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- Linear Dependence and Independence:

For a set of n given objects (vectors) $x_1, \dots, x_n$ we will say that they are Linearly dependent if at least one of them can be represented as a linear combination of the other $n-1$ vectors, e.g.

$$x_n = a_1 x_1 + \dots + a_{n-1} x_{n-1},$$

wher $a_1, \dots, a_{n-1}$ are some <u>constants</u>.

- This is equivalent to the following definitio

Set of n vectors $x_1, \dots, x_n$ is said to be linearly dependent if there are <u>constants</u> $a_1, \dots, a_n$ (at least one of them <u>must</u> be different from zero) so that

$$a_1 x_1 + \dots + a_n x_n = 0.$$

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- The set of vectors is said to be linearly independent if it is not linearly dependent.

- Example:

1) Vectors $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ are linearly dependent, because $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, so $x_2 = 2x_1$.

2) Vectors $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ are linearly independent, because there is no constant a such that $x_2 = a x_1$.

- Theorem: A rank of a matrix $A \in \mathbb{R}^{m \times n}$ is equal to the number of linearly independent rows or, equivalently, to the number of linearly independent columns.

- Exercise: Find the rank of given matrices using the above theorem.

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 1 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 & 10 \\ 2 & 1 & 5 \\ 1 & 3 & 15 \end{bmatrix}$$

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- A sequence is mapping

$$a : \mathbb{N} \to \mathbb{R},$$

i.e. we have a sequence of real numbers

$$a(1) = a_1,\ a(2) = a_2, \dots, a(n) = a_n, \dots$$

- Example: $a_n = \frac{1}{n}$ is a sequence

$$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3}, \dots$$

- A sequence is said to converge to a number c if for each $\varepsilon > 0$ we can find the number N so that

$$(\forall n \geq N)\ |a_n - c| < \varepsilon$$

- If a sequence is convergent with the limit c, we will write

$$\lim_{n \to \infty} a_n = c.$$

- Example: $\lim_{n \to \infty} \frac{1}{n} = 0$

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- A sequence which is not convergent is said to be divergent or to diverge.
- Example: $a_n = 1+\frac{2}{n}$ is convergent and $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right) = 1$. Let us prove that:

  $|a_n - c| = |1+\frac{2}{n} - 1| = |\frac{2}{n}| = \frac{2}{n} < \varepsilon \Rightarrow$

  $n > \frac{2}{\varepsilon}$; so, if we choose $N = [\frac{2}{\varepsilon}] + 1$,

  for all $n \geq N$ we will have $|a_n - c| < \varepsilon$,

  i.e. $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right) = 1$. QED

- Example: $a_n = n$. This sequence is divergent.
- Cauchy's convergence principle: A sequence a_n is convergent <u>iff</u> for every $\varepsilon > 0$ we can find a number N such that

  $$(\forall m, n \geq N)\ |a_m - a_n| < \varepsilon.$$

- Here we don't have to know actual value of a limit to be able to prove the convergence!

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- A sequence is bounded if there exists a real number B such that

$$(\forall n \in \mathbb{N})\ |a_n| \leq B$$

B is then said to be the bound of a sequence

- Theorem: Convergent sequence is bounded. Consequently, if a sequence is not bounded, it diverges.

- Example: A sequence $a_n = n$ is not bounded since for every $B \in \mathbb{R}$ we can find sufficiently large n so that

$$|a_n| = n > B$$

(for example we can just choose $n = [B] + 1$). Therefore, $a_n = n$ diverges.

- A sequence is said to be monotone increasing if

$$a_1 \leq a_2 \leq a_3 \leq \ldots$$

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- A sequence is said to be monotone decreasing if

$$a_1 \geq a_2 \geq a_3 \geq \dots$$

- A sequence is said to be monotone if it is monotone increasing or monotone decreasing.

- Theorem: If a real sequence is bounded and monotone, it converges.

- Homework III: Are the following sequences bounded or not, convergent or divergent? Determine their limits if they converge.

