0.2 Partial Derivatives

Let $z = f(x, y)$ be a real function of two independent real variables, x and y. If we hold y constant, say, $y = y_1$, and think of x as a variable, then $f(x, y_1)$ depends on x alone. If the derivative of $f(x, y_1)$ with respect to x for a value $x = x_1$ exists, then the value of this derivative is called the *partial derivative of $f(x, y)$ with respect to x at the point* (x_1, y_1) and is denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1)} \quad \text{or by} \quad \left.\frac{\partial z}{\partial x}\right|_{(x_1,y_1)}.$$

Other notations are

$$f_x(x_1, y_1) \quad \text{and} \quad z_x(x_1, y_1);$$

these may be used when subscripts are not used for another purpose and there is no danger of confusion.

We thus have, by the definition of the derivative,

$$\text{(1)} \qquad \left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1)} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x, y_1) - f(x_1, y_1)}{\Delta x}.$$

The partial derivative of $z = f(x, y)$ with respect to y is defined similarly; we now hold x constant, say, equal to x_1, and differentiate $f(x_1, y)$ with respect to y. Thus

$$\text{(2)} \qquad \left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1)} = \left.\frac{\partial z}{\partial y}\right|_{(x_1,y_1)} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

Other notations are $f_y(x_1, y_1)$ and $z_y(x_1, y_1)$.

It is clear that the values of those two partial derivatives will in general depend on the point (x_1, y_1), and so the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ at a variable point (x, y) are functions of x and y. The function $\partial z/\partial x$ is obtained as in ordinary calculus by differentiating $z = f(x, y)$ with respect to x, *treating y as a constant*, and $\partial z/\partial y$ is obtained by differentiating z with respect to y, *treating x as a constant*.

Example 1. Let $z = f(x, y) = x^2y + x \sin y$. Then

$$\frac{\partial f}{\partial x} = 2xy + \sin y, \qquad \frac{\partial f}{\partial y} = x^2 + x \cos y.$$

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of a function $z = f(x, y)$ have a very simple geometric interpretation. The function $z = f(x, y)$ can be represented by a surface in space. The equation $y = y_1$ then represents a vertical plane intersecting the surface in a curve, and the partial derivative $\partial z/\partial x$ at a point (x_1, y_1) is the slope of the tangent (i.e., tan α where α is the angle shown in Fig. 13) to the curve. Similarly, the partial derivative $\partial z/\partial y$ at (x_1, y_1) is the slope of the tangent to the curve $x = x_1$ on the surface $z = f(x, y)$ at (x_1, y_1).

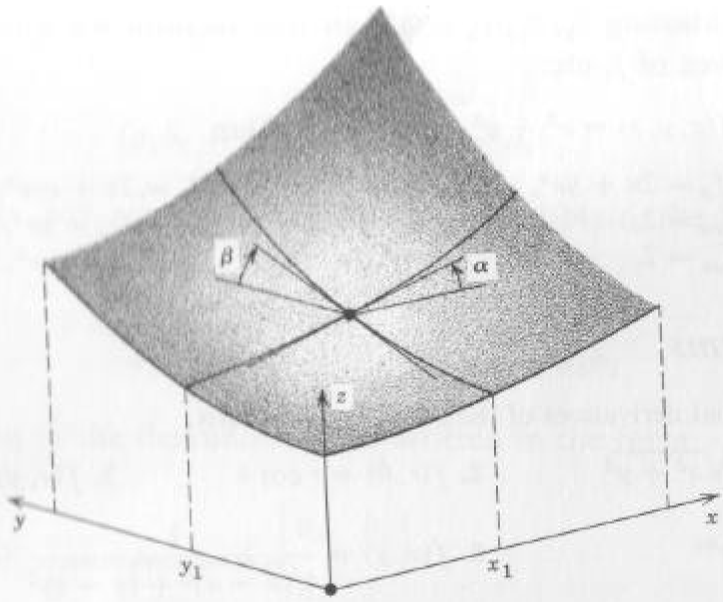

Fig. 13. Geometrical interpretation of first partial derivatives.

