Hypothesis Testing

- Goals to learn how to:
  - Define a hypothesis
  - Define and interpret the null and alternative hypothesis
  - Define and interpret a test statistics
  - Special applications: mean value and variance
- Important market questions are answered through a formulation of hypotheses and their testing, e.g.
  - Did the volatility of returns on some stock change after the stock was added to a stock market index
  - Are a security's bid-ask spreads related to the number of dealers making a market in the security?
  - Is the expected rate of return of a stock different when using the data for the last year and the last month?
  - ...
- This is the basis of the scientific method: define a hypothesis, test it using the data, accept if tests are positive, try to integrate it into previously acquired knowledge — this is how science is made!

Hypothesis Testing

- Previously we discussed how estimates are made using the data
- Hypothesis testing question can be "Is the value of some parameter (e.g. population mean) 45?"
- A hypothesis is precisely defined statement about one or more populations
- Theory should be able to make accurate (meaning, testable and consistent with the observed data) predictions
- Phenomenological theory is predecesor to the real theory – it just describes the observed data
- Necessary steps in hypothesis testing:
  - stating the hypothesis
  - identifying the appropriate test statistics and its probability distribution
  - specifying the significance level
  - stating the decision rule
  - Collecting the data and calculating the test statistics
  - making the statistical decision
  - making the economic or investment decision

Hypothesis Testing

- Stating the hypothesis: we always state two hypotheses, the null hypothesis and the alternative hypothesis (H_0 and H_a)
  - H_0 is the hypothesis to be tested
  - H_a is the hypothesis accepted when H_0 is rejected
- Example:
  - H_0: mean value of the return of some stock is ≥ 45
  - H_a: mean value of that stock is < 45
- Two_sided hypothesis formulation (two-tailed):
  - $H_0: \theta = \theta_0$, $H_a: \theta \neq \theta_0$ $(\theta < \theta_0 \vee \theta > \theta_0)$
- One-sided hypothesis formulation (one-tailed):
  - $H_0: \theta \leq \theta_0$, $H_a: \theta > \theta_0$
  - $H_0: \theta \geq \theta_0$, $H_a: \theta < \theta_0$
- How hypotheses are motivated?
  - Experience
  - Knowledge
  - Data (preliminary analysis)
  - Hope, ...

Hypothesis Testing

- Identifying the appropriate test statistics and its probability distribution:
  - Test statistics is a quantity (calculated based on a sample) whose value is the basis for all decisions regarding the hypothesis
  - Usually: Test statistics =

$$= \frac{(\text{Sample statistics}) - (\text{Value of the par. in } H_0)}{(\text{Standard error of the sample})}$$

  - Probability distribution of the test statistics; we will cover
    - * t-distribution (t-test)
    - * normal or z-distribution (z-test)
    - * χ^2 distribution (χ^2-test)
    - * F-distribution (F-test)
- Specifying the significance level: reflects how much sample evidence we require to reject H_0
- Possible errors:
  - Rejecting the false H_0 (no error)
  - Rejecting the true H_0 (Type I error)
  - Accepting the false H_0 (Type II error)
  - Accepting the true H_0 (no error)

Hypothesis Testing

- The probability of Type I error is denoted by α — this is the level of significance
- The probability of Type II error is denoted by β
- Although they are related (e.g. when α decreases, β increases), the relation is hard to quantify
- Usually the following significance levels are used:
  - $\alpha = 0.1$
  - $\alpha = 0.05$
  - $\alpha = 0.01$
- The power of a test is the probability of <u>correctly rejecting</u> the H_0

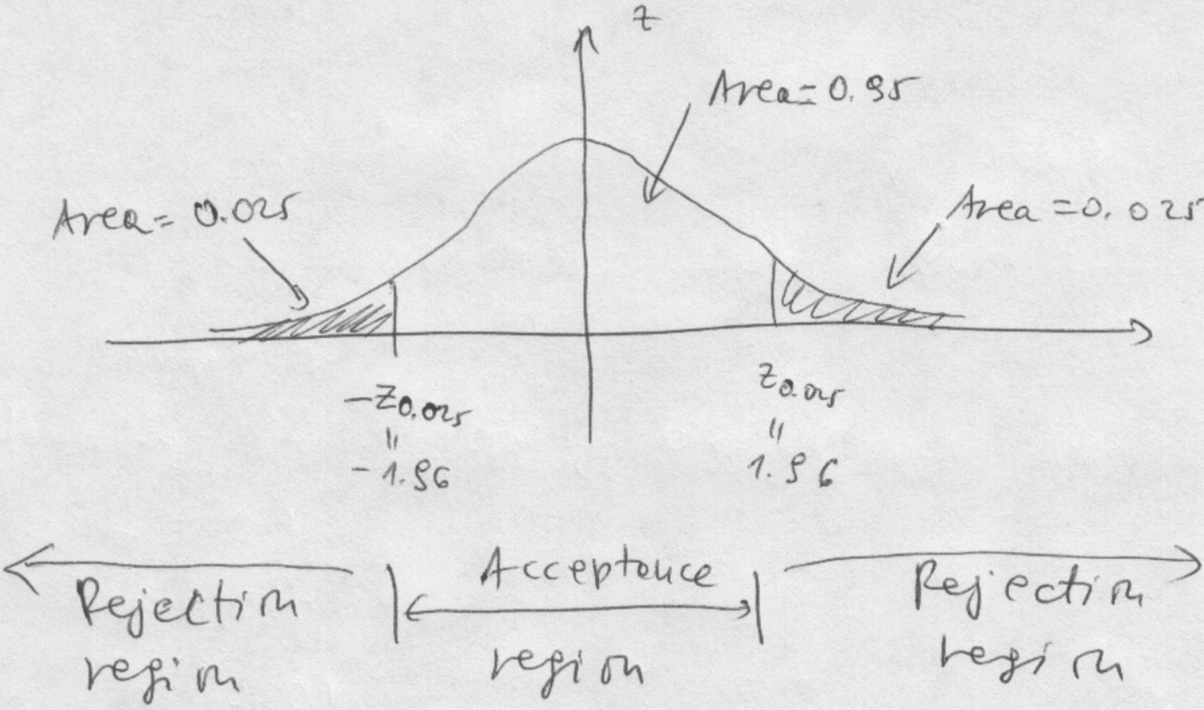
- Stating the decision rule:
  - If the test statistics is statistically significant (i.e. probability $> \alpha$), we reject the H_0
  - Otherwise, we do not reject H_0 (result is not statistically significant)

