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FACULTY OF ECONOMICS

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QM, Dec 23, 2004. (HW 1)

Solutions to the Homework Problems

assigned on Dec 23, 2004.

1) Equality constrained optimization

$$f(x,y) = x^2 - y^2, \quad g(x,y) = 1 - x - y = 0$$

constraint

Lagrangian is $L = x^2 - y^2 + \lambda(1 - x - y)$

$$\frac{\partial L}{\partial x} = 2x - \lambda = 0 \Rightarrow x^* = \frac{\lambda^*}{2}$$

$$\frac{\partial L}{\partial y} = -2y - \lambda \Rightarrow y^* = -\frac{\lambda^*}{2} = -x^*$$

$$\frac{\partial L}{\partial \lambda} = 1 - x - y = 0 \Rightarrow x^* + y^* = 1, \text{ but } x^* + y^* = \frac{\lambda^*}{2} - \frac{\lambda^*}{2} = 0$$

This system does not have a solution!

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(HW2)

2) Equality constrained optimization

$$f(x,y) = \frac{x^3}{3} - \frac{3y^2}{2} + 2x \quad , \quad g(x,y) = x - y = 0$$

$$L = \frac{x^3}{3} - \frac{3y^2}{2} + 2x + \lambda(x-y)$$

$$\frac{\partial L}{\partial x} = x^2 + 2 + \lambda = 0$$

$$\frac{\partial L}{\partial y} = -3y - \lambda = 0 \Rightarrow y^* = -\frac{\lambda^*}{3}$$

$$\frac{\partial L}{\partial \lambda} = x - y = 0 \Rightarrow x^* = y^*$$

So, from the second and third eq. we have $x^* = y^* = -\frac{\lambda^*}{3}$. From the first eq. we have

$$x^{*2} + 2 + \lambda^* = 0 \Rightarrow \frac{\lambda^{*2}}{9} + \lambda^* + 2 = 0 \ / \cdot 9$$

$$\lambda^{*2} + 9\lambda^* + 18 = 0 \Rightarrow \lambda^*_{1,2} = \frac{-9 \pm \sqrt{81 - 4 \cdot 18}}{2} =$$

$$= -\frac{9 \pm 3}{2} \Rightarrow \lambda^*_1 = -3 \ , \ \lambda^*_2 = -6$$

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HW3

The matrix $\left[\frac{\partial f_j}{\partial x_i}(x^*) \right]$ is

$$\left[\frac{\partial f}{\partial x}(x^*, y^*) \quad \frac{\partial f}{\partial y}(x^*, y^*) \right] = \left[1 \quad -1 \right],$$

and rank $[1 \quad -1] = 1$ = number of constraints, so this condition is fullfilled.

Hessian matrix is

$$H = \left[\frac{\partial^2 L}{\partial x_i \partial y_j}(x^*, y^*) \right] = \begin{bmatrix} 2x^* & 0 \\ 0 & -3 \end{bmatrix}$$

For $\lambda_1^* = -3$ we have the solution

$(x_1^*, y_1^*, \lambda_1^*) = (1, 1, -3)$, so

$$H_1 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \quad H_{11} > 0, \det H = -6$$

this is not max, nor min

For $\lambda_2^* = -6$ we have $(x_2^*, y_2^*, \lambda_2^*) = (2, 2, -6)$,

$H = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$, and again, since $H_{11} > 0$, but $\det H < 0$, this solution is neither max nor min.

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HW4

3) inequality constrained optimization

$$f(x,y) = x^2 - y \ , \quad h(x,y) = 1 - x^2 - y^2 \geq 0$$

$$L = x^2 - y + \lambda(1 - x^2 - y^2)$$

$$\frac{\partial L}{\partial x} = 2x - 2x\lambda = 2x(1-\lambda) = 0$$

$$\frac{\partial L}{\partial y} = -1 - 2y\lambda = 0 \Rightarrow y^* \lambda^* = -\frac{1}{2} , \quad y^* \neq 0, \lambda^* \neq 0$$

$$\frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 \geq 0$$

If $x^* = 0$, then the first equation is satissfied. For $\lambda^*(1 - x^2 - y^2) = 0$ we have $y^* = \pm 1$, and, from the second eq. we have $\lambda^* = \mp \frac{1}{2}$.

