

Matrix Algebra

- What is matrix? An collection of numbers, a rectangularly ordered array
- The order of a matrix can be $r \times c$, where r is the number of its rows, and c being the number of its columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} \\ a_{21} & a_{22} & \dots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rc} \end{bmatrix}$$

- The order is also called the type
- Precise mathematical definition

$$A: \mathbb{N}^{r \times c} \rightarrow \mathbb{R}^{r \times c} \text{ (or } \mathbb{C}^{r \times c}, \text{ or...)}$$

- We will introduce mathematical operations with matrices: addition, subtraction, multiplication, inversion, etc.
- Matrices of the type $r \times 1$ are usually called vectors:

$$x = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{r1} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_r \end{bmatrix}$$

Matrix Algebra

(2)

- One important operation that can be performed on any matrix is transposition: its rotation around the main diagonal

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1c} \\ a_{21} & a_{22} & \dots & a_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \dots & a_{rc} \end{bmatrix}_{r \times c}^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{r1} \\ a_{12} & a_{22} & \dots & a_{r2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1c} & a_{2c} & \dots & a_{rc} \end{bmatrix}_{c \times r}$$

- The easiest way to write down a transposed matrix of some given matrix A is to take its rows and put them one by one, starting from row one, as columns in another matrix.
- The matrix A^T will have c rows and r columns

- Example: (in Excel as well)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$



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Matrice Algebra

- Matrice addition and subtraction:
can be performed iff both matrices
are of the same type; the resulting
matrice is then ^{also} of the same type

$$+, - : (\mathbb{R}^{r \times c}, \mathbb{R}^{r \times c}) \longrightarrow \mathbb{R}^{r \times c}$$

($\mathbb{R}$ can be $\mathbb{C}$, or some other set)

- If A is represented as $[a_{ij}]$
and B as $[b_{ij}]$, then $C = A + B$ is
represented as $[c_{ij}] = [a_{ij} + b_{ij}]$, and
 $D = A - B$ is represented as $[d_{ij}] = [a_{ij} - b_{ij}]$

— Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad A - B = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Matrix Algebra

4

— Matrix can be multiplied by a number (scalar): $\lambda A = \lambda [a_{ij}] = [\lambda a_{ij}]$

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

— Examples in Excel

— Matrix multiplication can be performed iff the number of columns of the 1st matrix is ~~the~~ same as the number of rows of the second.

$$\bullet : (\mathbb{R}^{r_1 \times c_1}, \mathbb{R}^{r_2 \times c_2}) \rightarrow \mathbb{R}^{r_1 \times c_2}, \quad \underline{\underline{c_1 = r_2}}$$

— Let us first see how we can multiply matrices of the type $1 \times n$ and $n \times 1$:

$$\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \end{bmatrix}_{1 \times 1}$$

Matrix Algebra

- Excel's function `SUMPRODUCT` does this (although you don't have to take care if vectors are entered as $1 \times n$ or $n \times 1$ type)
- Example: $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4 = 11 = [11]$
(in Excel as well)
- Now we can define how matrices are generally multiplied: if the condition for the multiplication is satisfied (the # of c-s of the 1st matrix is equal to the # of r-s of the 2nd), then we will do the following:

$$\begin{bmatrix} \text{---} & r_1 \\ \text{---} & r_2 \\ \vdots & \\ \text{---} & r_n \end{bmatrix}_{n \times R} \cdot \begin{bmatrix} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{bmatrix}_{R \times M} = \begin{bmatrix} r_1 c_1 & r_1 c_2 & \dots & r_1 c_m \\ r_2 c_1 & r_2 c_2 & \dots & r_2 c_m \\ \vdots & \vdots & \ddots & \vdots \\ r_n c_1 & r_n c_2 & \dots & r_n c_m \end{bmatrix}_{n \times m}$$

Matrix Algebra

— The mathematical formula: $C = A \cdot B$

$$A = [a_{ij}]_{n \times k}, \quad B = [b_{ij}]_{k \times m}, \quad C = [c_{ij}]_{n \times m}$$

$$c_{ij} = \sum_{s=1}^k a_{is} b_{sj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj}$$

$$= r_i C_j$$

— Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \cdot \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2} =$

$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & 4 \cdot 4 + 5 \cdot 5 + 6 \cdot 6 \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$$

— Excel's function for this is MMULT

— Examples in Excel

— MMULT can be used also as SUMPRODUCT

— Operations with matrices can be applied several times

$$A(B+C)D = (AB+AC)D = ABD+ACD$$

Matrix Algebra

– Note that $AB \neq BA$ (generally) !!