Hypothesis Testing

- Example: z-test, 0.05 significance level

$z_{\alpha/2} = 1.96$; if $-1.96 \le z \le 1.96$, we do not reject H_0,

if $z > 1.96$ or $z < -1.96$, we do reject H_0



- Collecting the data and calculating the test statistics:
  - Quality of the data
  - Amount of the data
  - Biases
  - errors

- Decision making:
  - Statistical decision
  - Economic (investment) decision

Hypothesis Testing

- Economic decision may involve
  - Testing other hypotheses
  - Insights from other theories
  - Knowledge about other economic issues
- Definition of p-value: the smallest level of significance at which H_0 can be rejected (aka the marginal significance level)
- Using the p-value simplifies comparison of different calculations: instead of using some predefined value of α (which may be different for different analyses), p-value can be calculated and stated (p-value approach to hypothesis testing)
- Hypothesis tests concerning the mean
  - usually t-statistics encountered (more spread out than the normal distribution, symmetrical, when # of degrees of freedom $\to \infty$, t-distr. tends to the normal distribution, i.e. t-test becomes z-test)

Hypothesis Testing

- Example: If population variance is not known, and the following conditions are met
    - the sample is large, <u>or</u>
    - the sample is small, but the population is known to be normally distributed (or close)

then the test statistics can be defined as

$$t_{n-1} = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \qquad (n = \text{sample size})$$

where $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$, μ_0 is the hypothesized value of the mean, s is the sample standard deviation, n is the sample size, and t_{n-1} is the t-statistics with $n-1$ degrees of freedom

- Example: If the variance is known to be σ (or the population is sufficiently large so that we know that $\sigma \approx s$), then z-statistics is

$$z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \left(= \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \right)$$

Hypothesis Testing

- Practical: population variance not known

Population	Large sample ($n \geq 30$)	Small sample ($n < 30$)
Normal	t-test (z-test)	t-test
Non-normal	t-test (z-test)	—

- Tests concerning differences between means:
  - test statistics for the difference between two population means; normally distributed populations, variances unknown but assumed equal

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}}$$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

# of degrees of freedom is $n_1 + n_2 - 2$

  - variances not equal or not known

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}, \quad df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{s_1^4}{n_1^3} + \frac{s_2^4}{n_2^3}}$$

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Hypothesis Testing

- Tests concerning mean differences: paired comparison test

$$\bar{d} = \sum_{i=1}^{n} \frac{d_i}{n}, \quad s_d^2 = \frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}$$

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

- Hypothesis tests concerning variance ($H_0: \sigma^2 = \sigma_0^2$)

  - $$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}, \quad s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

  - χ^2 is asymmetrical

- Tests concerning the equality (or $\neq$) of two variances ($H_0: \sigma_1^2 = \sigma_2^2$)

  - normally distributed populations

$$F = \frac{s_1^2}{s_2^2}, \quad df_1 = n_1 - 1, \quad df_2 = n_2 - 2$$

(practice: $s_1^2 \geq s_2^2$)

Hypothesis Testing

- Example for t-test: Analysis of an equity fund reveals that for the past 24 months it achieved a mean monthly return of 1.5% with a sample standard deviation of 3.6%. Given its level of systematic (market) risk and according to a pricing model, this mutual fund was expected to have earned a 1.10% mean monthly. Assuming returns are normally distributed, are the actual returns consistent with an underlying (population) mean of 1.1%?

Solution: $H_0: \mu = \mu_0 = 1.1\%$, $H_a: \mu \neq \mu_0$

t-test used with $n-1 = 24-1 = 23$ d.o.f

For $\alpha = 0.1$, we have $t_{\alpha/2, n-1} = t_{0.05, 23} = 1.714$

For the sample:

$$t = \frac{1.5 - 1.1}{3.6/\sqrt{24}} = 0.544$$

Since $-1.714 \le 0.544 \le 1.714$, we <u>do not</u> reject H_0

Confidence interval $(1-\alpha)$ for μ

$$[\bar{X} - t_{\alpha/2} S_{\bar{X}}, \bar{X} + t_{\alpha/2} S_{\bar{X}}]$$

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For 90% confidence interval ($\alpha = 0.1$) we have $t_{\alpha/2} S_{\bar{x}} = t_{0.05} \frac{S}{\sqrt{n}} = 1.26$, so the interval is $[0.24, 2.76] \ni 1.1$

- Example for χ^2: For the same fund, test if $\sigma < 4\% = \sigma_0$.

Solution: $H_0: \sigma^2 \geq 16$, $H_a: \sigma^2 < 16$

χ^2-test, $n-1 = 24-1 = 23$ d.o.f

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = \frac{23 \cdot 3.6^2}{4^2} = 18.63$$

For $\alpha = 0.05$ we have $\chi^2_{1-\alpha, 23} = 13.091$

Since $18.63 > 13.091$, we do not reject H_0.

So, we cannot conclude that $\sigma_0 < 4\%$.