0.4 Complex Numbers

The fact that there are equations such as

$$x^2 + 3 = 0, \qquad x^2 - 10x + 40 = 0,$$

which are not satisfied by any real number, leads to the introduction of complex numbers.[4] These are of the form $a + ib$ (or $a + bi$) where a and b are real numbers. The symbol i is called the **imaginary unit;** the number a is called the **real part** and b the **imaginary part** of $a + ib$. For example, the real part of $4 - 3i$ is 4, and the imaginary part is -3. To each ordered pair of real numbers a, b there corresponds one complex number $a + ib$, and conversely.

Complex numbers can be represented as points in the plane. For this purpose we choose two perpendicular coordinate axes, the horizontal x-axis and the vertical y-axis and on both axes the same unit of length (Fig. 15).

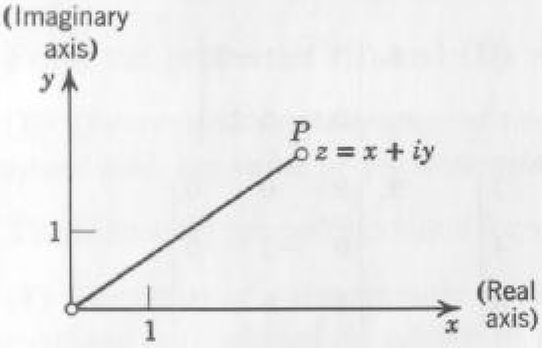

Fig. 15. The complex plane.

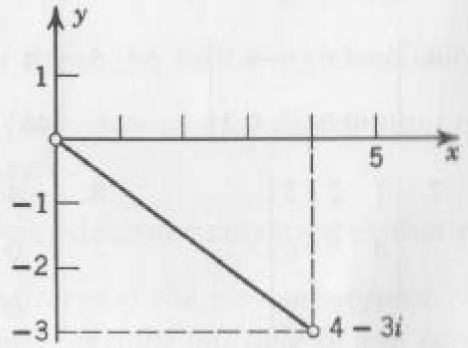

Fig. 16. The number $4 - 3i$ in the complex plane.

The xy-coordinate system thus obtained is called a **Cartesian coordinate system.**[5] To the complex number $z = x + iy$ there corresponds the point P with Cartesian coordinates (x, y). The xy-plane in which the complex numbers are represented geometrically in this fashion is called the **complex plane** or *Argand diagram.*[6] The x-axis is called the *real axis*, and the y-axis the *imaginary axis*.

Two complex numbers

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2$$

[4] First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by the great German mathematician CARL FRIEDRICH GAUSS (1777–1855) whose work was of basic importance in algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, numerical analysis, and theoretical mechanics. He also paved the way for a general and systematic use of complex numbers.

[5] Named after the French philosopher and mathematician RENATUS CARTESIUS (latinized for RENÉ DESCARTES (1596–1650)), who invented analytic geometry.

[6] JEAN ROBERT ARGAND (1768–1822), French mathematician. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818).

are defined to be **equal** if, and only if, their real parts are equal and their imaginary parts are equal; that is,

$$z_1 = z_2 \quad \textit{if, and only if,} \quad x_1 = x_2 \text{ and } y_1 = y_2.$$

If z_1 and z_2 are different (not equal), then we may write $z_1 \neq z_2$.

Inequalities between complex numbers, such as $z_1 < z_2$ or $z_1 \geqq z_2$, have no meaning. (Inequalities may hold between the *absolute values* of complex numbers; see Sec. 0.5.)

Addition. The sum $z_1 + z_2$ of z_1 and z_2 is defined as the complex number obtained by adding the real parts and the imaginary parts of z_1 and z_2, that is,

$$(1) \qquad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

We see that addition of complex numbers is in accordance with the "*parallelogram law*" by which forces are added in mechanics (Fig. 17).

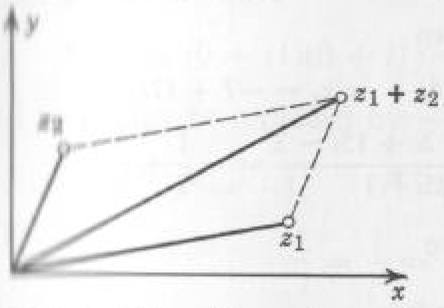

Fig. 17. Addition of complex numbers.