– Let us now consider quadratic ($n \times n$) matrices

– $I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}$ – Unity matrix

– For any $A \in \mathbb{R}^{nn} (\mathbb{C}^{nn})$ we have

$$AI = IA = A$$

– Inverse matrix: can be defined just for quadratic matrices with non-vanishing determinant.

$$(\forall A \in \mathbb{R}^{nn}) \det A \neq 0 \Rightarrow \begin{pmatrix} \begin{matrix} A^{-1} \\ 1 \end{matrix} \end{pmatrix} \cdot AA^{-1} = I$$

$$A^{-1}A = I$$

– In Excel there are functions MDETERM and MINVERSE that can help us in those calculations

Matrix Algebra

— Let us apply matrix algebra to portfolio construction

— For a 2-asset portfolio, let R_1 and R_2 be asset returns, and w and $(1-w)$ weights of these assets in the portfolio; so, we have

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad W = \begin{bmatrix} w \\ 1-w \end{bmatrix}$$

— The portfolio return is

$$R_p = w R_1 + (1-w) R_2 = W^T R$$

— The portfolio variance is

$$\begin{aligned} \text{Var}(R_p) &= \text{Var}(w R_1) + \text{Var}((1-w) R_2) + \\ &\quad + 2 \text{Cov}(w R_1, (1-w) R_2) \end{aligned}$$

$$\begin{aligned} \text{Var}(w R_1) &= \overline{(w R_1)^2} - \overline{w R_1}^2 = w^2 (\overline{R_1^2} - \overline{R_1}^2) = \\ &= w^2 \text{Var}(R_1) \end{aligned}$$

Matrix Algebra

— Therefore,

$$\text{Var}(R_p) = w^2 \text{Var}(R_1) + (1-w)^2 \text{Var}(R_2) + 2w(1-w) \text{Cov}(R_1, R_2)$$

— If we introduce $w_1 = w$, $w_2 = 1-w$,
then $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and

$$\text{Var}(R_p) = w_1^2 \text{Var}(R_1) + w_2^2 \text{Var}(R_2) + 2w_1 w_2 \text{Cov}(R_1, R_2)$$

Now, we can introduce covariance matrix of assets

$$\begin{bmatrix} \text{Cov}(R_1, R_1) & \text{Cov}(R_1, R_2) \\ \text{Cov}(R_2, R_1) & \text{Cov}(R_2, R_2) \end{bmatrix} = V$$

$\begin{matrix} = \text{Var}(R_1) & & \\ & & \text{Var}(R_2) \end{matrix}$

— We see that

$$\text{Var}(R_p) = W^T V W$$

— PORTFOLIO STANDARD DEVIATION $\sigma_p = \sqrt{\text{Var}(R_p)}$

Matrix Algebra

- Example in Excel: Find the volatility of a portfolio containing three assets with variances $S_1^2 = 0.01887$, $S_2^2 = 0.02763$, $S_3^2 = 0.00866$ and covariances $S_{12} = 0.001722$, $S_{23} = 0.00609$, $S_{13} = 0.00721$, and with weights $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.3$
- We will apply matrix algebra on minimum risk portfolio problems later.

- Let us now introduce quadratic forms

- General form

$$Q(x_1, x_2, \dots, x_n) = [x_1 \dots x_n] Q \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $Q \in \mathbb{R}^{n \times n}$ is a quadratic matrix

If we introduce x as a vector, then

$$Q(x) = x^T Q x$$

Matrix Algebra

- Each term can be expressed in several ways: $L_{ij} x_i x_j = L_{ji} x_j x_i = \frac{1}{2}(L_{ij} + L_{ji}) x_i x_j$,

i.e. for a given i and j we can set

$$L_{ij} = a, \quad L_{ji} = 0; \quad \text{or} \quad L_{ij} = 0, \quad L_{ji} = a,$$

$$\text{or} \quad L_{ij} = \frac{1}{2}a, \quad L_{ji} = \frac{1}{2}a$$

The result is the same

- Example: $Q(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$

$$Q = [x_1 \ x_2] \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= [x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= [x_1 \ x_2] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 + 2x_1x_2 + x_2^2 = x_1^2 + 2x_2x_1 + x_2^2 =$$

$$= x_1^2 + x_1x_2 + x_2x_1 + x_2^2$$

Matrix Algebra

(12)

- That way, we can always replace matrix Q with the symmetric matrix

$$\frac{1}{2}(Q + Q^T)$$

- Symmetric matrix: $A = A^T$, $a_{ij} = a_{ji}$

- Example in Excel

- Definiteness: If Q satisfy one of the following conditions for all vectors $x \neq 0$, we say it has defined definiteness

$$x^T Q x > 0 : Q \text{ is positive definite}$$

$$x^T Q x < 0 : Q \text{ is negative definite}$$

$$x^T Q x \geq 0 : Q \text{ is positive semi-definite}$$

$$x^T Q x \leq 0 : Q \text{ is negative semi-definite}$$

- Example: $\text{Var}(R_p) = W^T V W$, where V is the covariance matrix of all assets in a portfolio with ~~weights~~ $W = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$

Matrix Algebra

- Variance of a portfolio can be written

as $V = D C D$

where $D = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$ is a diagonal

matrix containing volatilities (st. dev's) of individual assets, while C is the correlation matrix: $c_{ij} = \frac{\text{var}(R_i, R_j)}{\sigma_i \sigma_j}$

- The definition of Markowitz problem:

Find the portfolio weights W which minimize the portfolio variance, i.e. for a given variance matrix V and fixed portfolio return (rate) r

$W^T V W$ should be minimized subject to $\sum_{i=1}^n w_i = 1$ and $W^T R = r$,

- Consider example 14 Excel