UNIVERSITY OF BELGRADE
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SECCF
South European Center for Contemporary Finance

Numerical Methods II

- Numerical optimisation: Markowitz problem
- Given the covariance matrix V and expected rates of return of individual assets in our portfolio $\bar{r}_1, \dots, \bar{r}_n$, find portfolio weights so that the portfolio rate of return is $\bar{r}$ and portfolio variance is minimal

$$\text{minimize } \frac{1}{2}\sum_{i,j=1}^{n} w_i \operatorname{cov}(i,j) w_j \quad \text{subject to}$$

$$\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

- Analytical approach:

$$L = \frac{1}{2}\sum_{i,j=1}^{n} w_i c_{ij} w_j + \lambda\left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r}\right) + \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

$$\frac{\partial L}{\partial w_i} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial \mu} = 0$$

- Example:

$$V = \begin{bmatrix} 0.01887 & 0.01722 & 0.00721 \\ 0.01722 & 0.02763 & 0.00604 \\ 0.00721 & 0.00604 & 0.00966 \end{bmatrix}$$

$\bar{r}_1 = 8\%$, $\bar{r}_2 = 10\%$, $\bar{r}_3 = 7\%$, $\bar{r} = 9\%$.

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Numerical Methods II

- Numerical approach: Gradient method, i.e. generalisation of Newton-Raphson method
- Example: $f(x,y) = x^2 + 6xy + 2y^3$, to minimise it, we need to find zeros of

$$g(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}.$$ This is vector

- Using vector notation, we have $\vec{g}(\vec{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$,

and $\vec{x} = (x_1, x_2)$, $f(\vec{x}) = f(x_1, x_2)$

- If $\vec{g}(\vec{x}_*) = 0$ is the equation, then for a chosen $\vec{x}_0$, we have

$$\vec{g}(\vec{x}_0 + \vec{h}) \approx \vec{g}(\vec{x}_0) + \vec{H}(x_0) \cdot \vec{h},$$

where $\vec{H}(\vec{x}_0)$ is the Hessian matrix of f!

- Therefore, if $\vec{g}(\vec{x}_0 + \vec{h})$ is expected to be zero, we have $\vec{h} = H^{-1}(\vec{x}_0) \cdot \vec{g}(\vec{x}_0)$
- So, the solution is to use

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} - H^{-1}(\vec{x}_k) \cdot \vec{g}(\vec{x}_k)$$

Numerical Methods II

- Example in Excel
- This was unconstrained optimisation
- Constrained optimisation involves Kuhn-Tucker analysis
- Excel solver uses it
- Gradient represents the direction of the maximum rate of increase of the function
- Analogue: hill climbing
- Lagrangian method is used for equality-constrained problems (turning them into unconstrained problems)
- With inequality constraints, we can assume equality constraints, and then further analyze consequences of inequalities
- Solution of the Markowitz problem in Excel using Solver

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