The solutions are $(x^*, y^*, \lambda^*) = (0, \pm 1, \mp \frac{1}{2})$.

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HW5

If $x^* \neq 0$, then from the first eq. we have $\lambda^* = 1$, from the second $y^* = -\frac{1}{2}$, and from $\lambda^*(1 - x^{*2} - y^{*2}) = 0$ we have $x^{*2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow x^* = \pm\frac{\sqrt{3}}{2}$, so the solution is $(x^*, y^*, \lambda^*) = (\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$.

For all solutions we have $h(x^*, y^*) = 0$, so the matrix $\left[\frac{\partial h_{Ej}}{\partial x_i}(x^*)\right]$ is

$$\left[\frac{\partial h}{\partial x}(x^*, y^*) \quad \frac{\partial h}{\partial y}(x^*, y^*)\right] = \left[-2x^* \quad -2y^*\right]$$

For $(0, \pm 1, \mp\frac{1}{2})$ this matrix has rank $[0 \quad \mp 2] = 1$ = number of constraints h_E.

For $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1)$ is rank $[\mp\sqrt{3} \quad 1] = 1$ = number of constraints h_E.

Hessian matrix is

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(HW 6)

$$H = \begin{bmatrix} 2(1-\lambda^*) & 0 \\ 0 & -2\lambda^* \end{bmatrix}$$

For $\lambda^* = \mp \frac{1}{2}$ we have $H = \begin{bmatrix} 2(1 \pm \frac{1}{2}) & 0 \\ 0 & \pm 1 \end{bmatrix}$,

i.e. for $\lambda^* = -\frac{1}{2}$, $H = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} > 0$, so

this point $(0, +1, -\frac{1}{2})$ is minimum;

for $\lambda^* = +\frac{1}{2}$, we have $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which does not have defined definiteness.

For $\lambda^* = 1$ we have $H = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$, and this matrix also does not have defined definiteness (neither semi-definiteness!).

4) inequality constrained optimization

$$f(x,y) = 2x^3 - 3x^2, \quad h(x,y) = (3-x)^3 \geq 0$$

$$L = 2x^3 - 3x^2 + \lambda(3-x)^3$$

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HW7

$$\frac{\partial L}{\partial x} = 6x^2 - 6x + 3\lambda(3-x)^2 \cdot (-1) = 0$$

$$\frac{\partial L}{\partial y} = 0 \quad ; \quad \frac{\partial L}{\partial \lambda} = (3-x)^3 \geq 0$$

First eq. is $6x^2 - 6x - 3\lambda(9 - 6x + x^2) = 0$,

or $(2-\lambda)x^2 - (2-6\lambda)x - 9\lambda = 0$, so

$$x^*_{1,2} = \frac{2 - 6\lambda \pm \sqrt{(2-6\lambda)^2 + 36\lambda(2-\lambda)}}{2(2-\lambda)} =$$

$$= \frac{2 - 6\lambda \pm \sqrt{4 - 24\lambda + \cancel{36\lambda^2} + 72\lambda - \cancel{36\lambda^2}}}{2(2-\lambda)} =$$

$$= \frac{2 - 6\lambda \pm \sqrt{4 + 28\lambda}}{2(2-\lambda)} =$$

$$= \frac{1 - 3\lambda \pm \sqrt{1 + 7\lambda}}{2-\lambda}$$

We know that $\lambda^* \geq 0$ and that $\lambda^*(3-x^*)^3 = 0$

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HW8

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If $\lambda^*=0$, then $x^*_{1,2} = \frac{1\pm 1}{2}$, i.e. $x^*_1=1$ and $x^*_2=0$. The solutions are $(1, y^*, 0)$ and $(0, y^*, 0)$, where $y^* \in \mathbb{R}$. Here $h(x^*, y^*) > 0$, so we don't have conditions to satisfy.

If $\lambda^*>0$, then $x^*=3$, and from the first eq. we see that

$$6x^{*2} - 6x^* - 3\lambda^*(3-x^*)^2 = 0 \iff$$

$$6\cdot 9 - 6\cdot 3 - 3\lambda^* \cdot 0 = 0$$ is not satisfied,

so this solution does not exist.

