Differential Calculus

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- We will cover the following
  - the concept of differentiation
  - differentiation of simple functions
  - rules of differentiation
  - Taylor series and approximation
  - higher order derivatives
  - differentiation of functions of multiple variables
- Differentiation provides the information on the rate of change in one variable in relation to changes in one or more other variables
- The physics example: speed / velocity
- Financial example: the change of the price of the bond is a result of the change in the yield on that bond; we will apply differential calculus to quantitatively describe the connection between those two changes
- The differential calculus is also used in optimisation problems

Differential Calculus

- Let us ~~ca~~ draw graphs of several functions: $f_1(x) = 2x$, $f_2(x) = 5 + 2x$, $f_3(x) = 2x^2$ (in Excel)
- Looking at them, we can find the appropriate changes in f for a given change in x:

$$f(x+\Delta x) = f(x) + \Delta f(x)$$

- the rate of the change in f is the first derivative

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = f'(x)$$

- Example: For $f_1(x) = 2x$ we have

$$f_1(x+\Delta x) = 2(x+\Delta x) = 2x + 2\Delta x = f_1(x) + \Delta f_1(x)$$

$$\Rightarrow \Delta f_1(x) = 2\Delta x \Rightarrow \frac{\Delta f_1(x)}{\Delta x} = 2$$

Therefore $f_1'(x) = 2$ (limit is not needed)

The same applies to $f_2(x)$.

For $f_3(x) = 2x^2$ we have

$$f_3(x+\Delta x) = 2(x+\Delta x)^2 = 2x^2 + 4x\Delta x + 2\Delta x^2$$

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$$\Delta f_3(x) = 4x\Delta x + 2\Delta x^2 \Rightarrow \frac{\Delta f_3(x)}{\Delta x} = 4x + 2\Delta x$$

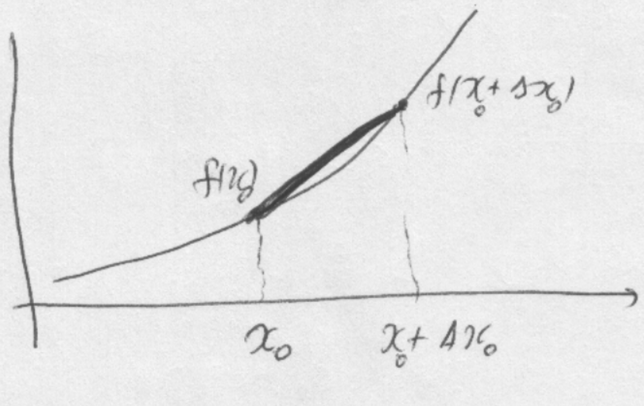
Limit is now needed:

$$\lim_{\Delta x \to 0} \frac{\Delta f_3(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[4x + 2\Delta x\right] = 4x$$

- First derivative represents the slope of the tangent to the function in a given point $(x, f(x))$.

- Notation: $f'(x) = \frac{df(x)}{dx}$ (Newton, Leibnitz)

- Finite difference method relies on $\frac{\Delta f(x)}{\Delta x}$: slope of the chord to the curve



- the equation of the cord passing through x_0

$$y - f(x_0) = \frac{\Delta f(x_0)}{\Delta x_0}(x - x_0)$$

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- Remember the equation of a line passing through a point (x_0, y_0) and has a slope k:

$$y - y_0 = k\,(x - x_0)$$

- Linear function has a slope k which is equal to its first derivative; the slope does not depend on x, i.e. it is the same everywhere
  - Constant function (special case of a linear function, slope $k=0$) has zero slope, i.e. its first derivative is zero! The function is always the same, so its rate of change is expected to be zero!

- General equation of a tangent passing through $(x_0, f(x_0))$:

$$y - f(x_0) = f'(x_0)\,(x - x_0)$$

Differential Calculus

– So far, we know that

$$\frac{d}{dx}(a+bx) = b, \quad \frac{d}{dx}(2x^2) = 4x$$

– More general rule

$$\frac{d}{dx}\left(\alpha f_1(x) + \beta f_2(x)\right) = \alpha f_1'(x) + \beta f_2'(x)$$

