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Unconstrained Optimization with Multiple Variables

$f: \mathbb{R}^n \to \mathbb{R}$, i.e. we have $f(x, y, z, \dots)$

Set S is some subset of $\mathbb{R}^n$

We want to find $\max_{x \in S} f$ or $\min_{x \in S} f$.

- Note that is $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ satisfies $f(x^*) = \min_{x \in S} f(x)$, then

$-f(x^*) = \max_{x \in S} (-f(x)) \Rightarrow$ we can exchange min for max or vice versa!

- We will here consider only minima and maxima at the interior of S.
- If there are no extremal points at the interior, we will have to look at the borders of S, but that

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can easily be done.

– The Algorithm:

1) Find all first derivatives of f and solve the system

$$\frac{\partial f}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$$

The solution is denoted by $x^* = (x_1^*, \ldots, x_n^*)$

2) For every solution of 1) calculate Hessian matrix (at x^*)

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

H is symmetric matrix!

If the corresponding quadratic form $x^T H x$ is:

positively defined, x^* is min

negatively defined, x^* is max

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- If H is semidefinite, we need to look closely at the function – we will not cover this case.

- Example: $f(x,y) = x^2 + y^2$

step 1:

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = 2x = 0 \\[2em] \dfrac{\partial f}{\partial y} = 2y = 0 \end{array}\right\} \Rightarrow x^* = y^* = 0$$

step 2:

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(0,0) & \dfrac{\partial^2 f}{\partial x \partial y}(0,0) \\[1em] \dfrac{\partial^2 f}{\partial y \partial x}(0,0) & \dfrac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H_{11} > 0$, $\det H > 0$ $\Rightarrow$

$(0,0)$ is minimum

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-Example: $f(x,y) = x^4 + x^2y^2 + y^4 - 4x^2 - 4y^2$

$$\frac{\partial f}{\partial x} = 4x^3 + 2xy^2 - 8x = 0$$

$$\frac{\partial f}{\partial y} = 2x^2y + 4y^3 - 8y = 0$$

If $x=0$ then $\frac{\partial f}{\partial x} = 0$ and

$\frac{\partial f}{\partial y} = 4y(y^2-2) = 0$, so we have

solutions $\boxed{(0,0),\ (0, \pm\sqrt{2})}$

If $x \neq 0$, we will divide $\frac{\partial f}{\partial x}$ with $2x$, and obtain

$$2x^2 + y^2 = 4$$

$$\frac{\partial f}{\partial y} = 2y(x^2 + 2y^2 - 4) = 0$$

If $y=0$, $\frac{\partial f}{\partial y} = 0$, and $2x^2 + 0^2 = 4 \Rightarrow x = \pm\sqrt{2}$,

so the solutions are $\boxed{(\pm\sqrt{2}, 0)}$

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If $y \neq 0$, then we have a system

$$2x^2 + y^2 = 4$$
$$x^2 + 2y^2 = 4$$

which has a unique solution $x^2 = y^2 = \frac{4}{3}$, so the solutions to the original system are all combination of signs

$$\left(\pm\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{4}{3}}\right), \left(\pm\sqrt{\tfrac{4}{3}}, \mp\sqrt{\tfrac{4}{3}}\right);$$

$$H = \begin{bmatrix} 12x^2 + 2y^2 - 8 & 4xy \\ 4xy & 2x^2 + 12y^2 - 8 \end{bmatrix}$$

For $(0,0)$ we have $H = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$

$\det H_{11} = -8 < 0$, $\det H = 64 > 0 \Rightarrow$

H is negatively defined $\Rightarrow$

$(0,0)$ is maximum of f

Check other solutions for HW!

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– Economic Example: Suppose a single firm faces two markets, A and B. Each market has its own demand function $d_i(q_i)$ $(i = A, B)$, where q_A and q_B is the number of products offered at the markets. Prices at the markets are different, say p_A and p_B ~~~~ per single product. The cost function is $C(q_A+q_B)$. How to maximize the profit

$$\pi(q_A, q_B) = p_A d_A(q_A) + p_B d_B(q_B) - C(q_A+q_B)$$

– We are searching for $\max\limits_{q_A, q_B} \pi(q_A, q_B)$

So

$$\frac{\partial \pi}{\partial q_A} = p_A d_A'(q_A) - \frac{\partial C}{\partial q_A} = 0$$

$$\frac{\partial \pi}{\partial q_B} = p_B d_B'(q_B) - \frac{\partial C}{\partial q_B} = 0$$

These are conditions for extremal point

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Hessian matrix is:

$$H = \begin{bmatrix} p_A d_A''(q_A) - \frac{\partial^2 C}{\partial q_A^2} & -\frac{\partial^2 C}{\partial q_A \partial q_B} \\ -\frac{\partial^2 C}{\partial q_A \partial q_B} & p_B d_B''(q_B) - \frac{\partial^2 C}{\partial q_B^2} \end{bmatrix},$$

and for a found solutions of

$\frac{\partial \Pi}{\partial q_A} = 0$, $\frac{\partial \Pi}{\partial q_B} = 0$ the neccessary conditions are (it has to be maximum).

$$H_{11} < 0 \iff p_A d_A''(q_A^*) - \frac{\partial^2 C}{\partial q_A^2}(q_A^*, q_B^*) < 0$$

$$\det H > 0 \iff \left(p_A d_A''(q_A^*) - \frac{\partial^2 C}{\partial q_A^2}(q_A^*, q_B^*) \right) \cdot$$

$$\cdot \left(p_B d_B''(q_B^*) - \frac{\partial^2 C}{\partial q_B^2}(q_A^*, q_B^*) \right) - \left(\frac{\partial^2 C}{\partial q_A \partial q_B}(q_A^*, q_B^*) \right)^2 > 0$$

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