UNIVERSITY OF BELGRADE
FACULTY OF ECONOMICS

SECCF
South European Center for Contemporary Finance

QM, Dec 22, 2004.

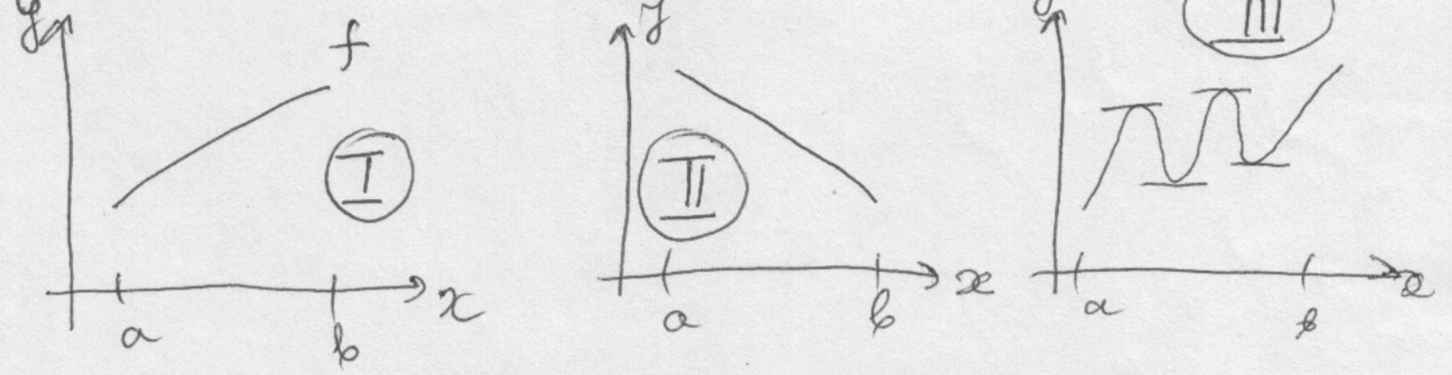
Unconstrained Optimization with one Variable

$f : \mathbb{R} \to \mathbb{R}$ and we want to find

$\max_{x \in S} f(x)$, or $\min_{x \in S} f(x)$, where

$S \subseteq \mathbb{R}$, i.e. $S \subseteq D(f)$.

- We will assume that f is differentiable, i.e. that f' exist for $\forall x \in S$
- Let us also assume that S is an interval, $S = [a, b]$ (or $S = (a, b)$, or any other combination of borders)
- We can have different possibilities:



QM, Dec 22, 2004.

Case I: Function f is monotone increasing, so

$$\min_{x\in[a,b]} f(x) = f(a), \quad x_{min} = a$$

$$\max_{x\in[a,b]} f(x) = f(b), \quad x_{max} = b$$

– This is equivalent to the condition

$$(\forall x \in (a,b))\ f'(x) \geq 0$$

(At the borders f' can be zero, i.e. we can have $f'(a)=0$ or $f'(b)=0$, but $f'(a)$ and $f'(b)$ must be nonnegative)

Example: $f(x) = x^2 \quad S = [0,3]$

$f'(x) = 2x, \quad \forall x \in (0,3) \ f'(x) > 0 \Rightarrow$

$$\max_{x\in[0,3]} x^2 = 3^2 = 9 \qquad \min_{x\in[0,3]} x^2 = 0^2 = 0$$

QM, Dec 22, 2004.

Example: $f(x) = x^2$, $S = [-1, 1]$

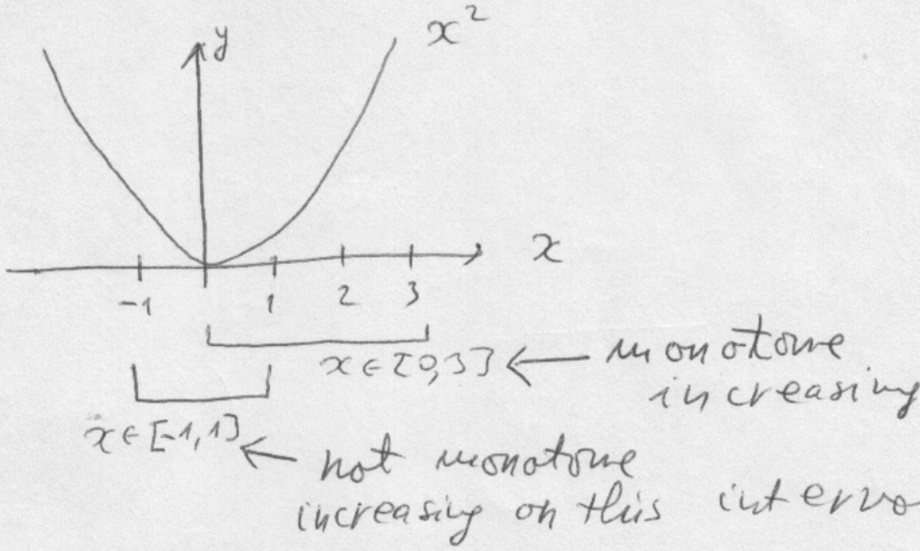
$f'(x) = 2x$, if $x \in (-1, 0)$, $f'(x) < 0$,

and if $x \in (0, 1)$, $f'(x) > 0$

So, this function on $[-1, 1]$ is

not monotone increasing.

We will use other methods to find min and max of this function.



Example: $f(x) = e^x$, $S = \mathbb{R}$

$f'(x) = e^x$, $(\forall x \in \mathbb{R})$ $f'(x) > 0$, monotone increasing

But, S does not have borders, so f does not have min and max!

