QS, Jan 9, 2004. (1)

- We will discuss today Single-Period Random Cash flows.

- Typically, when making an investment, the initial outlay of capital is known, but the amount to be returned is uncertain. We will restrict our attention to the case of a single investment period: money is invested at the initial time, and payoff is attained at the end of the period.

- Suppose that you purchase an asset at time zero, and one year later you sell the asset. The <u>total return</u> on your investment is defined to be

$$\text{total return} = R = \frac{\text{amount received}}{\text{amount invested}}$$

$$R = \frac{X_1}{X_0}$$

- The rate of return $r = \dfrac{X_1 - X_0}{X_0}$

QS, Jan 9, 2004.

- The relation is $R = 1 + r$,

  and also, $X_1 = R X_0 = (1+r) X_0$.

- Now suppose that n different assets are available. We can form a portfolio of these n assets. Suppose that we have to invest amount X_0 of money. We then select amounts X_{0i}, where $i = 1, 2, .., n$, such that

$$\sum_{i=1}^{n} X_{0i} = X_0,$$

and invest X_{0i} in the i-th asset. So, all X_{0i}'s are nonnegative.

- We have for i-th asset $X_{0i} = w_i X_0$, and w_i is the corresponding <u>weight</u> or <u>fraction</u> of asset <u>i</u> int the portfolio. Clearly

$$\sum_{i=1}^{n} w_i = 1. \quad (w_i \geq 0)$$

QS, Jan 9, 2004.

- Let R_i denote the total return of asset i. The amount of money generated at the end of the period by the i-th asset is $R_i X_{0i} = R_i w_i X_0$.

- The total amount received by this portfolio will be

$$\sum_{i=1}^{n} R_i w_i \cdot X_0 ,$$

so the total return will be

$$R = \frac{\sum_{i=1}^{n} R_i w_i \cdot X_0}{X_0} = \sum_{i=1}^{n} w_i R_i$$

- The same applies to rate of return:

$$r = \sum_{i=1}^{n} w_i r_i .$$

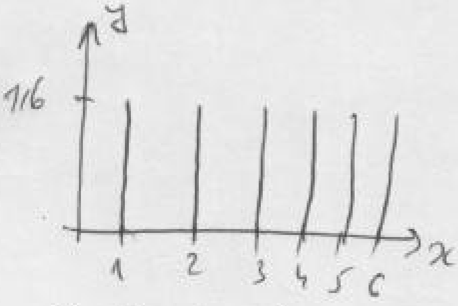
QS, Jan 9, 2004.

4

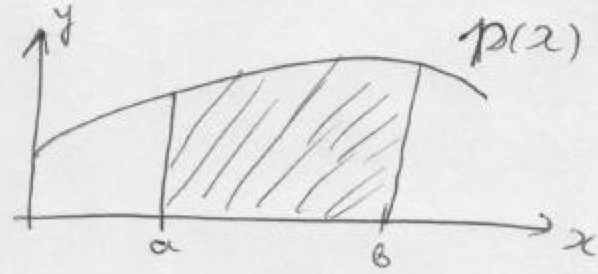
- Now we will introduce random variables.
- Suppose x is a random quantity that can take on any one of a finite number of specific values, say, $x_1, x_2, \dots, x_m$. Assume further that associated with each possible x_i, there is a probability p_i that represents the relative chance of an occurance of x_i
- The p_i's satisfy $\sum_{i=1}^{m} p_i = 1$ and $p_i \geq 0$
- Each p_i can be thought of as the relative frequency with which x_i would occur if an experiment of observing x were repeated infinitely often.
- The variable x characterized in this way is <u>random variable</u>.
- Example: Rolling of an ordinary six-sided die. Possibilities for the numer of spots $\{1,2,3,4,5,6\}$. Probabilities: $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

QS, Jan 9, 2004.

