Second Edition

ADVANCED ENGINEERING MATHEMATICS

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INTRODUCTION

Review of Some Topics from Algebra and Calculus

This introductory chapter includes some topics which are usually covered in elementary algebra and calculus. We shall refer to sections of this chapter whenever we need some of these topics as a prerequisite for our further consideration.

References: Appendix 1, Ref. [A13].
Answers to problems: Appendix 2.

0.1 *Elementary Functions*

In this section we shall present a collection of some basic formulas for reference.

Figure 1 shows the graph of the **exponential function** e^x, where

$$e = 2.7\,1828\,1828\,45\,90\,45\,2\cdots.$$

Basic identities are

$$\text{(1)}\qquad e^x e^y = e^{x+y}, \quad e^x/e^y = e^{x-y}, \quad (e^x)^y = e^{xy}.$$

The inverse of e^x is the **natural logarithm** $\ln x$ (Fig. 2). It satisfies the identities

$$\text{(2)}\qquad \ln(xy) = \ln x + \ln y, \quad \ln\frac{x}{y} = \ln x - \ln y, \quad \ln(x^a) = a\ln x.$$

Furthermore,

$$e^{\ln x} = x, \qquad e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}.$$

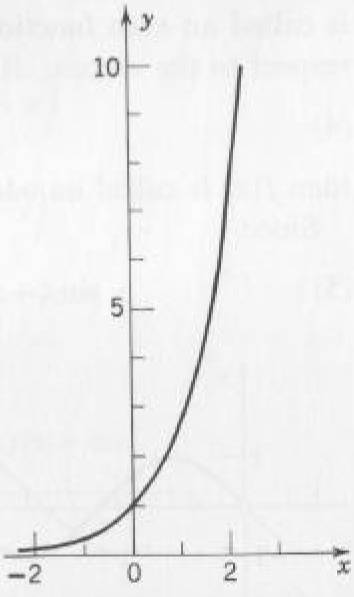


Fig. 1. Exponential function e^x.

The inverse of the exponential function 10^x is the *logarithm of base* 10 which is denoted by

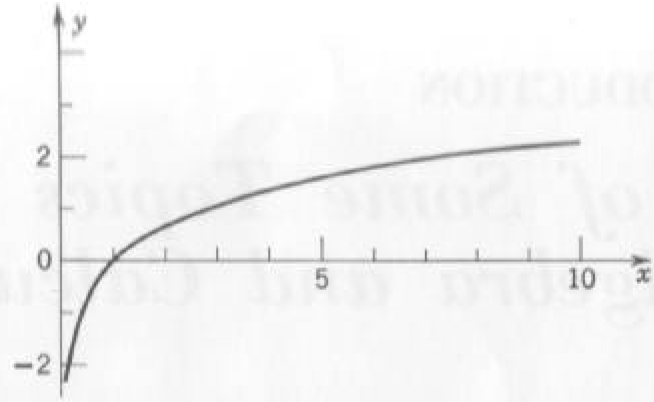


Fig. 2. Natural logarithm $\ln x$.

$\log_{10} x$ or simply by $\log x$. We have

$$\log x = M \ln x$$

where

$$M = \log e = 0.43429\ 44819\ 03251\ 82765$$

and conversely

$$\ln x = \frac{1}{M} \log x$$

where

$$\frac{1}{M} = \ln 10 = 2.30258\ 50929\ 94045\ 68402.$$

The **sine** and **cosine functions** $\sin x$ and $\cos x$ are defined in trigonometry for all values of x. *Throughout calculus, angles are measured in radians* so that both functions have the period 2π.

A function $w = f(x)$ which is defined for all x and has the property

(3) $$f(-x) = f(x) \qquad \text{for all } x$$

is called an **even function**. The graph of such a function is symmetric with respect to the w-axis. If $f(x)$ is defined for all x and

(4) $$f(-x) = -f(x) \qquad \text{for all } x,$$

then $f(x)$ is called an **odd function**. These are two quite important concepts.

Since

(5) $$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x$$

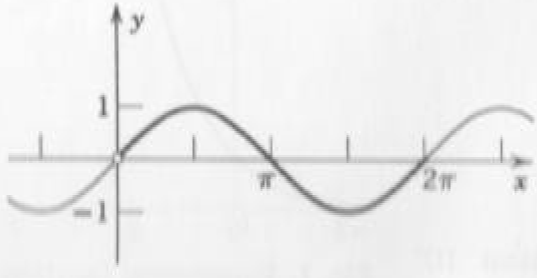


Fig. 3. $\sin x$.

