16. In Prob. 13 the approximate values of the positive root will always be somewhat smaller than the exact values of the root. Why?

17. Find the root of $x^5 = x + 0.2$ near $x = 0$ by the iteration method, starting from $x_1 = 0$.

18. The equation in Prob. 17 has a root near $x = 1$. Find this root by the iteration method, starting from $x_1 = 1$. *Hint:* Write the equation in the form $x = \sqrt[5]{x + 0.2}$.

19. What happens in Prob. 18 if you write the equation in the form $x = x^5 - 0.2$ and start from $x_1 = 1$?

20. Using the iteration method, show that the smallest positive root of the equation $x = \tan x$ is 4.49, approximately. *Hint:* Conclude from the graphs of x and $\tan x$ that a root lies close to $x_1 = 3\pi/2$; write the equation in the form $x = \text{arc} \tan x$. (Why?)

0.8 *Approximate Integration*

If $f(x)$ is a given function and we can find a function $F(x)$ whose derivative is $f(x)$, then we may evaluate definite integrals of $f(x)$ by using the familiar formula

$$(1) \qquad \int_a^b f(x)\, dx = F(b) - F(a) \qquad [F'(x) = f(x)].$$

Tables of integrals[9] may be helpful for this purpose. However, in engineering applications there frequently occur integrals which cannot be evaluated by familiar methods known from calculus. These may be integrals of two types, namely, integrals such as

$$\int e^{-x^2}\, dx, \qquad \int \frac{\sin x}{x}\, dx, \qquad \int \cos(x^2)\, dx,$$

which cannot be represented in terms of finitely many elementary functions, or integrals whose integrands are empirical functions, given by a table of numerical values which are obtained from a physical experiment or in some other way. In many cases, integrals of the first type may be evaluated by one of the advanced methods (complex integration, use of power series, or asymptotic expansions), and in the case of an integral of the second type we may try to approximate the empirical function by a polynomial or some other elementary function.

Nevertheless, in various situations it will be preferable to apply one of the standard methods of numerical integration which we shall consider in the present section. These methods use the fact that the definite integral in (1) equals the area A of the shaded region R in Fig. 26 on p. 32.

To determine A we may cut R from cardboard, weight the piece, and divide the result by the weight of a square of this cardboard whose side is 1.

Another simple method is to draw R on a graph paper and count squares.

[9] Cf. for example, Ref. [A3] in Appendix 1.

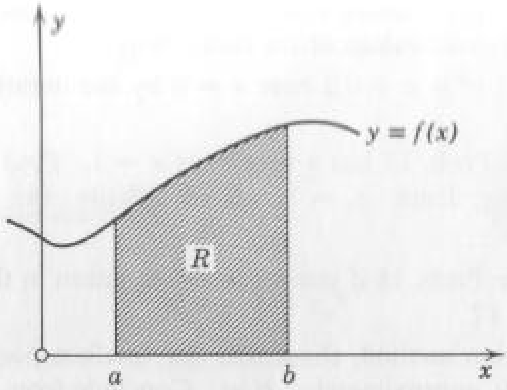

Fig. 26. Geometrical interpretation of a definite integral.

More accurate results are obtained by using a **planimeter**. So let us briefly discuss this instrumental method. Planimeters are inexpensive precision instruments which measure the area of any region R bounded by a closed curve C. There are several types of planimeters.[10] Figure 27 shows a so-called *polar planimeter*. This instrument has a rod, the tracing arm (I), on one end of which is mounted the tracing pin (II) which follows the curve C. The other end of the tracing arm is moved on a circular path by means of a second rod, the pole arm (III), which can be turned about the end point 0 (the pole) and is joined to the other end of the tracing arm by means of a hinge. An integrating wheel (IV), mounted on the tracing arm, measures the area of the region R when the pin traces once around the entire boundary curve C, starting from any point of C and returning to that point; the area is obtained as the difference of the initial and final reading on the scale of the integrating wheel. The accuracy can be increased by repeating this procedure m times, adding the m results, and dividing their sum by m.