- If the variable can take any real value in an interval, we have continuous random variable, and probability density function $p(x)$ that describes the probability. The probability that the variable's value will lie in any segment of the line is equal to the area of the vertical region bounded by this segment and the density function.



Probability distribution for the die.



Probability density

$$P(x \in [a,b]) = \int_a^b p(x)\,dx$$

- Of course, $p(x) \geq 0$ and $\int_{-\infty}^{\infty} p(x)\,dx = 1$.

- Expected value $E(x) = \sum_{i=1}^{n} p_i x_i = \int_{-\infty}^{\infty} p(x)\,x\,dx$

Sometimes we will use $E(x) = \overline{x} = \langle x \rangle$.

QS, Jan 9, 2004.

- Example: For the roll of a die, we have

$$\bar{x} = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 3.5$$

<u>Note</u> that $\bar{x}$ is not necessarily a possible outcome of a roll.

- Properties:
  1) If y is a known value (not random), then $E(y) = y$.
  2) If x and y are random variables and α and β are real numbers, then
  $$E(\alpha x + \beta y) = \alpha E(x) + \beta E(y).$$
  3) If x is random but $x \geq 0$, then $E(x) \geq 0$.

- Variance: Although the expected value provides a useful summary of the probabilistic nature of random variable, typically one wants to have a <u>measure of possible deviation</u> from the mean.

QS, Jan 9, 2004.

- Since $E(x-\bar{x})=0$, we will use variance:

$$\sigma_x^2 = Var(x) = E((x-\bar{x})^2) = E(x^2) - (E(x))^2$$

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2$$

- Standard deviation $\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\overline{x^2} - \bar{x}^2}$

- We will now use Excel to find mean and standard deviation of the rate of return for several big companies, using the data from the stock Market (obtained from the Yahoo! Finance/ Historical Quotes).

- When considering two or more random variables, their mutual dependence can be summarized conveniently by their covariance.

QS, Jan 9, 2004.

- Let x_1 and x_2 be two random variables with $E(x_1) = \overline{x_1}$ and $E(x_2) = \overline{x_2}$. The covariance of these variables is defined as

$$\sigma_{x_1, x_2} = Cov(x_1, x_2) = E\Big((x_1 - \overline{x_1})(x_2 - \overline{x_2}) \Big) =$$

$$= E(x_1 x_2) - E(x_1) E(x_2) =$$

$$= E(x_1 x_2) - \overline{x}_1 \overline{x}_2 = \overline{x_1 x_2} - \overline{x}_1 \overline{x}_2$$

- We have $\sigma_{x,x} = \overline{x^2} - \overline{x}^2 = \sigma_x^2$.

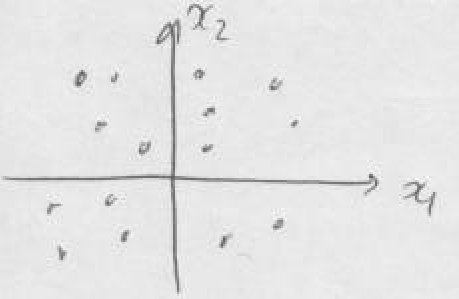
- Theorem $|\sigma_{x_1, x_2}| \leq \sigma_{x_1} \sigma_{x_2}$

- If $\sigma_{x_1, x_2} = 0$, variables x_1 and x_2 are uncorrelated.

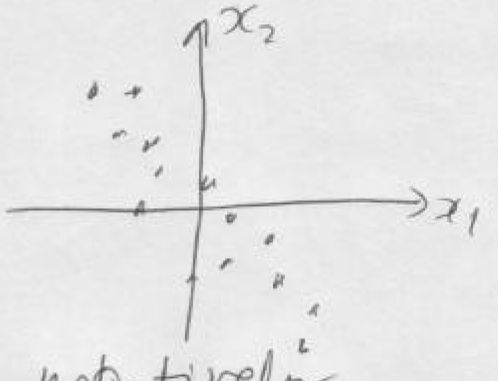
- If $\sigma_{x_1, x_2} < 0$, they are negatively correlated