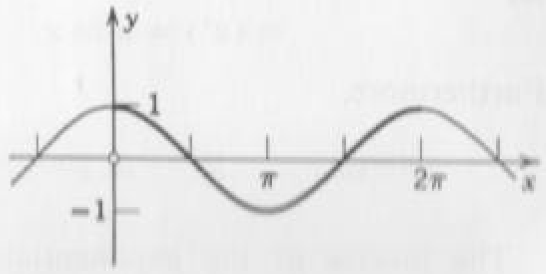


Fig. 4. $\cos x$.

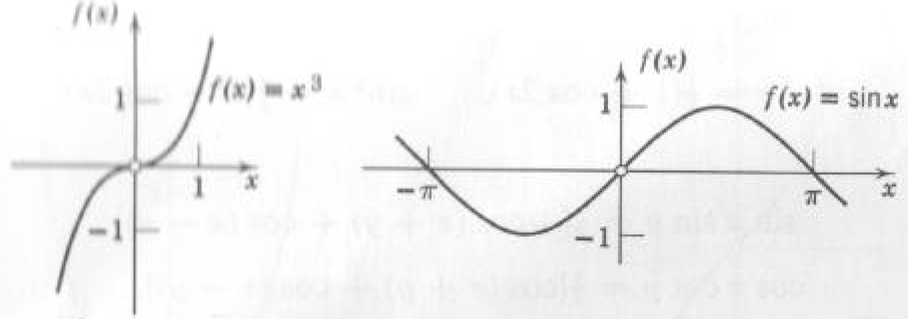


Fig. 5. Odd functions.

sin x is odd while cos x is even. The exponential function e^x is neither odd nor even.

The functions sin x and cos x are related by the identity

$$\sin^2 x + \cos^2 x = 1. \tag{6}$$

The addition formulas of the sine function are

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y. \end{aligned} \tag{7}$$

In particular,

$$\sin 2x = 2 \sin x \cos x. \tag{7*}$$

The addition formulas of the cosine function are

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y. \end{aligned} \tag{8}$$

In particular,

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{8*}$$

From (7) and (8),

$$\begin{aligned} \sin x &= \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) \\ \cos x &= \sin\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right) \end{aligned} \tag{9}$$

$$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos x.$$

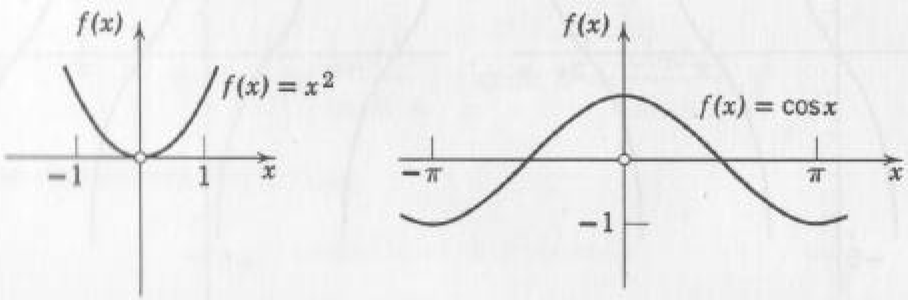


Fig. 6. Even functions.

From (8*) and (6),

(10) $$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

From (8),

(11a) $$\begin{aligned}\sin x \sin y &= \tfrac{1}{2}[-\cos(x + y) + \cos(x - y)]\\ \cos x \cos y &= \tfrac{1}{2}[\cos(x + y) + \cos(x - y)],\end{aligned}$$

and from (7),

(11b) $$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)].$$

From this, by setting $x + y = u$ and $x - y = v$,

(12) $$\begin{aligned}\sin u + \sin v &= 2 \sin \frac{u + v}{2} \cos \frac{u - v}{2}\\ \cos u + \cos v &= 2 \cos \frac{u + v}{2} \cos \frac{u - v}{2}\\ \cos v - \cos u &= 2 \sin \frac{u + v}{2} \sin \frac{u - v}{2}.\end{aligned}$$

The other trigonometric functions are defined by the identities

(13) $$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

The addition formulas for the tangent are

(14) $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

In applications it is sometimes required to write $A \cos x + B \sin x$ where A and B are given constants, in the form $C \cos(x - \delta)$ where C and δ are