1) $a_n = \frac{n^2 - 2n + 2}{n^3}$ 2) $a_n = \frac{\sin n}{n}$

3) $a_n = \frac{e^n}{n}$ 4) $a_n = \frac{n-1}{n} + \frac{n^2+2}{n^2}$

- Important properties of sequences:

$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ (if all three limits exist)

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$$\lim_{n\to\infty}(a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$$ (if all three limits exist)

$$\lim_{n\to\infty}\frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$$ (if both limits exist)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$ (if all three limits exist)

- We will now demonstrate several limits in Excel.
- Important example:

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$$

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- We will now define derivation of a function (first derivative)

- First derivative of the function f at the point x is defined as

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}, \quad \text{or}$$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\frac{\varepsilon}{2}) - f(x-\frac{\varepsilon}{2})}{\varepsilon}$$

We can also use some other, similar formula.

- If the limit in the above definition exists, the function is said to be differentiable at the point x.

- The meaning of the first derivative: the slope of the tangent to the function f at the point x.

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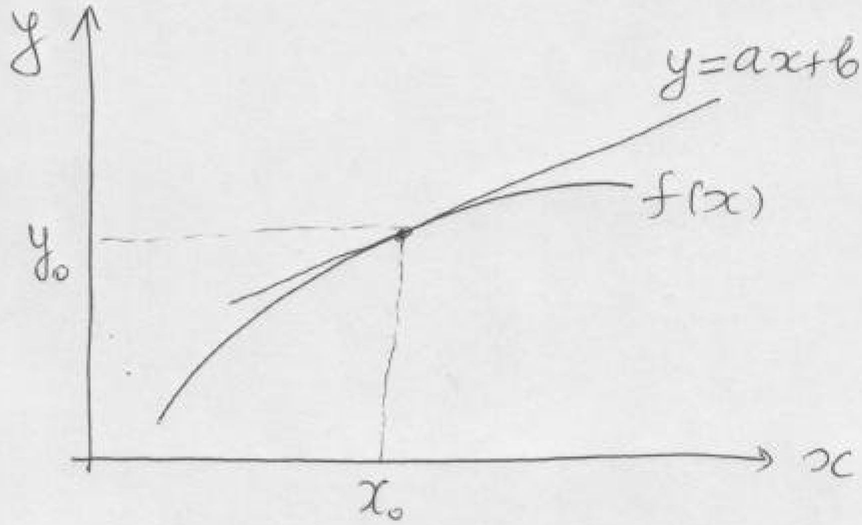

$a=f'(x_0)$
slope

- If we want to plot a linear function (line) through the point (x_0, y_0) with the slope a, the corresponding equation is given by

$$y-y_0=a(x-x_0),$$

i.e.

$$y=a(x-x_0)+y_0$$

It is obvious that the slope is a, and that

$$y(x_0)=a(x_0-x_0)+y_0=y_0,$$

as desired.

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- So, the equation of the tangent to the function f at the point x_0 is given by

$$y = f'(x_0)(x - x_0) + f(x_0)$$

At $x = x_0$ we have $y(x_0) = f'(x_0)(x_0 - x_0) + f(x_0) =$
$= f(x_0)$,
which is correct, and the slope is $f'(x_0)$.

- If we calculate the value of the first derivative $f'(x_0)$ for some function at different values of x_0 ($x_0 \in$ some set), then we have a new function, $f'(x)$, which we will call also the first derivative.

- Let us try to find the first derivative of a linear function:

$$f(x) = ax + b$$

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$$f'(x_0) = \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{a(x_0+\varepsilon)+b - ax_0 - b}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{a\varepsilon}{\varepsilon} = a$$

as we expected.

- First derivative of a quadratic function

$$f(x) = x^2$$

$$f'(x_0) = \lim_{\varepsilon \to 0} \frac{f(x_0+\varepsilon) - f(x_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(x_0+\varepsilon)^2 - x_0^2}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{x_0^2 + 2x_0\varepsilon + \varepsilon^2 - x_0^2}{\varepsilon} = \lim_{\varepsilon \to 0} \left(2x_0 + \varepsilon\right) = 2x_0$$

- We will now give a table of first derivatives for some elementory functions.