- If $\sigma_{x_1, x_2} > 0$, they are positively correlated

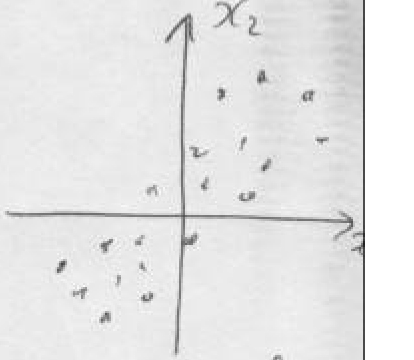
QS, Jan 9, 2004.



uncorrelated



negatively correlated



positively correlated

- Variance of the sum of two random variables:

$$Var(x+y) = \sigma^2_{x+y} = \sigma^2_x + 2\sigma_{xy} + \sigma^2_y$$

- If x and y are uncorrelated, then

$$Var(x+y) = \sigma^2_{x+y} = \sigma^2_x + \sigma^2_y .$$

- When an asset is originally acquired, its rate of return is usually uncertain. Accordingly, we consider its rate of return to be a random variable.

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade; and
SECCF, Andrićev Venac 2, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: 011 32 44 953, 011 32 40 345

QS, Jan 9, 2004.

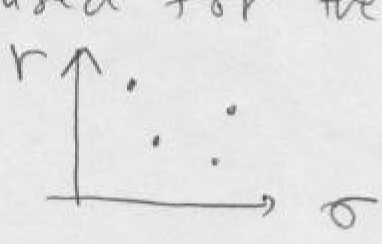
- The expected value of the rate of return is $\bar{r} = E(r)$, and its variance is $\sigma_r^2 = E((r-\bar{r})^2)$.

- If there are several assets, we will also use the covariance of appropriate rates of return, $\mathrm{cov}(r_1, r_2) = E((r_1 - \bar{r}_1)(r_2 - \bar{r}_2)) = \sigma_{r_1, r_2}$.

- Of course, $\mathrm{cov}(r_1, r_2) = \mathrm{cov}(r_2, r_1)$.

- We will now calculate all these quantities for the data from the stock market for Boeing (BA), Hewlett-Packard (HPQ), and IBM (IBM).

- The random rates of return of assets can be represented on a two-dimensional diagram: the horizontal axis is used for standard deviation, and the vertical axis is used for the mean.



$\bar{r}-\sigma$ diagram, or mean-standard deviation diagram.

QS, Jan 9, 2004.

- We will now draw $\bar{r}-\sigma$ diagram for our example (BA, HPQ, IBM).

- Portfolio Mean and Variance:
  - Suppose that there are n assets with random rates of return $r_1, \ldots, r_n$, and suppose that these have expected values $E(r_1)=\bar{r}_1, \ldots, E(r_n)=\bar{r}_n$.
  - Suppose further that we form a portfolio of these n assets using the weights $w_1, \ldots, w_n$. The rate of return of the portfolio is given by
$$r = w_1 r_1 + \cdots + w_n r_n = \sum_{i=1}^{n} w_i r_i$$
  - The expected value (mean) of the rate of return of ~~the~~ the portfolio is given by
$$E(r) = \bar{r} = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \bar{r}_i .$$
  - Variance of portfolio return
$$\sigma_r^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}, \quad \text{where } \sigma_{ij} = \operatorname{cov}(r_i, r_j).$$

UNIVERSITY OF BELGRADE
FACULTY OF ECONOMICS

SECCF
South European Center for Contemporary Finance

QS, Jan 9, 2004.

