Numerical Methods I



- We use numerical approach when analytical approach is not available or it is too complex

- Example 1: An investment opportunity will return 5,000 at the end of year 1, 4,000 at the end of year 2, 4,000 at the end of year 3, 2,500 at the end of year 4 and 53,000 at the end of year 5, the latter including the realisation of assets. The investment will cost 50,000. What is the yield?

If we denote yield by $y$ [%], and introduce $x = 1 + \frac{y}{100}$, then we have

$$50,000 = \frac{5000}{x} + \frac{4000}{x^2} + \frac{4000}{x^3} + \frac{2,500}{x^4} + \frac{53000}{x^5}$$

or, by multiplying with $x^5$,

$$50,000\, x^5 - 5,000\, x^4 - 4,000\, x^3 - 4,000\, x^2 - 2,500\, x - 53,000 = 0.$$

Numerical Methods I



- Example 2: A loan of 50,000 is to be repaid by 25 equal annual payments of 3,500. What is the interest rate?

If $y$ is the interest rate and $x = 1+y$, then $x$ is the annual compounding factor, and the value of the loan after the first year is $50{,}000 \cdot x$. After the payment, the loan value is $50{,}000x - 3{,}500$. After the second year's payment, we have $(50{,}000x - 3{,}500)x - 3{,}500$. At the end we have

$$0 = (\cdots(((50{,}000x - 3{,}500)x - 3{,}500)\cdots)x - 3{,}500$$

where factor $x$ must appear exactly 25 times. This simplifies to

$$50{,}000x^{25} - 3{,}500x^{24} - 3{,}500x^{23} - \cdots - 3{,}500x - 3{,}500 = 0$$

Note here that $x > 1$. The above equation can be written as

$$50{,}000x^{25} - 3{,}500(x^{24} + x^{23} + \cdots + x + 1) = 0, \quad \text{or}$$

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080

Numerical Methods I



$$50\,000\, x^{25} - 3500 \frac{x^{25}-1}{x-1} = 0$$

If we multiply it with $(x-1)$, we get

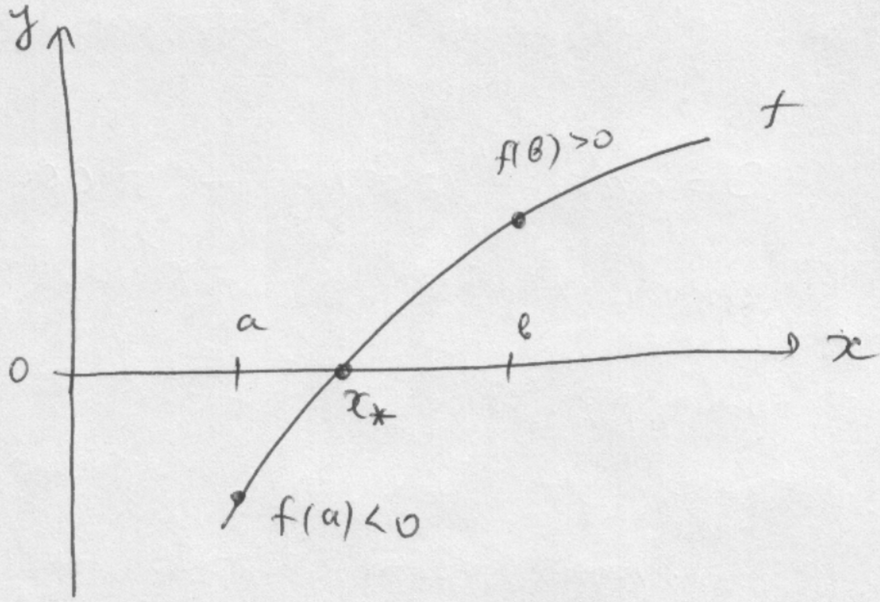
$$50000\, x^{26} - 53\,500\, x^{25} + 3500 = 0.$$

Note that $x=1$ is the solution of the above equation, but not of the original problem! It appeared when we multiplied the equation with $(x-1)$, and thus should be ignorred.