– Polynomials: $\frac{d}{dx}(x^n) = n x^{n-1}$

– Examples: $3x^5$, x^{-4}, $x^{4/3}$, $\sqrt[3]{x}$

– Examples: $2x^4 + 3x^2$, $3x^3 + 2x$

– General rule for the first derivative of the product of two functions

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \qquad \text{(Leibnitz)}$$

$$(uv)' = u'v + uv' \qquad \text{(Newton)}$$

– Examples: $x^2(2x^2+3)$, $x(3x^2+2)$

Differential Calculus

- General rule for the quotient of two f's

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

- Examples: $\frac{4x^2}{5x+2}$, $\frac{1}{x}$

- General rule for the combination of f's

$$\frac{d}{dx}\left(f_1(f_2(x))\right) = f_1'(f_2(x)) \cdot f_2'(x)$$

- Example: $(2x^3+3)^6$

- Exponential function: $\frac{d}{dx} e^x = e^x$

- Examples: e^{3x}, e^{x^2}, 5^x

- Logarithmic function: $\frac{d}{dx} \ln x = \frac{1}{x}$

- Examples: $\ln(x^3+2)$, $\log_a x$

- Table II. C. 1 in PRM II

Differential Calculus

– Example: Modified duration of a bond

The dirty price of a bond defined by by $(CF_1, .., CF_n)$ is its present value at the rate (yield) y:

$$PV(y) = \frac{CF_1}{1+y} + \cdots + \frac{CF_n}{(1+y)^n}$$

First derivative of PV with respect to y is

$$\frac{dPV}{dy} = -\frac{CF_1}{(1+y)^2} - 2\frac{CF_2}{(1+y)^3} - \cdots - n\frac{CF_n}{(1+y)^{n+1}} =$$

$$= -\frac{1}{1+y}\left[\frac{CF_1}{1+y} + 2\frac{CF_2}{(1+y)^2} + \cdots + n\frac{CF_n}{(1+y)^n}\right]$$

The proportional change in the bond price for a small change in the yield is

$$\underbrace{\frac{1}{PV}\frac{dPV}{dy}}_{\text{Modified Duration}} = -\frac{1}{(1+y)}\underbrace{\sum_{k=1}^{n} k\left[\frac{CF_k}{(1+y)^k}\cdot\frac{1}{PV}\right]}_{\text{Macaulay's Duration}}$$

Do we expect $\frac{dPV}{dy}$ to be negative?

Differential Calculus

- Example: Three year bond paying annual coupons of 4 and trading on a yield-to-maturity of 5%. (Excel)

- Often used: PVBP or PV01 – percentage price change for a basis point change in yield
  - this is (negative) modified duration multiplied by the corresponding change in yield (one basis point = 0.01%), i.e.

  Modified duration × 0.0001

- Example: $\sigma_p = MDP \cdot \sigma_y$, where MDP is the modified duration of the portfolio (negative)

- Higher order derivatives:

$$f^{(n)}(x) = \frac{d^n f(x)}{dx^n} = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

- Example: Find second and third derivative of e^x, $2x^3 + 3x^2$

Differential Calculus

- Taylor Series: $f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n =$

$$= f(x) + f'(x)h + \frac{1}{2} f''(x) h^2 + \cdots$$

- Zeroth-Order approximation

$$f(x+h) \approx f(x)$$

- Linear approximation

$$f(x+h) = f(x) + f'(x) h$$

- Second order (harmonic, quadratic) approximation

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(x) h^2$$

- Example in Excel
- What happens if f depends on more than one variable? Partial differentiation:

$$\frac{\partial}{\partial x} f(x,y,z), \frac{\partial}{\partial y} f(x,y,z), \frac{\partial}{\partial z} f(x,y,z)$$

- When calculating $\frac{\partial}{\partial x}$ just x is assumed to change, other variables taken as constants

Differential Calculus

- That way, partial derivatives are like the ordinary derivatives; it is just important to remember what is the variable we are differentiating over

- Examples: $f(x,y) = x^2 + 6xy + 2y^3$

$$f(x,y) = f_1(x)\, f_2(y)$$

- Example: Black - Sholes PDE

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

V - value of any derivative security
S - underlying security price
σ - volatility of S
r - risk-free rate

- Mixed partial derivatives / higher partial derivatives

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y)$$

$$\frac{\partial^2}{\partial x^2} f(x,y) \quad , \quad \frac{\partial^2}{\partial y^2} f(x,y)$$

Differential Calculus

- Hessian matrix: matrix of second partial derivatives

Example:
$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & \frac{\partial^2}{\partial z^2} \end{bmatrix} f(x,y,z)$$

- Total differentiation: If f depends on multiple variables, then

$$\Delta f(x,y,z) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$$

- Example: $f(x,y,z) = x^4 + xz^3 - 4y^3$

$\Delta x = 0.01$, $\Delta y = 0.05$, $\Delta z = -0.02$

$x = 2$, $y = 1$, $z = 3$

- Example: change of option value

$W(S, \sigma)$ is the value of an option

It depends on the underlying security price S and its volatility (and ...)

$$\Delta W(S,\sigma) = \underbrace{\frac{\partial W}{\partial S}}_{delta}\Delta S + \underbrace{\frac{\partial W}{\partial \sigma}}_{vega}\Delta\sigma + \frac{1}{2}\underbrace{\frac{\partial^2 W}{\partial S^2}}_{gamma}(\Delta S)^2 + \cdots$$