UNIVERSITY OF BELGRADE
FACULTY OF ECONOMICS

SECCF
South European Center for Contemporary Finance

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- Ordinary Differential Equation is a relation which involves one or several Derivatives of an unspecified function f; this relation may also involve the function f itself, the independent variable x, and some constants.
- Example:

$$f' = \cos x \qquad (1)$$

$$f'' + 4f = 0 \qquad (2)$$

$$x^2 f''' f' + 2 e^x f'' = (x^2+2) f^2 \qquad (3)$$

These are all differential equations.

- The order of differential equation n is equal to the order of highest derivative of function f in the equation.
- Example: (1) is of order 1, (2) is of order 2, and (3) is of order 3.

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade; and
SECCF, Andrićev Venac 2, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: 011 32 44 953, 011 32 40 345

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- The solution of the differential equation is a function that satisfies the equation.

- General solution is a solution that contains n unspecified constants, and all particular solutions of the given equation can be obtained by the certain choice of these n constants (here n is the order of equation).

- Particular solution is a solution without any unknown constants in it; It is fully specified

- The ~~usd~~ usual way for fixing the particular solution is giving the initial conditions in the following form:

$$f(x_0) = A_0, \quad f'(x_0) = A_1, \ \dots, \ f^{(n-1)}(x_0) = A_{n-1}$$

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- So, the equation is accompanied by these conditions, and we are searching for a function which satisfies the differential equation and all these n conditions.

- If we can, we will first find the general solution and then we will calculate all unknown constants in it by solving the n conditions

$$f(x_0) = A_0, \ldots, f^{(n-1)}(x_0) = A_{n-1}$$

in unknown constants.

- Of course, we can have a system of differential equations, involving several unknown functions, their derivatives, and independent variable x.

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- Example:

$$\left.\begin{aligned} f' + g' &= x^2 \\ f'' g' &= x \end{aligned}\right\}$$ system of two differential equations in functions f and g

- First order differential equation:

$$F(f, f', x) = 0$$ general form

- Linear first order diff. equation:

$$f' + f\, g(x) = h(x),$$

where $g(x)$ and $h(x)$ are some given functions.

This equation is linear in f' and f

- Linear n-order diff. equation

$$f^{(n)} + f^{(n-1)} g_1(x) + f^{(n-2)} g_2(x) + \cdots f\, g_n(x) = h(x)$$

where $g_1(x), \ldots, g_n(x), h(x)$ are given functions.

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- Partial Differential Equations are relations which involve one or several partial derivatives of one or more unspecified functions; these relations may also involve the funltions itself, the independent variables $x, y, z, \dots,$ and some constants

- Example: For $f = f(x,y)$ we may have

1) $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2\frac{\partial^2 f}{\partial x \partial y}$ (second order)

2) $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ (first order

- The order is defined similar to the order of ordinary diff. equations

- Partial diff. equation is linear if the function f and all its partial derivatives are of the first degree.

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- The famous Black-Scholes option pricing equation, which initiated the modern theory of finance, is also a partial diff. equation
- We have a security with the price S (we will call it stock)

$$dS = \mu S\,dt + \sigma S\,dz \leftarrow \text{the change of a price after } dt$$

$$dS = S(t+dt) - S(t)$$

dz is a Wiener process, roughly $dz \sim \epsilon(t)\sqrt{dt}$, $\epsilon(t)$ being a normal random variable

This is realistic approach; there are other, simpler models

(Additive model: $S(t+1) = aS(t) + u(t)$
Multiplicative model: $S(t+1) = u(t)S(t)$ etc.)

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- We have also a risk-free asset (a bond) with the interest rate r:

$$dB = rB\,dt \quad (B = B_0 e^{rt})$$

- If $f(S,t)$ is a derivative of S, i.e. its price depends on S and t, then f satisfies Black-Scholes equation.

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

- Examples: $f(S,t) = S$

$$0 + 1 \cdot r \cdot S + 0 = r \cdot S \quad \text{O.K.}$$

- Example: $f(S,t) = B = B_0 e^{rt}$

$$rB_0 e^{rt} + 0 + 0 = r \cdot B_0 e^{rt} \quad \text{O.K.}$$

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- We will now solve ordinary linear homogeneous differential equations of the first and second order, with constant coefficients

- First order:

$$f' + gf = 0$$

ordinary: $f = f(x)$, no partial derivativ

linear: f and f' are of the first degree

homogeneous

first order

constant coefficient

$$g \neq g(x)$$
$$g \in \mathbb{R}$$

- We will try to find solution in the form

$$f(x) = e^{\lambda x} : \quad f'(x) = \lambda e^{\lambda x} \Rightarrow$$

$$0 = f' + gf = (\lambda + g)\, e^{\lambda x} \Rightarrow \lambda = -g$$

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The solution is given by $f(x) = e^{-\delta x}$,
and general solution is

$$f(x) = C e^{-\delta x},$$

wher C is arbitrary constant.

- Example: $f' + 2\cancel{f} = 0 \Rightarrow f(x) = Ce^{-2x}$

Initial condition $f(0) = 2 \Rightarrow Ce^{-2\cdot 0} = C = 2$,
so the particular solution for this initial value is $f_p(x) = 2e^{-2x}$.

- Second order:

$$f'' + af' + bf = 0$$

- We will again try to find solution in the form $f(x) = e^{\lambda x}$; $f'(x) = \lambda e^{\lambda x}$, $f''(x) = \lambda^2 e^{\lambda x}$:

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \Rightarrow \lambda^2 + a\lambda + b = 0 \Rightarrow$$

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$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2-4b}}{2}$$

- We have here different possibilities:

$a^2-4b>0$: 2 roots, $\lambda_1, \lambda_2 \in \mathbb{R}$
general solution
$$f(x) = A e^{\lambda_1 x} + B e^{\lambda_2 x}$$

$a^2-4b=0$: 1 double root, $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$
general solution
$$f(x) = A e^{\lambda x} + B x e^{\lambda x}$$

$a^2-4b<0$: 2 complex conjugate roots
$$\lambda_1 = p + iq, \quad \lambda_2 = \lambda_1^* = p - iq$$
$$p, q \in \mathbb{R}$$
general solution:
$$f(x) = e^{px}(A \sin qx + B \cos qx)$$

- We will consider some other types of diff. equations after we finish Lecture on integration tomorrow.

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