, 1)$ $r = 10\%$.

$$FV = -2\cdot(1+0.1)^3 + 1\cdot(1+0.1)^2 + 1\cdot(1+0.1)^1 + 1 = 0.648.$$

- Present value of a stream

$$PV = x_0 + \frac{x_1}{1+r} + \dots + \frac{x_{n-1}}{(1+r)^{n-1}} + \frac{x_n}{(1+r)^n}.$$

- Theorem:

$$PV = \frac{FV}{(1+r)^n}.$$

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13

— Generalizations:

1) Suppose that r is nominal annual interest rate and interest is compounded at m equally spaced periods per year. Then,

$$PV = \sum_{k=0}^{n} \frac{x_k}{\left(1+\frac{r}{m}\right)^k}$$

$$FV = \sum_{k=0}^{n} x_k \left(1+\frac{r}{m}\right)^{n-k}$$

2) Suppose now that the nominal interest rate r is compounded continuously and cash flows occur at times $t_0, t_1, .., t_n$. ($t_k = k/m$)

In this case,

$$PV = \sum_{k=0}^{n} x(t_k)\, e^{-rt_k} = \sum_{k=0}^{n} x_k\, e^{-rt_k}$$

$$FV = \sum_{k=0}^{n} x(t_k)\, e^{r(t_n - t_k)}$$

Here, $PV = \dfrac{FV}{e^{rt_n}}$.

14

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- Main theorem on present value: The cash flow streams $x=(x_0,\dots,x_n)$ and $y=(y_0,\dots,y_n)$ are equivalent for a constant ideal bank with interest rate r iff the present values of these two streams are equal.

- This result implies that present value is the only number needed to characterize a cash flow stream.

- This allows us to compare different cash flow streams, and to choose most desirable one.

- Homework IX:

1) Check the Seven-Ten rule.
2) Find the FV and PV of cash flow streams $(-1,1,1,1,2)$ and $(-2,1,1,1,3)$ at the rate $r=10\%$.
3) Are the cash flow streams $(-1,1,1,2,1)$ and $(-1,0,1.1,1,2.1)$ equivalent at the rate $r=10\%$?

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(15)

- Solutions of HW $\overline{VI}$; assigned on December 30, 2003.

1) $\int 2x^2 \ln x \, dx = 2\int \ln x \, d\frac{x^3}{3} = \frac{2}{3}\int \ln x \, dx^3 =$

$= \frac{2}{3}\left(x^3 \ln x - \int x^3 \cdot \frac{1}{x} dx\right) = \frac{2}{3} x^3 \ln x - \frac{2}{3}\int x^2 dx =$

$= \frac{2}{3} x^3 \ln x - \frac{2}{3} \cdot \frac{x^3}{3} + C = \frac{2}{3} x^3 \ln x - \frac{2}{9} x^3 + C$

2) $\int \sin(x+1) dx = \int \sin(x+1) \, d(x+1) = -\cos(x+1) + C$

3) $\int x \sin(x+1) \, dx = \int x \sin(x+1) \, d(x+1)$

$y = x+1 \Rightarrow dy = dx$

$\int x \sin(x+1) dx = \int (y-1) \sin y \, dy = \int y \sin y \, dy -$

$- \int \sin y \, dy = \int y \, d(-\cos y) + \cos y =$

$= -y\cos y + \int \cos y \, dy + \cos y =$

$= -y\cos y + \sin y + \cos y + C =$

$= -(x+1)\cos(x+1) + \sin(x+1) + \cos(x+1) + C =$

$= -x\cos(x+1) + \sin(x+1) + C$

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4) $\int \sqrt{y}\,\sinh(\alpha\sqrt{y})\,dy = \int \frac{x}{\alpha}\sinh x \cdot \frac{2x\,dx}{\alpha^2} =$

$\alpha\sqrt{y} = x \Rightarrow y = \frac{x^2}{\alpha^2} \Rightarrow dy = \frac{2x\,dx}{\alpha^2}$

$= \frac{2}{\alpha^3}\int x^2 \sinh x\,dx = \frac{2}{\alpha^3}\int x^2\, d\cosh x = \frac{2}{\alpha^3}\Big(x^2\cosh x -$

$- \int \cosh x \cdot 2x\,dx\Big) = \frac{2}{\alpha^3}x^2\cosh x - \frac{4}{\alpha^3}\int x\cosh x\,dx =$

$= \frac{2}{\alpha^3}x^2\cosh x - \frac{4}{\alpha^3}\int x\,d\sinh x = \frac{2}{\alpha^3}x^2\cosh x - \frac{4}{\alpha^3}x\sinh x +$

$+ \frac{4}{\alpha^3}\int \sinh x\,dx = \frac{2}{\alpha^3}x^2\cosh x - \frac{4}{\alpha^3}x\sinh x + \frac{4}{\alpha^3}\cosh x + C$