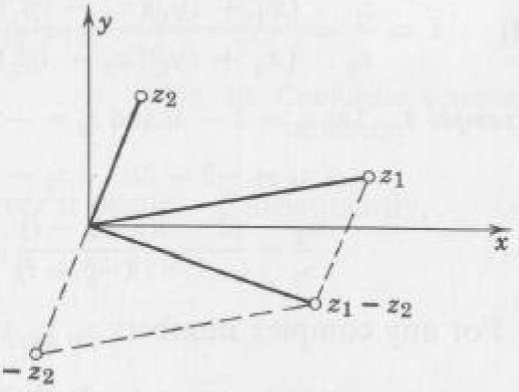

Fig. 18. Subtraction of complex numbers.

Subtraction. This operation is defined as the inverse operation of addition; that is, the difference $z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Obviously,

$$(2) \qquad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

Multiplication. The product z_1z_2 is defined as the complex number

$$(3) \qquad z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

which is obtained formally by applying the ordinary rules of arithmetic for real numbers, treating the symbol i as a number, and replacing $i^2 = ii$ by -1.

Division. This operation is defined as the inverse operation of multiplication; that is, the quotient $z = z_1/z_2$ is the complex number $z = x + iy$ for which

$$z_1 = zz_2 = (x + iy)(x_2 + iy_2).$$

By using (3) we see that this can be written

$$x_1 + iy_1 = (x_2x - y_2y) + i(y_2x + x_2y).$$

By definition of equality the real parts and the imaginary parts on both sides must be equal:

$$x_1 = x_2 x - y_2 y$$
$$y_1 = y_2 x + x_2 y.$$

This is a system of two linear equations in the unknowns x and y. The determinant of the coefficients has the value $x_2^2 + y_2^2$. Assuming that $z_2 = x_2 + iy_2 \neq 0$ that determinant is not zero, and Cramer's rule (Sec. 0.3) yields the unique solution

$$x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \quad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0).$$

From this we see that the number $z = x + iy$ can be obtained formally by multiplying both the numerator and the denominator of the quotient z_1/z_2 by $x_2 - iy_2$; thus

$$\text{(4)} \qquad z = \frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}.$$

Example 1. Let $z_1 = 2 - 3i$ and $z_2 = -5 + i$. Then

$$z_1 + z_2 = -3 - 2i, \qquad z_1 - z_2 = 7 - 4i, \qquad z_1 z_2 = -7 + 17i,$$
$$\frac{z_1}{z_2} = \frac{(2-3i)(-5-i)}{(-5+i)(-5-i)} = \frac{-10 - 2i + 15i - 3}{25 + 1} = -\frac{1}{2} + \frac{i}{2}.$$

For any complex numbers z_1, z_2, z_3 we have

$$\text{(5)} \qquad \begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1 z_2 &= z_2 z_1 \end{aligned} \qquad \text{(Commutative laws)}$$
$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3) \end{aligned} \qquad \text{(Associative laws)}$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad \text{(Distributive law).}$$

These laws follow immediately from the corresponding laws for real numbers and the preceding definitions of the algebraic operations for complex numbers.

Let $z = x + iy$ be any complex number; then $x - iy$ is called the **conjugate** of z and is denoted by $\bar{z}$ (Fig. 19). Thus

$$z = x + iy, \qquad \bar{z} = x - iy.$$

For example, the conjugate of $4 + 7i$ is $4 - 7i$. Now

$$z + \bar{z} = 2x, \qquad z - \bar{z} = 2iy$$

and therefore

$$\text{(6)} \qquad \text{Re}\, z = x = \frac{1}{2}(z + \bar{z}), \qquad \text{Im}\, z = y = \frac{1}{2i}(z - \bar{z})$$

where the symbol Re denotes the real part and Im denotes the imaginary part. Obviously,

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

In the fundamental operations of arithmetic the complex number $x + i0$ behaves like the real number x. Indeed, equality, sum, and product as defined for two such numbers $x_1 + i0$ and $x_2 + i0$ carry over, respectively, into equality, sum, and product as defined for two real numbers x_1 and x_2. For this reason we may say that a complex number whose imaginary part is zero is a real number, and write simply x instead of $x + i0$. The complex number $0 + iy$ is written iy and is called a **pure imaginary number**. Defining i to mean $0 + i1$ we have

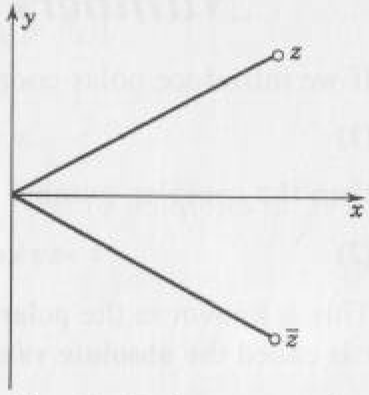

Fig. 19. Conjugate complex numbers.