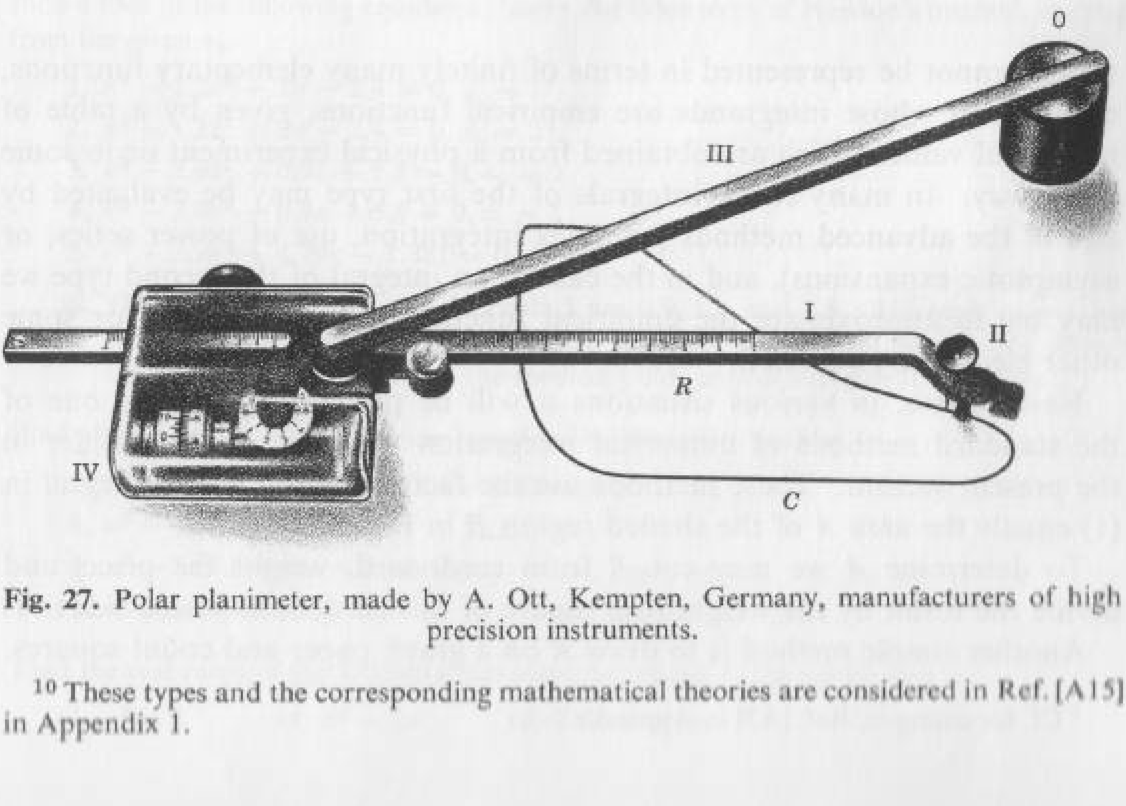

Fig. 27. Polar planimeter, made by A. Ott, Kempten, Germany, manufacturers of high precision instruments.

[10] These types and the corresponding mathematical theories are considered in Ref. [A15] in Appendix 1.

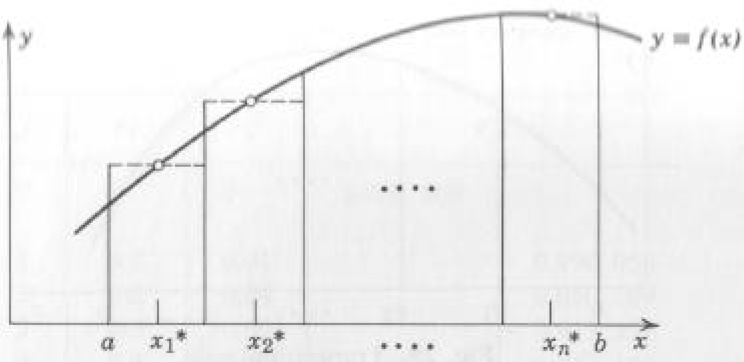

Fig. 28. Rectangular rule.

We shall now consider the simplest formulas of numerical integration, assuming that $b > a$ in (1).

We subdivide the interval of integration, whose length is $b - a$, into n equal parts of length

$$\Delta x = \frac{b - a}{n}.$$

Let $x_1^*, \cdots, x_n^*$ be the midpoints of these n intervals. Then the n rectangles in Fig. 28 have the areas $f(x_1^*)\,\Delta x, \cdots, f(x_n^*)\,\Delta x$. Therefore,

$$\textbf{(2)} \qquad \int_a^b f(x)\,dx \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)].$$

This simple formula is called the **rectangular rule**.

