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- Determinant is a function
  $\det : \mathbb{R}^{nn} \to \mathbb{R}$
  It maps a quadratic matrix into a real number
- Definition through examples

$\det[a] = a \quad (\mathbb{R}^{11})$

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad (\mathbb{R}^{22})$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} -$$

$$- d \det\begin{bmatrix} b & c \\ h & i \end{bmatrix} + g \det\begin{bmatrix} b & c \\ e & f \end{bmatrix} =$$

$$= a(ei - fh) - d(bi - ch) + g(bf - ce)$$

$(\mathbb{R}^{33})$

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$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = a_{11}\det\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} -$$

$$- a_{21}\det\begin{bmatrix} a_{12} & a_{12} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} + a_{31}\det\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix} -$$

$$- a_{41}\det\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}$$

- Properties of determinants:

1) if two columns are proportional, $\det = 0$

2) if two rows are proportional, $\det = 0$

3) $$\det\begin{bmatrix} \lambda a & b & c \\ \lambda d & e & f \\ \lambda h & i & j \end{bmatrix} = \lambda \det\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$$

- We will now calculate determinants in Excel

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- Inverse Matrix A^{-1} of a quadratic matrix $A \in \mathbb{R}^{nn}$ is defined by

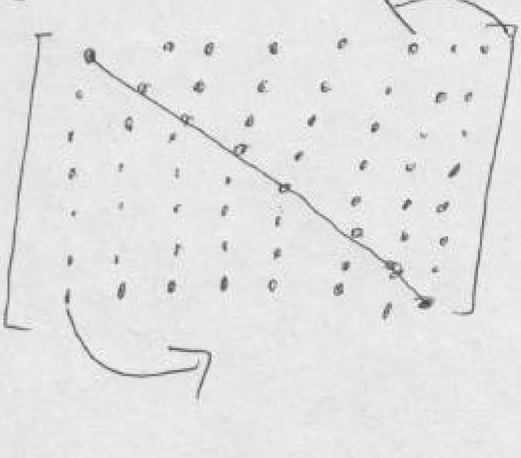
$$A^{-1} \cdot A = I_{nn} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \equiv \underset{n \times n}{\text{Identity matrix}}$$

also, $A \cdot A^{-1} = I_{nn}$

- How to calculate inverse matrix?
- First, we will define transposition:

$$A^T = B \iff b_{ij} = a_{ji}$$

i.e. we need to rotate the matrix A along the main diagonal



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– Examples:

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$ $\det A = 1\cdot \begin{vmatrix} 5 & 6 \\ 4 & 6 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} +$

$+ 2 \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 5\cdot 6 - 6\cdot 4 - 4(2\cdot 6 - 3\cdot 4) +$

$+ 2(2\cdot 6 - 3\cdot 5) = 30 - 24 - 4\underbrace{(12-12)}_{0} +$

$+ 2(12-15) = 6 + 2\cdot(-3) = 0$

as we expected

$A^T = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 6 \end{bmatrix}$

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- Determinant of a transposed matrix is equal to the determinant of the original matrix

$$\det A^T = \det A$$

- Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad \det A = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} +$$

$$+ 3 \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = -5 - 2 \cdot (-4) + 3 \cdot 3 =$$

$$= 12$$

$$\det A^T = \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} +$$

$$+ 3 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -5 - 2 \cdot (-7) + 3 (\cdot 1) =$$

$$= 12 \quad \text{OK}$$

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- Inverse matrix

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}^T ,$$

where

$$C_{ij} = (-1)^{i+j} \det \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} =$$

$= (-1)^{i+j} \det($Matrix A <u>without</u> column j and row i $)$

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— Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det A = 1\cdot 4 - 2\cdot 3 = -2$$

$$C_{11} = (-1)^{1+1} \det[4] = 4$$

$$C_{12} = (-1)^{1+2} \det[3] = -3$$

$$C_{21} = (-1)^{2+1} \det[2] = -2$$

$$C_{22} = (-1)^{2+2} \det[1] = 1$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1} \cdot A = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2\cdot 1 + 1\cdot 3 & -2\cdot 2 + 1\cdot 4 \\ \frac{3}{2}\cdot 1 - \frac{1}{2}\cdot 3 & \frac{3}{2}\cdot 2 - \frac{1}{2}\cdot 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{OK}$$

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- We will now calculate inverse matrices in Excel
- Let us calculate this matrix multiplication

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix}$$

- We can see that the System of Linear Equations can be written as

$$\begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad i.e.$$

$A \cdot x = \alpha$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

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- Now, the solution of the system is given by

$$x = A^{-1}\alpha$$

(A$x = \alpha$ and we multiply this equation with A^{-1} from the left)

- There is a theorem that says that the system has ~~a~~ unique solution <u>iff</u> $\det A \neq 0$ (in that case A^{-1} exists, so everything is OK with the above definition)

- Examples in Excel

1) Q2 from the test, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 8 \\ 1 & 3 & 2 \end{bmatrix}$, $\alpha = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$,