Hessian matrix is

$$H = \begin{bmatrix} 12x^* - 6 + 6\lambda^*(3-x^*) & 0 \\ 0 & 0 \end{bmatrix}.$$

For $(1, y^*, 0)$ we have $H = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$,

and for $(0, y^*, 0)$ we have $H = \begin{bmatrix} -6 & 0 \\ 0 & 0 \end{bmatrix} \leq 0$.

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HW9

5) inequality constrained optimization

$$f(x,y) = x^2 - x \quad , \quad h(x,y) = x \geq 0$$

$$L = x^2 - x + \lambda x$$

$$\frac{\partial L}{\partial x} = 2x - 1 + \lambda = 0 \Rightarrow x^* = \frac{1-\lambda^*}{2}$$

$$\frac{\partial L}{\partial y} = 0 \quad ; \quad \frac{\partial L}{\partial \lambda} = x \geq 0 \quad , \quad \lambda^* \geq 0$$

$$\lambda^* x^* = 0 \Rightarrow \quad \lambda^* = 0 , \; x^* = \frac{1}{2} \quad \text{or}$$

$$x^* = 0 \; , \; \lambda^* = 1$$

If $(x^*, y^*, \lambda^*) = (\frac{1}{2}, y^*, 0)$, $h(x^*) = x^* = \frac{1}{2}$, and we don't have additional (rank) constraints.

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \geq 0 .$$

If $(x^*, y^*, \lambda^*) = (0, y^*, 1)$, $h(x^*) = x^* = 0$, so we have to check is rank $\left[\frac{\partial h}{\partial x}(x^*) \quad \frac{\partial h}{\partial y}(x^*) \right] =$

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$= \text{rank}\begin{bmatrix} 1 & 0 \end{bmatrix} = 1 =$ number of h_i, which is satisfied.

Hessian is the same $H = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \geq 0$

d) Uncorrelated assets $\Rightarrow \sigma_{ij} = 0$ if $i \neq j$.

The equations are

$$\sigma_1^2 w_1 + \lambda \bar{r}_1 + \mu = 0$$
$$\sigma_2^2 w_2 + \lambda \bar{r}_2 + \mu = 0$$
$$\sigma_3^2 w_3 + \lambda \bar{r}_3 + \mu = 0$$
$$w_1 \bar{r}_1 + w_2 \bar{r}_2 + w_3 \bar{r}_3 = \bar{r}$$
$$w_1 + w_2 + w_3 = 1$$

, or, using $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$ and $\bar{r}_1 = 1$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$,

$$w_1 + \lambda + \mu = 0 \Rightarrow w_1 = -\lambda - \mu$$
$$w_2 + 2\lambda + \mu = 0 \Rightarrow w_2 = -2\lambda - \mu$$
$$w_3 + 3\lambda + \mu = 0 \Rightarrow w_3 = -3\lambda - \mu$$

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HW 11

Fourth eq. is

$$w_1 + 2w_2 + 3w_3 = \bar{r}, \text{ or}$$

$$-\lambda - \mu - 4\lambda - 2\mu - 9\lambda - 3\mu = \bar{r} \Rightarrow$$

$$-14\lambda - 6\mu = \bar{r}. \quad (1)$$

Fifth eq. is $w_1 + w_2 + w_3 = 1$, or

$$-\lambda - \mu - 2\lambda - \mu - 3\lambda - \mu = 1 \Rightarrow$$

$$-6\lambda - 3\mu = 1 \quad (2)$$

The system (1)+(2) has a unique solution

$\lambda = 1 - \frac{\bar{r}}{2}$, $\mu = -\frac{7}{3} + \bar{r}$, so we obtain

$$w_1 = \frac{4}{3} - \frac{\bar{r}}{2}, \quad w_2 = \frac{1}{3}, \quad w_3 = \frac{\bar{r}}{2} - \frac{2}{3}.$$

Standard deviation (minimal) for this portfolio is

$$\sigma = \sqrt{w_1^2 + w_2^2 + w_3^2} = \sqrt{\frac{7}{3} - 2\bar{r} + \frac{\bar{r}^2}{2}}.$$

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