Example 1. Use (2) to find an approximate value of ln 2 (= 0.693 15). We have

$$\ln 2 = \int_1^2 \frac{dx}{x}.$$

We choose $n = 5$ and $n = 10$ and arrange the work in tabular form.

j	x_j^*	$1/x_j^*$	j	x_j^*	$1/x_j^*$	j	x_j^*	$1/x_j^*$
1	1.1	0.909 091	1	1.05	0.952 381	6	1.55	0.645 161
2	1.3	0.769 231	2	1.15	0.869 565	7	1.65	0.606 061
3	1.5	0.666 667	3	1.25	0.800 000	8	1.75	0.571 429
4	1.7	0.588 235	4	1.35	0.740 741	9	1.85	0.540 541
5	1.9	0.526 316	5	1.45	0.689 655	10	1.95	0.512 821
Sum $S = 3.459\ 540$			Sum $S_1 = 4.052\ 342$			Sum $S_2 = 2.876\ 013$		
$\ln 2 \approx \frac{1}{5}S = 0.691\ 91$ Error: 0.001 24 (0.2%)			$\ln 2 \approx \frac{1}{10}(S_1 + S_2) = 0.692\ 84$ Error: 0.000 31 (0.05%)					
Computation for $n = 5$			Computation for $n = 10$					

We see that the result for $n = 10$ is more accurate than that for $n = 5$.

We shall now derive another integration formula. We subdivide the interval $a \leqq x \leqq b$ into n equal parts of length $\Delta x = (b - a)/n$, as before, and denote the end points of these subintervals by $a, x_1, x_2, \cdots x_{n-1}, b$.

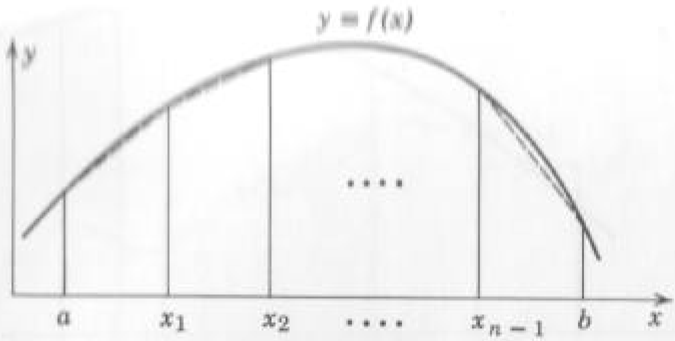

Fig. 29. Trapezoidal rule.

Then the n trapezoids in Fig. 29 have the areas

$$\tfrac{1}{2}[f(a) + f(x_1)]\,\Delta x, \qquad \tfrac{1}{2}[f(x_1) + f(x_2)]\,\Delta x, \cdots, \qquad \tfrac{1}{2}[f(x_{n-1}) + f(b)]\,\Delta x.$$

Thus

$$(3) \quad \int_a^b f(x)\,dx \approx \Delta x\left[\frac{f(a)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(b)}{2}\right]$$

where $\Delta x = (b - a)/n$. This formula is called the **trapezoidal rule.**

Let A be the exact value of the integral under consideration, and let A^* be the approximate value obtained by the trapezoidal rule (3). Then the difference

$$E_T = A^* - A$$

is called the *error* of A^*, and

$$(4) \qquad A = \int_a^b f(x)\,dx = A^* - E_T.$$

Clearly, if $f(x)$ is a *linear* function, its graph is a straight line, and the trapezoidal rule will yield the exact value of the integral, that is, $E_T = 0$. Now in this case, f' is constant and f'' is zero for all x. Hence the error results from the fact that in general $f'' \neq 0$, and its seems to be plausible that its magnitude will depend on the values of f'' in the interval of integration and also on the number n of subintervals in (3). In fact, it can be shown that E_T always lies in the interval

$$(5) \qquad KM^* \leqq E_T \leqq KM$$

where M^* and M are the smallest and largest values of the second derivative of f in the interval of integration, and

$$K = \frac{(b - a)^3}{12n^2}.$$

Example 2. Compute the integral

$$I = \int_0^1 e^{-x^2}\,dx$$

by means of the trapezoidal rule (3), taking $n = 10$, and estimate the error.