QM, Dec 22, 2004. (4)

Case II: Function is monotone decreasing, so

$$\min_{x\in[a,b]} f(x) = f(b), \quad x_{min} = b$$

$$\max_{x\in[a,b]} f(x) = f(a), \quad x_{max} = a$$

This is equivalent to the condition

$$(\forall x \in (a,b))\ f'(x) < 0$$

(At the borders we can have $f'(a)=0$ or $f'(b)=0$, but they must be nonnegative)

Example: $f(x) = x^4$, $S = [-4, -2]$

$f'(x) = 4x^3$, $(\forall x \in [-4,-2])\ f'(x) < 0 \Rightarrow$

$$\max_{x\in[-4,-2]} x^4 = (-4)^4 = 256, \quad \min_{x\in[-4,-2]} x^4 = (-2)^4 = 16$$

QM, Dec 22, 2004.

- If f is monotone increasing at some interval, then $-f$ is monotone decreasing on the same interval and vice versa; that exchanges min's and max's

<u>Case III</u>: Function is not monotone and have max or min at some point $x^* \in (a,b)$ (interior of S)

- At min or max tangent is parallel to x-axis and have slope $= \emptyset$, i.e.

$$\boxed{f'(x^*) = 0}$$

- This is neccessary condition!
- It is not sufficient to tell if x^* is min or max

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080

QM, Dec 22, 2004.

- Sufficient conditions:

If $f''(x^*) > 0$, then x^* is min

If $f''(x^*) < 0$, then x^* is max

- The algorithm:

1) Find all zeros of the $f'(x)$ on $x \in (a, b)$, i.e. all x^*'s so that $f'(x^*) = 0$

2) Check the sign of $f''(x^*)$ for all x^*'s; those with $f''(x^*) > 0$ are minima, and those with $f''(x^*) < 0$ are maxima

3) If there are no zeros of f' on (a, b), then f' is either positive (case I) or negative (case II), and maximum and minimum are on the borders

QM, Dec 22, 2004.

- Example: $f(x) = x^2, \quad S = \mathbb{R}$

Step 1: $f'(x) = 2x, \quad f'(x^*) = 2x^* = 0 \Rightarrow$

$x^* = 0$ unique solution!

step 2: $f''(x) = 2 > 0 \Rightarrow f''(x^*) = 2 > 0$

$\Rightarrow x^* = 0$ is minimum

since there are no other zeros of f', this is the only extremal point.

- Example: $f(x) = x^2 \sin x - \frac{x^3}{3} \cos x, \quad S = \mathbb{R}$

step 1: $f'(x) = 2x \sin x + \cancel{x^2 \cos x} -$

$- \cancel{\frac{3x^2}{3} \cos x} + \frac{x^3}{3} \sin x =$

$= x\left(2 + \frac{x^2}{3}\right) \sin x$

QM, Dec 22, 2004.

since $2+\frac{x^2}{3}>0$, and

$\sin x = 0$ for $x_k = k\pi$, $k \in \mathbb{Z} = \{\ldots,-2,-1,0,1,2\ldots\}$

We have $f'(x^*)=0$ for

$x^*=0$ and $x_k^* = k\pi$, $k\in\mathbb{Z}$

(for $k=0$ we get $x_0^*=x^*$, so $x_k^*=k\pi$, $k\in\mathbb{Z}$ contains all solutions)

<u>Step 2</u>: $f''(x) = (2+\frac{x^2}{3})\sin x + x\cdot\frac{2x}{3}\sin x + x(2+\frac{x^2}{3})\cos x$

$f''(x_k^*) = (2+\frac{k^2\pi^2}{3})\sin k\pi + \frac{2}{3}k^2\pi^2 \sin k\pi + k\pi(2+\frac{k^2\pi^2}{3})\cos k\pi$

Since $\sin k\pi = 0$, $\cos k\pi = (-1)^k$, we have

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080

QM, Dec 22, 2004.

$$f''(x_{k}^{*}) = (-1)^k k\pi\left(2 + \frac{k^2\pi^2}{3}\right)$$

$2 + \frac{k^2\pi^2}{3} > 0$ for all $k \in \mathbb{Z}$

We can consider three cases:

1° $k = n \in \mathbb{N} = \{1, 2, 3, \dots\}$

2° $k = -n$, $n \in \mathbb{N}$

3° $k = 0$

For $k = n \in \mathbb{N}$, we have

$k\pi = n\pi > 0$, and $(-1)^k = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

$$\Rightarrow \quad f''(x_k^*) = f''(x_n^*) \begin{cases} < 0, & n \text{ odd} \Rightarrow \max \\ > 0, & n \text{ even} \Rightarrow \min \end{cases}$$

For $k = -n$, $n \in \mathbb{N}$, we have $(-1)^k = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

$k\pi = -n\pi < 0$, and

$$\Rightarrow \quad f''(x_k^*) = f''(x_{-n}^*) \begin{cases} > 0, & n \text{ odd} \Rightarrow \min \\ < 0, & n \text{ even} \Rightarrow \max \end{cases}$$

QM, Dec 22, 2004.

– For $k = 0$ we have $f''(x_0^*) = f''(0) = 0$.

– If this is the case, we need to calculate further derivatives and find first one different from zero at $x_0^* = 0$ (say it is $f^{(n)}(x_0^*) = f^{(n)}(0) \neq 0$). That means

$f^{(m)}(0) = 0$ for $m = 1, \dots, n-1$, and

$f^{(n)}(0) \neq 0$

– If n is even, then

if $f^{(n)}(0) > 0 \Rightarrow x_0^* = 0$ is <u>min</u>

if $f^{(n)}(0) < 0 \Rightarrow x_0^* = 0$ is max

– If n is odd, $x_0^* = 0$ is not min nor max! (it is saddle point of f)

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080