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are called *first partial derivatives* or *partial derivatives of the first order*. By differentiating these derivatives once more we obtain the four *second partial derivatives* (or *partial derivatives of the second order*):

$$(3)\quad \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x, y), \quad \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x, y), \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y).$$

Thus $\partial^2 z/\partial x^2$ is the partial derivative of $\partial z/\partial x$ with respect to x, $\partial^2 z/\partial x\,\partial y$ is the partial derivative of $\partial z/\partial x$ with respect to y, etc.

It can be shown[1] that if all the derivatives concerned are continuous, then

$$(4)\qquad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}\,;$$

i.e., the order of differentiation is then immaterial.

Example 2. For the function in Ex. 1,

$$f_{xx} = 2y, \qquad f_{xy} = 2x + \cos y = f_{yx}, \qquad f_{yy} = -x \sin y.$$

By differentiating the second partial derivatives again with respect to x and y, respectively, we obtain the *third partial derivatives* or *partial derivatives of the third order* of f, etc.

If we consider a function $f(x, y, z)$ of three independent variables, then we have the three first partial derivatives $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$. Here f_x is obtained by differentiating f with respect to x, *treating both y and z as constants*. Thus, in analogy to (1), we now have

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1, y_1, z_1)} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x, y_1, z_1) - f(x_1, y_1, z_1)}{\Delta x},$$

[1] Cf. Ref. [A1] in Appendix 1.

etc. By differentiating f_x, f_y, f_z again in this fashion we obtain the second partial derivatives of f, etc.

Example 3. Let $f(x, y, z) = x^2 + y^2 + z^2 + xye^z$. Then

$$f_x = 2x + ye^z, \quad f_y = 2y + xe^z, \quad f_z = 2z + xye^z,$$
$$f_{xx} = 2, \quad f_{xy} = f_{yx} = e^z, \quad f_{xz} = f_{zx} = ye^z,$$
$$f_{yy} = 2, \quad f_{yz} = f_{zy} = xe^z, \quad f_{zz} = 2 + xye^z.$$

Problems

Find the first partial derivatives of the following functions.

1. $f(x, y) = \sqrt{x^2 + y^2}$ **2.** $f(r, \theta) = r\cos\theta$ **3.** $f(x, y) = \arctan\dfrac{y}{x}$

4. $f(x, y) = e^{xy}$ **5.** $f(x, y) = \dfrac{1}{\sqrt{(x - a)^2 + (y - b)^2}}$ (a, b constant)

6. $f(r, h) = \dfrac{\pi}{3}r^2h$ **7.** $f(R_1, R_2) = \dfrac{R_1 + R_2}{R_1R_2}$

8. Find $f_x + g_y$ where $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$.

9. Find $f_x + g_y$ where $f(x, y) = \ln(x^2 + y^2)$ and $g(x, y) = 2\arctan\dfrac{y}{x}$.

Find $f_{xx} + f_{yy}$ where

10. $f(x, y) = x^2 - y^2$ **11.** $f(x, y) = e^x \cos y$ **12.** $f(x, y) = \sin x \cosh y$

Sketch the surfaces corresponding to the following functions.

13. $z = x^2 + y^2$ **14.** $z = \ln(x^2 + y^2)$ **15.** $z = e^{xy}$

16. The curves $z = f(x, y) = const$ are called **level curves** of $f(x, y)$. Draw the level curves of the functions in Probs. 13–15.

Find the first partial derivatives of the following functions at the given points.

17. $f(x, y) = \sqrt{1 - x^2 - y^2}$, at (0, 0) **18.** $f(x, y) = (x^2 + y^2)^2$, at (1, 2)

Find the first and second partial derivatives of the following functions.

19. $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ **20.** $f(x, y, z) = \dfrac{x + y + z}{xyz}$

0.3 Second and Third Order Determinants

Consider the system

$$\text{(1)} \qquad \begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}$$

consisting of two linear equations in the unknowns x and y. To solve this system we may multiply the first equation by b_2, the second by $-b_1$, and add, finding

$$(a_1b_2 - a_2b_1)x = k_1b_2 - k_2b_1.$$