j	x_j	x_j^2	$e^{-x_j^2}$	
0	0	0	1.000 000	
1	0.1	0.01		0.990 050
2	0.2	0.04		0.960 789
3	0.3	0.09		0.913 931
4	0.4	0.16		0.852 144
5	0.5	0.25		0.778 801
6	0.6	0.36		0.697 676
7	0.7	0.49		0.612 626
8	0.8	0.64		0.527 292
9	0.9	0.81		0.444 858
10	1.0	1.00	0.367 879	
		Sums	1.367 879	6.778 167

We have $\Delta x = 0.1$, and

$$I \approx 0.1\left[\frac{1.363\ 879}{2} + 6.778\ 176\right] = 0.746\ 211.$$

Estimate of error.

$$f''(x) = 2(2x^2 - 1)e^{-x^2}, \qquad M^* = f''(0) = -2, \qquad M = f''(1) = 0.735\ 758.$$

From this and (5),

$$-2K \leqq E_T \leqq 0.735\ 758\ K$$

where $K = 1/1200$. Thus

$$-0.001\ 667 \leqq E_T \leqq 0.000\ 614.$$

From this and (4) it follows that the exact value of I must lie between

$$0.746\ 211 - 0.000\ 614 = 0.745\ 597$$

and

$$0.746\ 211 + 0.001\ 667 = 0.747\ 878.$$

(The exact value is $I = 0.746\ 824 \cdots$.)

The rectangular rule is obtained by approximating the integrand $f(x)$ by a step function (piecewise constant function). The trapezoidal rule results by approximating $f(x)$ by linear functions. We may expect to obtain a more accurate integration formula by approximating the curve of $f(x)$ by portions of parabolas.

For this purpose we subdivide the interval of integration $a \leqq x \leqq b$ into an *even* number of equal subintervals, say, into $2n$ subintervals of length $\Delta x = (b - a)/2n$, with end points $x_0(= a), x_1, \cdots, x_{2n-1}, x_{2n}(= b)$. In the first two intervals we approximate the curve of $f(x)$ by the parabola of the form $\alpha x^2 + \beta x + \gamma$ passing through the points A_0, A_1, A_2 of that curve

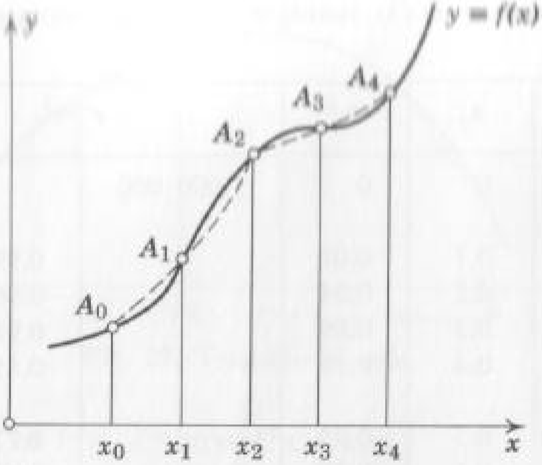

Fig. 30. Simpson's rule.

(Fig. 30). In the next two intervals we approximate that curve by another such parabola through A_2, A_3, A_4, and so on. Proceeding in this fashion, we obtain a curve consisting of n portions of parabolas, and the area under that curve is an approximation for the area under the curve of $f(x)$ between a and b. It can be shown that the integration formula thus obtained is

$$\textbf{(6)}\quad \int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

where $\Delta x = (b - a)/2n$ (cf. Ref. [A13] in Appendix 1). This important formula for approximate integration is called **Simpson's rule.**

If we write (6) in the form

$$(7)\qquad \int_a^b f(x)\,dx = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + \cdots + f(x_{2n})] - E_S,$$

then E_S is the error of the approximation. It can be shown that E_S lies in the interval

$$(8)\qquad CM_4^* \leqq E_S \leqq CM_4$$

where M_4^* and M_4 are the smallest and the largest value of the fourth derivative of f in the interval of integration, and

$$C = \frac{(b-a)^5}{180\,(2n)^4}.$$

From (8) we see that Simpson's rule yields the exact result not only for polynomials of the second degree but even for polynomials of the third degree.