$= \frac{2}{\alpha^3}\cdot y\alpha^2\cosh(\alpha\sqrt{y}) - \frac{4}{\alpha^3}\alpha\sqrt{y}\sinh(\alpha\sqrt{y}) + \frac{4}{\alpha^3}\cosh(\alpha\sqrt{y}) + C$

$= \frac{2y}{\alpha}\cosh(\alpha\sqrt{y}) - \frac{4\sqrt{y}}{\alpha^2}\sinh(\alpha\sqrt{y}) + \frac{4}{\alpha^3}\cosh(\alpha\sqrt{y}) + C =$

$= \left(\frac{2y}{\alpha} + \frac{4}{\alpha^3}\right)\cosh(\alpha\sqrt{y}) - \frac{4\sqrt{y}}{\alpha^2}\sinh(\alpha\sqrt{y}) + C.$

5) $\int \frac{\ln x}{x}\,dx = \int \ln x\,d\ln x = \frac{(\ln x)^2}{2} + C = \frac{\ln^2 x}{2} + C.$

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6) $\int \frac{x^2-3x+2}{x+1}\,dx =$ $\quad$ $x+1=y$, $dx=dy$, $x=y-1$

$$= \int \frac{(y-1)^2-3(y-1)+2}{y}\,dy = \int \frac{y^2-2y+1-3y+3+2}{y}\,dy =$$

$$= \int \frac{y^2-5y+6}{y}\,dy = \int \left(y-5+\frac{6}{y}\right) dy =$$

$$= \int y\,dy - 5\int dy + 6\int \frac{dy}{y} = \frac{y^2}{2} - 5y + 6\ln y + C =$$

$$= \frac{(x+1)^2}{2} - 5(x+1) + 6\ln(x+1) + C =$$

$$= \frac{1}{2}x^2 - 4x - \frac{9}{2} + 6\ln(x+1) + C$$

7) $\int x^3 \cos x\,dx = \int x^3\,d\sin x = x^3 \sin x - \int \sin x \cdot 3x^2\,dx =$

$$= x^3 \sin x - 3\int x^2\,d(-\cos x) = x^3\sin x + 3\int x^2\,d\cos x =$$

$$= x^3 \sin x + 3x^2\cos x - 3\int \cos x \cdot 2x\,dx =$$

$$= x^3\sin x + 3x^2\cos x - 6\int x\,d\sin x = x^3 \sin x +$$

$$+ 3x^2\cos x - 6x\sin x + 6\int \sin x\,dx =$$

$$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + C.$$

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8) $\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int x \cdot \frac{2x}{1+x^2}\,dx$,

since $d\ln(1+x^2) = \frac{1}{1+x^2}\cdot 2x \cdot dx \implies$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2\int \frac{x^2}{1+x^2}\,dx =$$

$$= x\ln(1+x^2) - 2\int \frac{1+x^2-1}{1+x^2}\,dx = x\ln(1+x^2) -$$

$$-2\int\left(1 - \frac{1}{1+x^2}\right)dx = x\ln(1+x^2) - 2x + 2\int \frac{dx}{1+x^2} =$$

$$= x\ln(1+x^2) - 2x + 2\,\mathrm{Arc}\tan x + C.$$

– Solution of HW $\underline{\text{VII}}$, assigned on December 30, 2003.

1) $\int_{-1}^{1} x\sin x\,dx = \int_{-1}^{1} x\,d(-\cos x) = -x\cos x\Big|_{-1}^{1} +$

$+\int_{-1}^{1} \cos x\,dx = -1\cdot\cos 1 + (-1)\cdot\cos(-1) + \sin x\Big|_{-1}^{1} =$

$\| \cos x = \cos(-x),\ \sin(-x) = -\sin x \|$

$$= -\cos 1 - \cos 1 + \sin 1 - \sin(-1) =$$

$$= -2\cos 1 + 2\sin 1 \approx 0.6$$

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2) $I=\int_0^{\infty} e^{-x}\,dx = ?$ $\qquad \int e^{-x}\,dx = -e^{-x} \Rightarrow$

$$I = \lim_{a\to\infty} \int_0^a e^{-x}\,dx = \lim_{a\to\infty}\left(-e^{-x}\Big|_0^a\right) =$$

$$= \lim_{a\to\infty}\left(-e^{-a}+e^{-0}\right) = \lim_{a\to\infty}\left(-e^{-a}+1\right) = 1.$$