(8) $$i^2 = (0 + i1)(0 + i1) = -1$$

and so i^2 may be replaced by -1 whenever it occurs. Consequently,

(9) $$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \cdots$$
$$\frac{1}{i} = \frac{i}{i^2} = -i, \qquad \frac{1}{i^2} = -1, \cdots.$$

Problems

Let $z_1 = 3 - 2i$ and $z_2 = 2 + 4i$. Reduce each of the following expressions to the form $a + ib$.

1. $z_1 z_2, z_1/z_2, z_1^2, z_1^2/z_2, (1 + i)z_1$
2. $z_1 + z_2, z_1 - z_2, z_2 - z_1, z_2^3, (z_1/z_2)^2, z_2/(1 - 2i)$

Reduce each of the following expressions to the form $a + ib$.

3. $(1 + i)^2, \left(\frac{1 + i}{1 - i}\right)^2$
4. $\left(\frac{1 + i}{1 - i}\right)^2 - \left(\frac{1 - i}{1 + i}\right)^2$

Solve the following equations.

5. $z^2 + 9 = 0$ 6. $z^2 - 2z + 2 = 0$ 7. $z^2 + 2z + 5 = 0$ 8. $z^2 + z + 9 = 0$
9. Prove the first two formulas of (7).
10. Prove the last two formulas of (7).
11. If $z = \bar{z}$, show that z is real. If $z = -\bar{z}$, show that z is pure imaginary or zero.
12. If $z^2 = \bar{z}^2$, show that z is either real or pure imaginary.
13. Prove that any number is equal to the conjugate of its conjugate.
14. Plot the numbers $2 + 4i, 2 - 4i, -2 + 4i, -2i$, and $1 - i$ as points in the complex plane.

15. If $z_1 z_2 = 0$, show that at least $z_1 = 0$ or $z_2 = 0$.

Let $z = x + iy$. Find

16. $\mathrm{Re}\,(1/z)$, $\mathrm{Im}\,(1/z)$ 17. $\mathrm{Im}\,(z^3)$ 18. $\mathrm{Re}\,(1/z^2)$, $\mathrm{Re}\,(z^2 + z)$
19. $\mathrm{Re}\,(-iz^2)$, $\mathrm{Im}\,(4iz^2 - 6z + 8i)$ 20. $\mathrm{Re}\,[1/(z - i)]$

0.5 *Polar Form of Complex Numbers*

If we introduce polar coordinates r, θ in the complex plane by setting

(1) $$x = r\cos\theta, \qquad y = r\sin\theta,$$

then the complex number $z = x + iy$ may be written

(2) $$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This is known as the **polar form** or *trigonometric form* of a complex number. r is called the **absolute value** or *modulus* of z and is denoted by $|z|$. Thus

(3) $$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \qquad (\geqq 0).$$

Obviously, r is the distance of the point P corresponding to z from the origin O in the complex plane (Fig. 20). The directed angle measured from the positive x-axis to OP is called the **argument** of z and is denoted by arg z; *angles will be measured positive in the counterclockwise direction and in terms of radians.* We have, then,

(4) $$\arg z = \theta = \arcsin\frac{y}{r} = \arccos\frac{x}{r} = \arctan\frac{y}{x},$$

as in trigonometry. Note that for given z, *the argument θ is determined only up to multiples of 2π.*

Example 1. Let $z = 1 + i$. Then $|z| = \sqrt{2}$ and $\arg z = \dfrac{\pi}{4} \pm 2n\pi$ where $n = 0, 1, 2, \cdots$

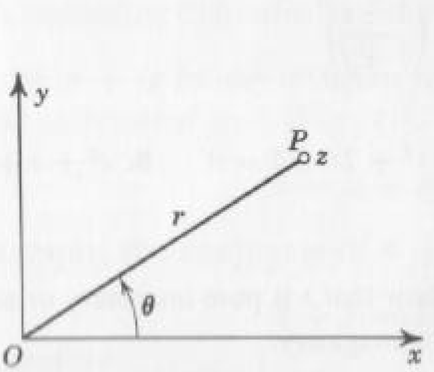

Fig. 20. Trigonometric form of complex numbers.

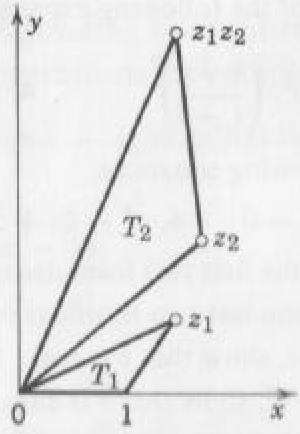

Fig. 21. Multiplication of complex numbers.