Example 3. Evaluate $I = \int_0^1 e^{-x^2}\,dx$ by Simpson's rule with $2n = 10$ and estimate the error.

j	x_j	x_j^2	$e^{-x_j^2}$		
0	0	0	1.000 000		
1	0.1	0.01		0.990 050	
2	0.2	0.04			0.960 789
3	0.3	0.09		0.913 931	
4	0.4	0.16			0.852 144
5	0.5	0.25		0.778 801	
6	0.6	0.36			0.697 676
7	0.7	0.49		0.612 626	
8	0.8	0.64			0.527 292
9	0.9	0.81		0.444 858	
10	1.0	1.00	0.367 879		
Sums			1.367 879	3.740 266	3.037 901

We find, since $\Delta x = 0.1$,

$$I \approx \frac{0.1}{3}(1.367\ 879 + 4 \cdot 3.740\ 266 + 2 \cdot 3.037\ 901) = 0.746\ 825.$$

Estimate of error. The fourth derivative of the integrand is

$$f^{\text{IV}}(x) = 4(4x^4 - 12x^2 + 3)e^{-x^2}.$$

By considering the derivative of f^{IV} we find that the smallest value of f^{IV} in the interval of integration occurs at $x = x^* = 2.5 + 0.5\sqrt{10}$ and the largest value occurs at $x = 0$. Computing the corresponding values of f^{IV} we obtain in (8)

$$M_4^* = f^{\text{IV}}(x^*) = -7.359 \cdots \quad \text{and} \quad M_4 = f^{\text{IV}}(0) = 12.$$

Furthermore, since $2n = 10$ and $b - a = 1$, in (8),

$$C = 1/1\ 800\ 000 = 0.000\ 000\ 55 \cdots.$$

Therefore

$$-0.000\ 004 \cdots \leqq E_S \leqq 0.000\ 006 \cdots.$$

This shows that the first four decimals of the above approximation are correct and, by (7),

$$0.746\ 818 < I < 0.746\ 830.$$

The exact value to six decimal places is $I = 0.746\ 824$, and we see that even five decimals or our result are correct. Our present result is much better than that obtained in Ex. 2 by the trapezoidal rule, while the amount of work is almost the same in both cases.

Problems

Review some integration formulas and methods by integrating

1. $\int \frac{dx}{a^2 + x^2}$ **2.** $\int \frac{dx}{\sqrt{a^2 - x^2}}$ **3.** $\int e^{ax} \cos bx\,dx$

4. $\int e^{ax} \sin bx\,dx$ **5.** $\int \ln x\,dx$ **6.** $\int \cos^2 x\,dx$

7. $\int \sin^2 \omega x \, dx$ 8. $\int \tan x \, dx$ 9. $\int \frac{dx}{x^2(x^2+1)^2}$

10. Compute $\int_0^1 x^3 \, dx$ by the rectangular rule (2) with $n = 5$. What is the error?

11. Compute the integral in Prob. 10 by the trapezoidal rule (3) with $n = 5$. What error bounds are obtained from (5)? What is the actual error of the result? Why is this result larger than the exact value?

12. Compute the integral in Ex. 2 by using (2) with $n = 5$.

Using the column of $\sin x$ in Table 1 in Sec. 0.1 evaluate $\int_0^1 \frac{\sin x}{x} \, dx$:

13. By the rectangular rule (2) with $n = 5$.

14. By the trapezoidal rule (3) with $n = 5$.

15. By (3) with $n = 10$.

16. By Simpson's rule with $2n = 2$.

17. By Simpson's rule with $2n = 10$.

18. Evaluate $\int_0^1 x^5 \, dx$ by Simpson's rule with $2n = 10$. What error bounds are obtained from (8)? What is the actual error of the result?

19. Find an approximate value of $\ln 2 = \int_1^2 \frac{dx}{x}$ by Simpson's rule with $2n = 4$. Estimate the error by (8).

20. If subintervals of different lengths $\Delta_1 x, \Delta_2 x, \cdots, \Delta_n x$ are chosen, show that the trapezoidal rule assumes the form

$$\int_a^b f(x) \, dx \approx \tfrac{1}{2}[f(a) + f(x_1)] \, \Delta_1 x + \cdots + \tfrac{1}{2}[f(x_{n-1}) + f(b)] \, \Delta_n x.$$