3) $$\int_{-1}^{1} x^2 e^x\,dx = \int_{-1}^{1} x^2\,de^x = x^2e^x\Big|_{-1}^{1} - \int_{-1}^{1} e^x\cdot 2x\,dx =$$

$$= e - e^{-1} - 2\int_{-1}^{1} x\,de^x = e - e^{-1} - 2xe^x\Big|_{-1}^{1} +$$

$$+2\int_{-1}^{1} e^x\,dx = e - e^{-1} - 2e + 2\cdot(-1)e^{-1} +$$

$$+2e^x\Big|_{-1}^{1} = -e - 3e^{-1} + 2e - 2e^{-1} = e - 5e^{-1} \approx$$

$$\approx 0.88.$$

4) $$\int_2^3 \ln^2 x\,dx = x\ln^2 x\Big|_2^3 - \int_2^3 x\cdot 2\ln x\cdot\frac{1}{x}\,dx =$$

$$= 3\ln^2 3 - 2\ln^2 2 - 2\int_2^3 \ln x\,dx = 3\ln^2 3 - 2\ln^2 2 -$$

$$-2x\ln x\Big|_2^3 + 2\int_2^3 x\cdot\frac{1}{x}\,dx = 3\ln^2 3 - 2\ln^2 2 -$$

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$$-6\ln 3 + 4\ln 2 + 2x\Big|_2^3 = 3\ln^2 3 - 2\ln^2 2 - 6\ln 3 + 4\ln 2 + 2 \approx$$

$$\approx 0.84$$

- solution of HW VIII, assigned on December 30, 2003.

1) $f' + \frac{1}{x} f = 2\sin x$, $f(1) = 1$.

~~Homogeneous equation~~

$g(x) = \frac{1}{x}$, $h(x) = 2\sin x$

$$\int g(x)\,dx = \int \frac{1}{x}\,dx = \ln x \Rightarrow e^{-\int g(x)\,dx} = e^{-\ln x} = \frac{1}{x}$$

$$\int e^{\int g(x)} h(x)\,dx = \int x \cdot 2\sin x\,dx = 2\int x\,d(-\cos x) =$$

$$= -2x\cos x + 2\int \cos x\,dx = -2x\cos x + 2\sin x \Rightarrow$$

general solution

$$f(x) = e^{-\int g(x)\,dx}\left[\int e^{\int g(x)\,dx} h(x)\,dx + C\right] =$$

$$= \frac{1}{x}\left[2\sin x - 2x\cos x + C\right] = \frac{2\sin x}{x} - 2\cos x + C$$

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QS, Jan 5, 2004. (21)

$$f(1) = \frac{2\sin 1}{1} - 2\cos 1 + C = 2\sin 1 - 2\cos 1 + C = 1 \Rightarrow$$

$$C = 1 - 2(\sin 1 - \cos 1) \approx 1 - 2 \cdot 0.3 = 1 - 0.6 = 0.4$$

$$f_P(x) = \frac{2\sin x}{x} - 2\cos x + 0.4$$

2) $f' + e^{\sin x} f = \cos x$

$g(x) = e^{\sin x}$, $h(x) = \cos x$

$\int g(x)\,dx = \int e^{\sin x}\,dx$ = we cannot find this integral, so we cannot find the solution of this differential equation!

3) $f'' - 2f' + f = x^2 + 1$

homogeneous eq: $f'' - 2f' + f = 0$

$$\lambda^2 - 2\lambda + 1 = 0 \Leftrightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$$

$$f_H(x) = A e^x + B x e^x$$

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We will try to find particular solution in a form $f_p(x) = ax^2 + bx + c$

$f_p'(x) = 2ax + b$, $f_p''(x) = 2a \Rightarrow$

$f_p'' + 2f_p' + f_p = 2a + 2(2ax + b) + ax^2 + bx + c =$

$= \underline{x^2 + 1}$ (from the equation) $\Rightarrow$

$a = 1$, $4a + b = 0 \Rightarrow b = -4a = -4$,

$2a + 2b + c = 1 \Rightarrow c = 1 - 2a - 2b = 1 - 2 - 2\cdot(-4) =$

$= 1 - 2 + 8 = 7 \Rightarrow$

$f_p = x^2 - 4x + 7 \Rightarrow$ general solution is

$f(x) = f_H(x) + f_p(x) = Ae^{x} + Bxe^{x} + x^2 - 4x + 7$

4) $f'' - 2f' + 2f = x + 2$

$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} =$

$= 1 \pm i \Rightarrow \lambda_1 = 1 + i$, $\lambda_2 = 1 - i$

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$\Rightarrow p=1,\ q=1$

$$f_H(x) = e^{x}(A\cos x + B\sin x)$$

$f_p(x) = ax + b \quad , f_p' = a, \quad f_p'' = 0$

$$f_p'' - 2f_p' + 2f_p = -2a + 2(ax+b) = 2ax - 2a + 2b =$$

$$= x + 2 \quad (\text{from eq.}) \Rightarrow$$

$$2a = 1 \Rightarrow a = \tfrac{1}{2} \quad , \quad -2a + 2b = 2 \Rightarrow$$

$$-a + b = 1 \Rightarrow b = 1 + a = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$

$$f_p(x) = \frac{x}{2} + \frac{3}{2} = \frac{x+3}{2}$$

general solution:

$$f(x) = f_H(x) + f_p(x) = e^{x}(A\cos x + B\sin x) + \frac{x+3}{2}$$

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