The polar form of complex numbers is particularly useful in analyzing multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then the product is

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

If we apply the familiar addition theorems of the sine and the cosine, this assumes the simple form

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \tag{5}$$

We thus obtain the important rules

$$|z_1z_2| = |z_1|\,|z_2| \tag{6}$$

and

$$\arg(z_1z_2) = \arg z_1 + \arg z_2 \quad \text{(up to multiples of } 2\pi). \tag{7}$$

Similarly, from the definition of division it follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{8}$$

and

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \quad \text{(up to multiples of } 2\pi). \tag{9}$$

Table 2. arc tan $\frac{y}{x}$

$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$
0	0.00000	1.0	0.78540	2.0	1.10715	4.0	1.32582
0.1	0.09967	1.1	0.83298	2.2	1.14417	4.5	1.35213
0.2	0.19740	1.2	0.87606	2.4	1.17601	5.0	1.37340
0.3	0.29146	1.3	0.91510	2.6	1.20362	5.5	1.39094
0.4	0.38051	1.4	0.95055	2.8	1.22777	6.0	1.40565
0.5	0.46365	1.5	0.98279	3.0	1.24905	7.0	1.42890
0.6	0.54042	1.6	1.01220	3.2	1.26791	8.0	1.44644
0.7	0.61073	1.7	1.03907	3.4	1.28474	9.0	1.46014
0.8	0.67474	1.8	1.06370	3.6	1.29985	10.0	1.47113
0.9	0.73282	1.9	1.08632	3.8	1.31347	11.0	1.48014

Example 2. From (6) and (7) we obtain

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

and from this so-called **formula of De Moivre**[7]

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{10}$$

From (6) and (7) it follows that the triangles T_1 and T_2 in Fig. 21 are similar. This can be used as the basis for a graphical construction of the product of complex numbers.

[7] ABRAHAM DE MOIVRE (1667–1754), French mathematician, who introduced imaginary quantities in trigonometry and contributed to the theory of mathematical probability.

Problems

Represent the following complex numbers in polar form.

1. $2 - 2i, i, 3 + 4i$ **2.** $5 + 5i, -5 + 5i, -5 - 5i$

3. Show that $\arg \bar{z} = -\arg z$ (up to multiples of 2π).

4. Show that $\arg (1/z) = -\arg z$ (up to multiples of 2π).

5. Prove that $|a - z| = |a - \bar{z}|$ where a is any real number. Interpret geometrically.

6. Prove that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane.

Find the absolute values of the following numbers.

7. $1 + i\sqrt{3}, -9i, 2 + i\sqrt{5}$ **8.** $2 - i\sqrt{5}, 2 + 3i, (4 + i)^3$

9. $\dfrac{(4 - 3i)(\frac{1}{2} + i)^4}{\left(1 - \dfrac{3i}{4}\right)^2(-3 + 4i)}$ **10.** $\left(\dfrac{1 + i}{1 - i}\right)^8, (3 + 4i)^3(-1 - i)^6$

What loci are represented by the following equations and inequalities? (Plot a graph in each case.)

11. $|z| = 1$ **12.** $|z - 1| = 1$ **13.** $\operatorname{Re} (z^2) = -1$ **14.** $\operatorname{Im} (2z) = -1$

15. $0 \leqq \arg z \leqq \pi/2$ **16.** $0 \leqq \arg (1/z) \leqq \pi/2$

17. Using (10), prove the identities

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta, \qquad \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

18. Find similar identities as in Prob. 17 for $\cos 2\theta$ and $\cos 4\theta$.

19. Find $(3 - i)(2 + i)$ by means of a graphical construction in the complex plane.

20. For given z find $1/z$ by a graphical construction.

0.6 *Some General Remarks About Numerical Computations*

The present section is devoted to a few simple but important general remarks about numerical computations.

It is clear that engineering mathematics ultimately comes down to numerical results, and the engineering student should, therefore, supplement his mathematical equipment with a definite knowledge of some fundamental numerical methods.

Various examples and problems in the text will help the student to learn how to arrange computations in a suitable form. However, numerical computation requires practical experience. It cannot be learned solely from books; practical training is needed, just as in swimming, driving a car, or playing the piano. Consequently, the student should not only work out book examples and problems, but also set up and calculate examples of his own. Active work is more important in numerical analysis than in many other branches of mathematics. Famous mathematicians like Gauss spent a considerable part of their time with numerical computations, and the engineering student will do well in developing a similar attitude.