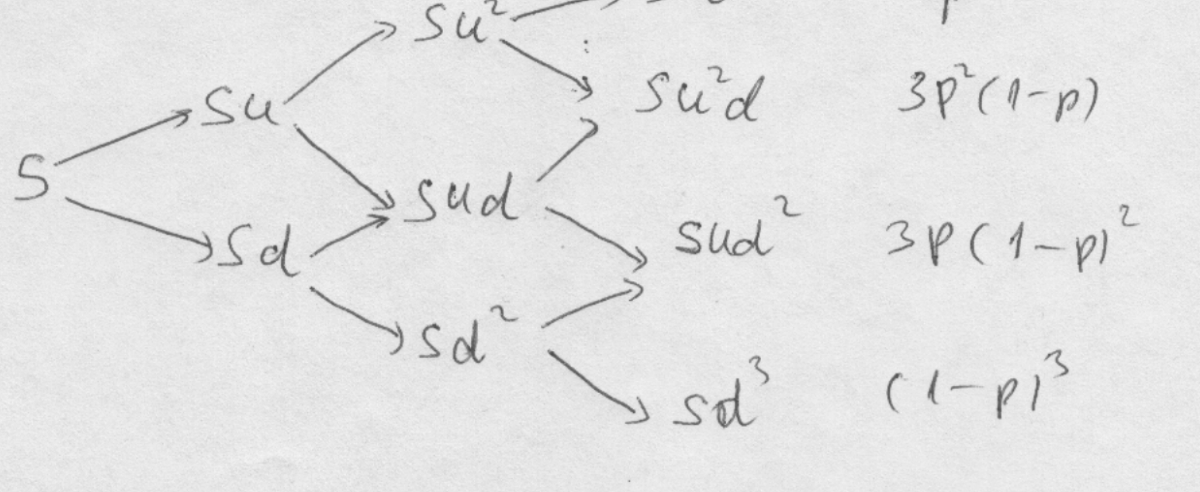
Numerical Methods III

- Valuing options using binomial lattices
- ~~An~~ European option (security can be bought only at maturity at a fixed price) can be described by a Black-Sholes equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

- If $\sigma = \text{const.}$, the above equation can be solved analytically; in other cases we must use numerical approach
- We will now implement numerical approach for valuing European option, and then adopt it for valuing an American (can be sold earlier; therefore the price depends on the path of security prices)
- European option can be valued as follows:



Numerical Methods II

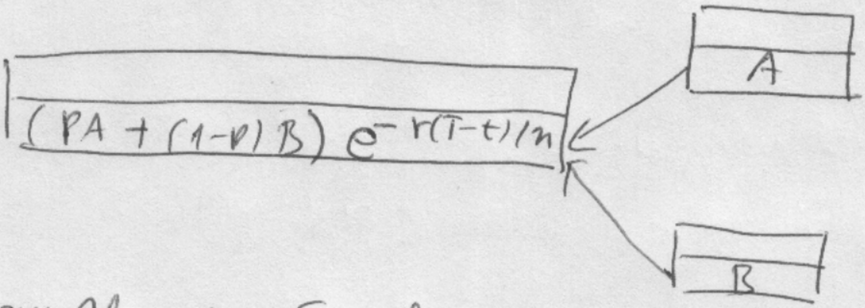
- Here S is the price of the underlying security, u and d are appropriately chosen factors ($u>1$, $0<d<1$) governing the behavior of the security: with the probability p the price will grow with the factor u, and with the probability $(1-p)$ the security price will decrease by a factor d.
- If u, d, and p are well chosen, the binomial latice with enough steps will give good estimate of the price of European option
- Conditions:
  - expected rate of return of asset over a time period dt should be rdt
  - the standard deviation over dt should be $\sigma\sqrt{dt}$ (σ^2 grows linearly with time)
- Many possible choices, Cox, Ross & Rubenstein parametrisation is convenient:

$$u=e^{\sigma\sqrt{\frac{T-t}{n}}}, \quad d=\frac{1}{u}, \quad p=\frac{e^{r\frac{T-t}{n}}}{u-d}$$

- Example: $S=100$, $X=102$, $\sigma=0.2$, $r=0.05$, $T-t=1$, $n=6$

Numerical Methods III

- This approach can be re-arranged so that it better suits other problems



- Example in Excel
- For an American, the above approach can be applied if we just replace $\boxed{\begin{array}{c}P\\ \hline Q\end{array}}$ with the $\boxed{\begin{array}{c}P\\ \hline M\end{array}}$, where $M = \max(Q, X-P)$
- Example in Excel
- However, if $\sigma \neq$ const. we may need to solve Black-Scholes eq. numerically, using finite difference method:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$f''(x) \approx \frac{f'(x+h) - f'(x)}{h} = \frac{\frac{f(x+2h) - f(x+h)}{h} - \frac{f(x+h) - f(x)}{h}}{h} =$$

$$= \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

- For Black-Scholes it is appropriate to use

$$\frac{\partial V}{\partial t} \approx \frac{V(t+\delta t, S) - V(t, S)}{\delta t}$$

$$\frac{\partial V}{\partial S} \approx \frac{V(t+\delta t, S+\delta S) - V(t+\delta t, S-\delta S)}{2\,\delta S}$$

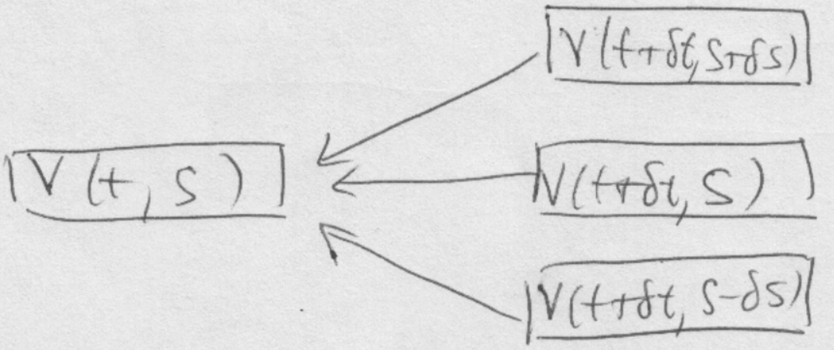
$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V(t+\delta t, S+\delta S) - 2V(t+\delta t, S) + V(t+\delta t, S-\delta S)}{\delta S^2}$$

- This way, we can transform Black-Scholes eq. into finite difference eq..

$$V(t,S) = \frac{1}{1+r\delta t}\Big[aV(t+\delta t, S+\delta S) + bV(t+\delta t, S) + cV(t+\delta t, S-\delta S)\Big]$$

$$a = \frac{S\delta t}{2\delta S}\left(\frac{S\sigma^2}{\delta S} + r\right), \quad b = 1 - \left(\frac{S\sigma}{\delta S}\right)^2 \delta t, \quad c = \frac{S\delta t}{2\delta S}\left(\frac{S\sigma^2}{\delta S} - r\right)$$

- How this works? Example in Excel



Numerical Methods III

- Monte Carlo simulation may be needed for valuing other options
- Example: an Asian option – call with life of 1 year, the strike being defined as the arithmetic mean of the underlying security prices at the end of Q1, Q2, Q3, Q4.
- Here return changes as $\left(r - \frac{\sigma^2}{2}\right)t$, and standard deviation is $\sigma\sqrt{t}$
- Example in Excel
- Standard deviation and estimate are not stable if the number of "samples" is too small
- Standard deviation is $\sim \frac{\sigma}{\sqrt{n}}$
- We can see stabilisation of estimated values when n increases