UNIVERSITY OF BELGRADE
FACULTY OF ECONOMICS

SECCF
South European Center for Contemporary Finance

Eigenvalues, Eigenvectors

- Operator is a function that can act on a vector and change it:

$$A(x) = y$$

- If vectors are matrices $(n \times 1)$, and we are interested just in linear functions, then each such function can be represented through a matrix:

$$A \cdot x = y$$

- Function acts on a vector x through matrix multiplication

- Example: For 2×2 matrices and 2×1 vectors, matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents anticlockwise rotation of 90°:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

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Eigenvalues, Eigenvectors

- If $A \cdot x = \lambda x$, where λ is some scalar, then x is the eigenvector of matrix A, and λ is the corresponding eigenvalue

- Example: Let us find eigenvectors and eigenvalues of a matrix $\begin{bmatrix} 4.6 & -1.2 \\ -1.2 & 1.4 \end{bmatrix}$:

$$\begin{bmatrix} 4.6 & -1.2 \\ -1.2 & 1.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow$$

$$\begin{aligned} 4.6x - 1.2y &= \lambda x \\ -1.2x + 1.4y &= \lambda y \end{aligned} \Leftrightarrow$$

$$\begin{aligned} (4.6-\lambda)x - 1.2y &= 0 \\ -1.2x + (1.4-\lambda)y &= 0 \end{aligned} \Leftrightarrow$$

$$\frac{4.6-\lambda}{-1.2} = \frac{-1.2}{1.4-\lambda} \Leftrightarrow$$

$$(4.6-\lambda)(1.4-\lambda) = 1.2^2$$

Eigenvalues, Eigenvectors

$$-4.6\lambda - 1.4\lambda + \lambda^2 + 6.44 = 1.44 \Rightarrow$$

$$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = \frac{6 \pm \sqrt{36 - 4\cdot 5}}{2} =$$

$$= \frac{6 \pm \sqrt{16}}{2} = \frac{6 \pm 4}{2} = \begin{cases} 5 \\ 1 \end{cases}$$

For $\lambda = 5$:

$$\begin{aligned} -0.4x - 1.2y &= 0 \\ -1.2x - 3.6y &= 0 \end{aligned} \iff x = -3y \text{ (both)}$$

Therefore, for the eigenvalue $\lambda = 5$, we have the following eigenvectors

$$\begin{bmatrix} -3y \\ y \end{bmatrix}, \quad y \in \mathbb{R}$$

Vectors are usually normalized so that $v^T v = 1$; if we require this, then

$$\begin{bmatrix} -3y \\ y \end{bmatrix}^T \begin{bmatrix} -3y \\ y \end{bmatrix} = \begin{bmatrix} -3y & y \end{bmatrix} \begin{bmatrix} -3y \\ y \end{bmatrix} = 10y^2 = 1 \Rightarrow y = \frac{1}{\sqrt{10}}$$

Eigenvalues, eigenvectors

- So, the normalized eigenvector for $\lambda = 5$ is

$$\begin{bmatrix} -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$$

- For $\lambda = 1$ do the math yourself

- Mathematical theorem states that if we normalize all eigenvectors, then:

$$x_i^T x_j = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

- Let us now define determinants: If A is a quadratic matrix, then

$$\det A : \mathbb{R}^{nn} \to \mathbb{R}$$

$$\det[a_{11}] = a_{11}, \quad \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} +$$

Eigenvalues, eigenvectors

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

– Determinant can be calculated by expressing the n-order determinant through (n-1)-order determinants (linear combination of n of them!), by using any column or row; coefficients in the resulting linear combination are the elements of the chosen column (row), the determinant standing to each element is obtained from the original one by removing the column and the row of the element used; signs are defined as:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & \cdots & \\ + & - & + & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

– More examples

eigenvalues eigenvectors

- Now we see that the equation $Ax = \lambda x$ can be written as $(A - \lambda I_n) \cdot x = 0$.
- It has non-zero solutions in x <u>iff</u> $\det(A - \lambda I) = 0$ (characteristic equation)
- We can calculate eigenvalues from this equation
- For the variance matrix to have Cholesky decomposition it is necessary and sufficient to be $V > 0$ (or $V \geq 0$)
- This can be checked using eigenvalues:

$$V > 0 \iff \text{all eigenvalues are} > 0$$

$$V \geq 0 \iff \text{all eigenvalues are} \geq 0$$

- Examples in Excel
- HW 2 due on October 25, 2006.