- How to solve the above two excamples? Mathematical theorem states that the solutions to the equations cannot be found analytically!
- The simplest method is the bisection method
- It relies on the knowledge of the interval that contains the solution, $[a, b]$
- How this can be verified? If we are looking for the solution of the equation $f(x)=0$, and we know that $f(a)f(b) < 0$ (i.e. $f(a)$ and $f(b)$ have different signs), then, if $f$ is a continuous function, there exists

Numerical Methods I

$x_*$ such that $f(x_*) = 0$.




- Therefore, if we know for such an interval,
we would proceed as follows:

1) $[a,b]$ satisfy $f(a)f(b) < 0$, say $f(a) < 0$, $f(b) > 0$

2) Calculate $f\left(\frac{a+b}{2}\right)$

3) If $f\left(\frac{a+b}{2}\right) < 0$, we narrowed the interval containing the solution to $\left[\frac{a+b}{2}, b\right]$; otherwise, if $f\left(\frac{a+b}{2}\right) > 0$, the solution is in $\left[a, \frac{a+b}{2}\right]$; our guess is $\frac{a+b}{2}$

4) Repeat the procedure until f(midpoint) is

Numerical Methods I

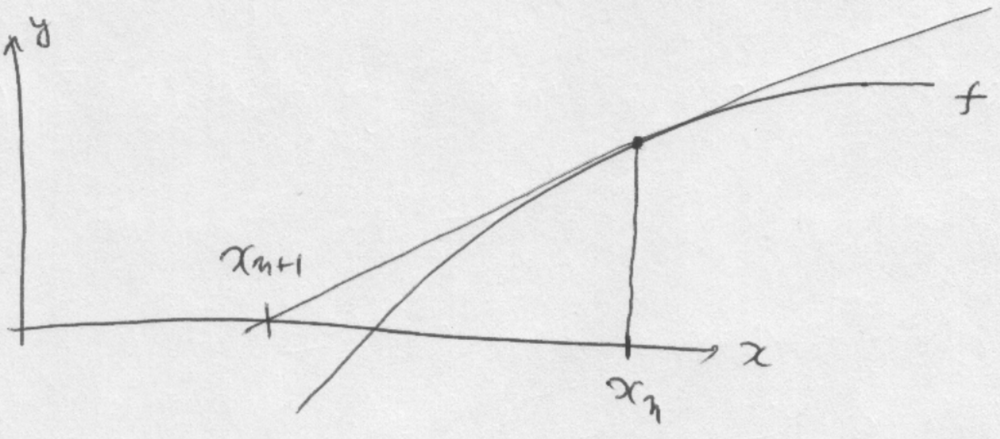
sufficiently small

- Solution of Example 1 and 2 in Excel

- Bisection method has the disadvantage that you need to know the interval containing the solution

- If not, others methods must be applied

- Newton-Raphson method uses predictions made by approximating the function with its tangent




- The equation of the tangent to the function f in $x_n$ is

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Numerical Methods I

- It has zero for $x_{n+1}$, i.e.

$$-f(x_n) = f'(x_n)(x_{n+1} - x_n) \Rightarrow$$

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)} \Rightarrow$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Another method to derive this formula: the slope of the tangent, $f'(x_n)$, can be also calculated as $\frac{f(x_n)}{x_n - x_{n+1}}$, so we have

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}} \Rightarrow x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)} \Rightarrow$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: (+381 11) 30 21 169. 30 21 080

Numerical Methods I

- Therefore, we would proceed as follows:

1) Choose appropriate $x_1$, calculate $f(x_1)$, $f'(x_1)$

2) Calculate $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $f(x_2)$, $f'(x_2)$

3) Repeat the procedure using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ until $f(x_{n+1})$ is sufficiently small

- Note that if you start with the solution of the equation, $x_1 = x_*$, you will always get $x_{n+1} = x_*$, i.e. you will not move from this fixed point; this is not desirable if $x_*$ is not the solution you seek; in this case, choose another starting point

- Solution of Example 1 and 2 in Excel
- Excel has built-in macro Goal Seek — Example 1 and 2