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Constrained Optimization

- Again we want to find $\max\limits_{x \in S} f(x)$ or $\min\limits_{x \in S} f(x)$ for some function $f : \mathbb{R}^n \to \mathbb{R}$
- Here we have some constraints:

$$g_1(x) = 0, \dots, g_M(x) = 0 \qquad \text{(equalities)}$$

$$h_1(x) \geq 0, \dots, h_p(x) \geq 0 \qquad \text{(inequalities)}$$

Equality-Constrained Optimization: Lagrange Method.

Theorem: If x^* is a local optimum, and

$$\operatorname{rank} \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \vdots & & \\ \frac{\partial g_M}{\partial x_1}(x^*) & \cdots & \frac{\partial g_M}{\partial x_n}(x^*) \end{bmatrix} = M,$$

then there exists a vector $\lambda^* = \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_M^* \end{bmatrix} \in \mathbb{R}^m$ such that

$$\frac{\partial f}{\partial x_i}(x^*) + \sum_{j=1}^{M} \lambda_j^* \frac{\partial g_j}{\partial x_i}(x^*) = 0. \qquad (i = 1, \dots, n)$$

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- This is equivalent to finding max or min of the Lagrangian:

$$L = f + \sum_{j=1}^{M} \lambda_j g_j : \quad \frac{\partial L}{\partial x_i} = 0 \quad \underline{\text{and}} \quad \overbrace{\frac{\partial L}{\partial \lambda_j} = 0}^{\text{constraints}}$$

- Additional conditions (maximality or minimality) are the same as in unconstrained case for f, i.e. we need to investigate definiteness of Hessian matrix for L

- Be aware that $\text{rank}\left[\frac{\partial g_j}{\partial x_i}(x^*)\right]$ <u>must</u> be equal to number of constraints M! If not, this procedure will not work, although max or min can exist.

- Example: $f(x,y) = x^2 - y^2$, $g(x,y) = 1 - x^2 - y^2$

$$L = f(x,y) + \lambda g(x,y) = x^2 - y^2 + \lambda(1 - x^2 - y^2)$$

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$$\frac{\partial L}{\partial x} = 2x - 2\lambda x = 0 \Rightarrow x^*(1-\lambda^*)=0$$

$$\frac{\partial L}{\partial y} = -2y - 2y\lambda = 0 \Rightarrow y^*(1+\lambda^*)=0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 = 0 \Rightarrow x^{*2} + y^{*2} = 1$$

If $x^*=0$, from the third equation follows $y^* = \pm 1$, and from the second $\lambda^* = -1$, so we have two solutions $(x^*, y^*, \lambda^*) = (0, \pm 1, -1)$.

If $x^* \neq 0$, then $\lambda^* = 1$ from the first, and $y^* = 0$ from the second equation. From the third equation we have $x^* = \pm 1$, so we have two additional solutions: $(\pm 1, 0, 1)$.

$\text{rank}\left[\frac{\partial g_j}{\partial x_i}\right] = \text{rank}\left[-2x^* \quad -2y^*\right]$, so for the first solution we have

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$$\operatorname{rank}\left[\frac{\partial f_j}{\partial x_i}\right] = \operatorname{rank}\begin{bmatrix} 0 & \mp 2 \end{bmatrix} = 1 = M,$$

so all conditions for the theorem are satisfied.

For the second pair is

$$\operatorname{rank}\left[\frac{\partial f_j}{\partial x_i}\right] = \operatorname{rank}\begin{bmatrix} \mp 2 & 0 \end{bmatrix} = 1 = M,$$

so everything is OK here also.

– Hessian matrix is

$$H = \begin{bmatrix} 2-2\lambda & 0 \\ 0 & -2-2\lambda \end{bmatrix}$$

For $(0, \pm 1, -1)$ is $H = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0$

For $(\pm 1, 0, 1)$ is $H = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \geqslant 0$

All solutions are minimums, although not strict ones.

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Inequality-Constrained Optimization: Kuhn-Tucker Method

Theorem: If x^* is a local maximum, and $\operatorname{rank}\left[\frac{\partial h_{Ej}}{\partial x_i}\right] = M_E$, where h_E are constraints with the equality sign valid for x^*, and M_E is the number of those constraints; then there exists a vector $\lambda^* = \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_N^* \end{bmatrix} \in \mathbb{R}^N$ such that

$$\frac{\partial f}{\partial x_i}(x^*) + \sum_{j=1}^{N} \lambda_j^* \frac{\partial h_j}{\partial x_i}(x^*) = 0 \qquad (i = 1, .., n)$$

$$h_i(x^*) \geq 0$$

$$\lambda_i^* \geq 0$$

$$\lambda_i^* h_i(x^*) = 0 .$$

- Note that you need to transform the problem of optimization into the

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maximization problem with all conditions of the form $h_i(x^*) \geq 0$, which can easily be done.

- We can again construct Lagrangian

$$L = f + \sum_{j=1}^{n} \lambda_j h_j ,$$

and the equations for x^* are

$$\frac{\partial L}{\partial x_i} = 0 \;, \quad \frac{\partial L}{\partial \lambda_j} \geq 0 \;, \quad \lambda_j^* \geq 0$$

$$\lambda_j^* h_j^*(x^*) = 0 .$$

- Be aware of the problems with rank!