– The Markowitz Model: To find a minimum-variance portfolio, we fix the mean value at some arbitrary value $\bar{r}$. Then we find the feasible portfolio of minimum variance that has this mean. Mathematical formulation:

$$\text{minimize}\left(\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}\right),$$

$$\text{subject to } \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r},$$

$$\sum_{i=1}^{n} w_i = 1, \quad \text{and} \quad w_i \geq 0, \text{ for all } i.$$

– After that, we can try to find minimum-variance portfolio for several fixed values of $\bar{r}$, draw them on a mean-standard deviation graph and choose the optimal portfolio.

– We will solve the Markowitz problem in Excel

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade; and
SECCF, Andrićev Venac 2, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: 011 32 44 953, 011 32 40 345

QS, Jan 9, 2004.

– Homework XIII:

1) Find the variance and standard deviation for the roll of the die (six-sided.).

2) Find the expected value, the variance, and standard deviation for a sum of spots obtained in two independent rolls of the six-sided die.

– Homework XIV:

1) Prove that the variance of a portfolio return is given by the formula $\sigma_r^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$, where $\sigma_{ij} = cov(r_i, r_j)$.

2) Solve the Markowitz problem for a portfolio containing stocks of BA, HPQ, IBM and another big company (with stocks available on the stock market). Download data from the Yahoo! Finance Historical Quotes and repeat the analysis done at the class. Find the optimal portfolio (calculate weights w_i, optimal values of $\bar{r}$ and $\sigma_{\bar{r}}$).

QS, Jan 9, 2004.

14

- Solutions of HW X, assigned on January 8, 2004.

1) PV of the prize is equal to the PV of a cash flow stream $(x_0, x_1, \ldots, x_{19})$, where $x_0 = x_1 = \ldots = x_{19} = \$500{,}000$, so at $r = 10\%$ we have

$$PV = \sum_{k=0}^{19} \frac{x_k}{(1+r)^k} = \$500{,}000 \cdot \sum_{k=0}^{19} \frac{1}{(1+r)^k} \cong \$4{,}682{,}460.05$$

We can use Excel's PV function. Since it calculates PV only for cash flow streams starting at the end of the first period (and then with cash flows occuring at the end of each period), the correct formula is

$$-\$500{,}000 + PV(10\%, 19, 500{,}000)$$

The minus sign is present since Excel assumes that in PV all cash flows are payments.

2) If the revenue obtained is a, then the correspon-ding cash flow stream is $(-1, 0, 0, a)$, and $NPV = -1 + \frac{a}{1.1^3}$. If it is not worthwile to

QS, Jan 9, 2004.

implement this plan, then this NPV is less than NPV of plan (b), which is prefered according to NPV criterion. So,

$$-1 + \frac{a}{1.1^3} < 1.48 \Rightarrow \frac{a}{1.1^3} < 2.48 \Rightarrow$$

$$a < 1.1^3 \cdot 2.48 = 3.3.$$

Since $a < 3.3$, we can conclude that $x = 3.3$.

- Solution of HW $\underline{XI}$, assigned on January 8, 2004.

1) The solution is given in Excel file InterdependentProjects.xls, which can be downloaded from seccf home page, Downloads section. The optimal choice corresponds to the following: accept projects 2, 5, and 10, reject all others. Total costs are \$4.5 milion, and total NPV is \$8 milion. All three goals are satisfied.

QS, Jan 9, 2004.

16

- Solution of HW XII, assigned on January 8, 2004.

1) The solution is given in Excel file CashMatching.xls, which can be downloaded from seccf home page, from Downloads section.
The optimal choice of bonds is:
11.2 bonds of type 2, 6.8 bonds of type 4, 6.3 bonds of type 8, and 0.3 bonds of type 9.
The cost of this investment is \$2,381.139.

SECCF, Faculty of Economics, Kamenička 6, room 409, IV floor, 11000 Belgrade; and
SECCF, Andrićev Venac 2, 11000 Belgrade
http://www.seccf.org http://www.ekof.bg.ac.yu e-mail: imqf@seccf.org
tel: 011 32 44 953, 011 32 40 345