UNIVERSITY OF BELGRADE
FACULTY OF ECONOMICS

SECCF
South European Center for Contemporary Finance

Joint Distributions

- Used when considering two or more random variables at the same time
- We alredy covered some of these issues
- If two variables are statistically independent, then the realisations of one of them do not influence realisations of others in any way, i.e. there is no causal connection between them
- The measure of this is the covariance

$$Cov(X,Y) = E\left((X - E(X))(Y - E(Y)) \right) =$$

$$= E(XY) - E(X)E(Y) = \overline{XY} - \bar{X}\,\bar{Y}$$

- Positive covariance ⇒ variables have the same trends
- Negative covariance ⇒ variables have the oposite trends
- Correlation: $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$$Cov(X,Y) = \sigma_X \sigma_Y \rho_{XY}$$

Joint Distributions

- For a linear combination of random variables, we have

$$Z = \alpha X + \beta Y$$

$$E(Z) = \alpha E(X) + \beta E(y)$$

$$\overline{\alpha X + \beta Y} = \alpha \bar{X} + \beta \bar{Y}$$

$$Var(Z) = Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y) + $$
$$+ 2\alpha\beta \, Cov(X, Y)$$

- Iff X and Y are uncorrelated ($Cov(X,Y)=0$), we have $Var(X+Y) = Var(X) + Var(Y)$
- This has profound effects

~~[illegible]~~ - Binomial, normal, and log-normal distributions already covered
- Here we will cover:
  - Poisson distribution
  - Uniform distribution
  - t-distribution
  - Bivariate normal distribution

Joint distributions

- Poisson distribution describes rare events

$$P(X=r) = \frac{(\lambda t)^r e^{-\lambda t}}{r!}$$

This is the probability that some event, that occurs with the rate λ, will occur exactly r times during the time t

- It can be obtained as a limit of binomial distributions, if this continuous time is discretized to several time intervals and then the number of intervals is allowed to tend to infinity
- Properties:

$$E(X) = \lambda t = \mu, \quad P(X=r) = \frac{\mu^r e^{-\lambda t}}{r!}$$

- Example: Some event occurs with an expected frequency $\lambda = 24 \frac{\text{events}}{\text{day}}$. If we divide a day into three-hour slots, what is the probability that this event will occur exactly, 0, 1, 2, ..., 10 times in a given time slot?

• Excel function POISSON(, ,)

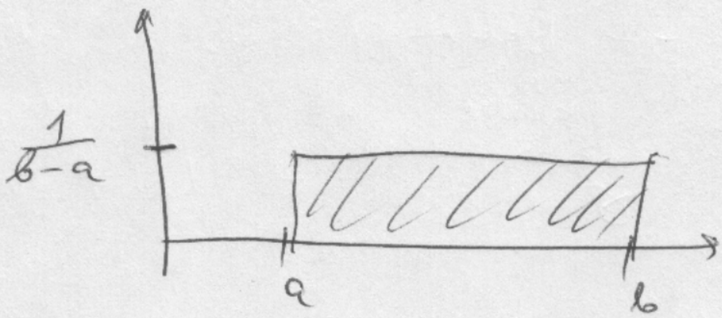
– Poisson distribution has a significant tail on the right hand side, i.e. it has positive skew

– Properties: $Var(X) = \mu$, $\sigma_X = \sqrt{\mu}$

$$E(X) = \mu$$

– Check this on the previous example

– The uniform continuous distribution on (a, b)



$$\text{Uniform}(a, b, x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}$$

– Example: check the above properties of a uniform distribution

- Why the uniform distribution is so important? For numerical simulations:
    - There are various methods for generating random variables according to this distribution
    - Other distributions can be obtained using the inversion method
- How the inversion method works?
    - Let X be distributed according to the uniform distribution on $(0, 1)$
    - Let f be some pdf
    - Let F be its cumulative distribution

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

    - Then, $F^{-1}(X)$ is distributed according to the pdf f
- Very extensively used in simulations!