- Example: $f(x,y) = x+y$, $h(x,y) = xy - 1$

$$L = f + \lambda h = x + y + \lambda(xy - 1)$$

$$\frac{\partial L}{\partial x} = 1 + \lambda y = 0 \Rightarrow \lambda^* y^* = -1 \Rightarrow \lambda^* \neq 0, \; y^* \neq 0$$

$$\frac{\partial L}{\partial y} = 1 + \lambda x = 0 \Rightarrow \lambda^* x^* = -1 \Rightarrow \lambda^* \neq 0, \; x^* \neq 0$$

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$$y^* = x^* = -\frac{1}{\lambda^*}$$

$$\frac{\partial L}{\partial \lambda} = xy - 1 \geq 0 \quad , \quad \lambda^* \geq 0 \quad , \quad \lambda^*(x^* y^* - 1) = 0$$

$$x^* y^* - 1 = \frac{1}{\lambda^{*2}} - 1 \geq 0 \Rightarrow \lambda^{*2} \leq 1$$

with $\lambda^* \geq 0 \Rightarrow 0 \leq \lambda^* \leq 1$

since $\lambda^* \neq 0$, we have $x^* y^* = 1 \Rightarrow \lambda^* = 1$,

$x^* = y^* = -1$, so the solution is $(-1, -1, 1)$.

$$\text{rank}\left[\frac{\partial h_{E_i}}{\partial x_i}(x^*)\right] = \left[x^* \;\; y^*\right] = \left[-1 \;\; -1\right] = 1,$$

so the conditions are OK.

- For homework you can check if this is a maximum or not (it cannot be a minimum!)

- For the definition and introduction to the Markowitz problem, you can consult lecture notes from Jan 9, 2004., or Ch. 6 of „Investment Science" by D. Luenberger.

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Economic Example: Markowitz Problem

If we have n assets with expected rates of return $\bar{r}_1, \dots, \bar{r}_n$ and the covariances σ_{ij}, $(i,j = 1, \dots, n)$, a portfolio is defined by a set of n weights w_i, $i = 1, \dots, n$, so that $\sum_{i=1}^{n} w_i = 1$. The problem is to

$$\text{minimize } \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij},$$

subject to constraints:

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}, \quad \sum_{i=1}^{n} w_i = 1.$$

- Lagrangian is

$$L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} + \lambda \left(\sum_{i=1}^{n} w_i r_i - \bar{r} \right) +$$

$$+ \mu \left(\sum_{i=1}^{n} w_i - 1 \right)$$

– To simplify things, we will work with $n=2$, so that

$$L = \frac{1}{2}\left(w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12} + w_2 w_1 \sigma_{21} + w_2^2 \sigma_2^2\right) +$$

$$+ \lambda\left(w_1 \bar{r}_1 + w_2 \bar{r}_2 - \bar{r}\right) + \mu\left(w_1 + w_2 - 1\right)$$

– The equations are

$$\frac{\partial L}{\partial w_1} = w_1 \sigma_1^2 + \frac{1}{2} w_2 \sigma_{12} + \frac{1}{2} w_2 \sigma_{21} + \lambda \bar{r}_1 + \mu = 0$$

$$\frac{\partial L}{\partial w_2} = \frac{1}{2} w_1 \sigma_{12} + \frac{1}{2} w_1 \sigma_{21} + w_2 \sigma_2^2 + \lambda \bar{r}_2 + \mu = 0$$

$$\frac{\partial L}{\partial \lambda} = w_1 \bar{r}_1 + w_2 \bar{r}_2 - \bar{r} = 0$$

$$\frac{\partial L}{\partial \mu} = w_1 + w_2 - 1 = 0$$

– Since $\sigma_{12} = \sigma_{21}$, we have a set of 4 equations in 4 unknowns $w_1, w_2, \lambda,$ and μ:

$$\begin{aligned}
w_1 \sigma_1^2 + w_2 \sigma_{12} + \lambda \bar{r}_1 + \mu &= 0 \\
w_1 \sigma_{12} + w_2 \sigma_2^2 + \lambda \bar{r}_2 + \mu &= 0 \\
w_1 \bar{r}_1 + w_2 \bar{r}_2 &= \bar{r} \\
w_1 + w_2 &= 1
\end{aligned}$$

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- And now we have a system of linear equations, which can be easily solved
- General set of equations is

$$\sum_{j=1}^{n} \sigma_{ij} w_j + \lambda \bar{r}_i + \mu = 0 \qquad (i=1,\dots,n)$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

- we have system of $n+2$ ~~unknowns~~ equations in $n+2$ unknowns (w_i, λ, μ).
- Home work Problem: Solve the Markowitz problem for a set of three uncorrelated assets, with $\sigma_1^2 = 1$, $\sigma_2^2 = 1$, $\sigma_3^2 = 1$, and $\bar{r}_1 = 1$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$.

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