Joint Distributions

- Student's t-distribution describes the mean of small samples of numbers from a normal distribution with an unknown variance
- Usually: samples with less than 30 elements

$$t(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi\sigma^2}}\left[1 + \frac{(x-\mu)^2}{(\nu-2)\sigma^2}\right]^{-\frac{\nu+1}{2}}$$

Γ is the Gamma function (here, $\Gamma(n) = n!$, $\Gamma(0) = 1$)

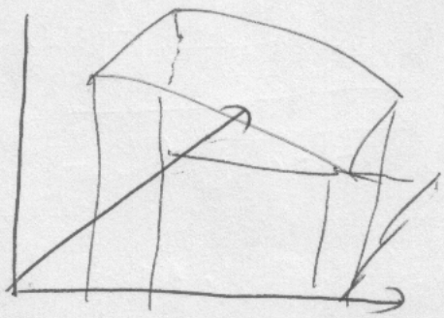
ν is the number of degrees of freedom

- Kurtosis is more prominent, i.e. tails are more significant than those of a normal distribution
- ν is usually one less than the number of sample elements (Sample size -1)
- If $\nu \to \infty$ then $t(x)$ tends to the normal distribution
- Example in Excel

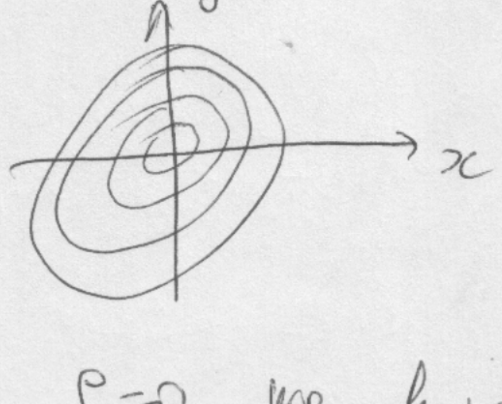
SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080

Joint Distributions

- Bivariate (Multivariate) Normal Distributions
- Here the pdf depends on two (or more) variables, hence to draw it we need three (or more) dimensional graphs



- For bivariate distributions we can use projections of this surface (for given values) on x-y axis (surface plots)



- If $\rho = 0$, we have circles; for $\rho \neq 0$ we have elipses

Joint Distributions

- For $\rho \to \pm 1$ the elipses degenerate into straight lines

- Generally

$$f(x,y) = \frac{1}{\sqrt{2\pi\sigma_x^2}\sqrt{2\pi\sigma_y^2}} \frac{1}{\sqrt{1-\rho^2}} \cdot$$

$$\cdot e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_x)^2}{\sigma_x^2} + 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right\}}$$

- So, if X and Y have a bivariate normal distribution, then the conditional density of Y given that X has value x is normal, with a mean

$$\mu_{Y|X=x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_x)$$

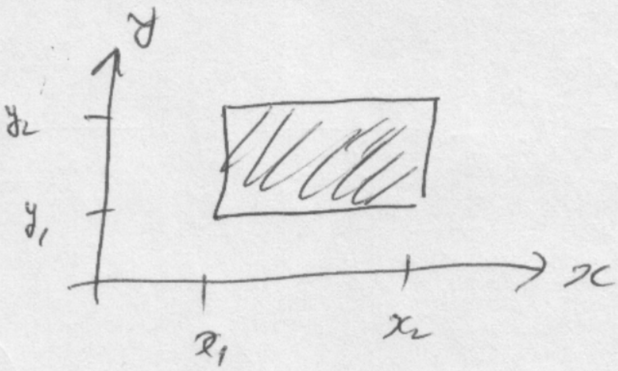
and a variance $\sigma^2_{Y|X=x} = \sigma_Y^2(1-\rho^2)$

Joint Distributions

- this lies at the heart of regression analysis
- The probability for the event

$x \in [x_1, x_2]$ AND $y \in [y_1, y_2)$

is the **volume** above the rectangle



This volume is bounded from above with the $f(x,y)$

- Be careful - not the surface of the rectangle, but the volume above it!