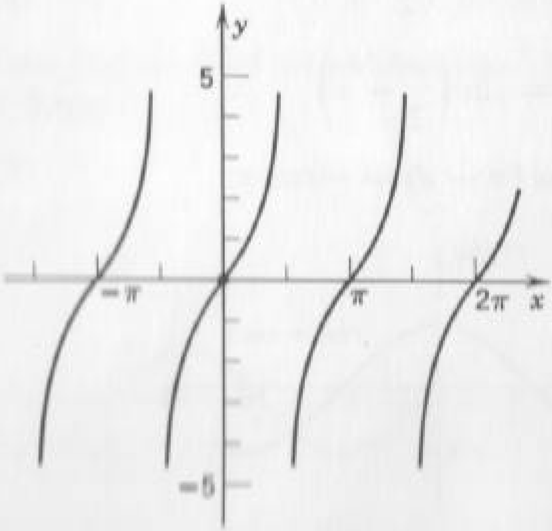


Fig. 7. tan x.

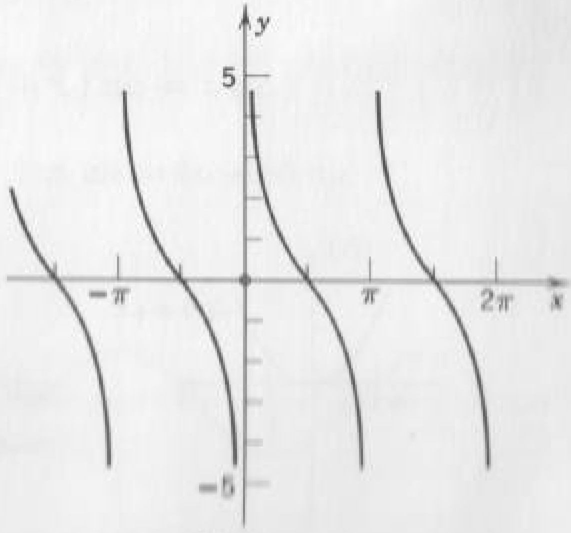


Fig. 8. cot x.

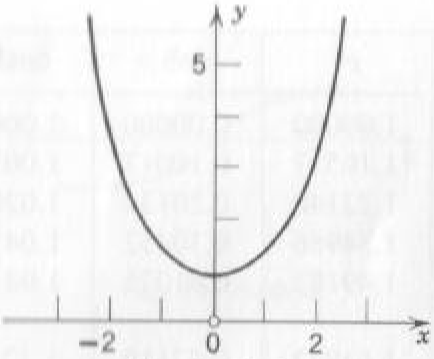


Fig. 9. Cosh x.

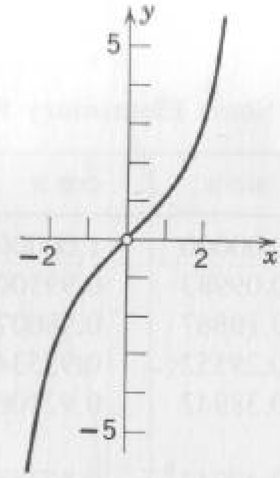


Fig. 10. Sinh x.

constants. From (8) we obtain

$$C\cos(x-\delta) = C\cos\delta\cos x + C\sin\delta\sin x$$

and this is equal to $A\cos x + B\sin x$, if $C\cos\delta = A$ and $C\sin\delta = B$. Using (6), we thus obtain

(15a) $$A\cos x + B\sin x = \sqrt{A^2+B^2}\cos(x-\delta)$$

where

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{B}{A}.$$

Similarly,

(15b) $$A\cos x + B\sin x = \sqrt{A^2+B^2}\sin(x\pm\delta)$$

where

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \pm\frac{A}{B}.$$

The **hyperbolic cosine** and **sine functions** are defined by the identities

(16) $$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

$\cosh x$ is even, while $\sinh x$ is odd. The other hyperbolic functions are defined by the identities

(17) $$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}.$$

From the definitions we obtain

(18) $$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}$$

Table 1. Some Elementary Functions

x	$\sin x$	$\cos x$	$\tan x$	e^x	$\sinh x$	$\cosh x$
0	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.1	0.09983	0.99500	0.10033	1.10517	0.10017	1.00500
0.2	0.19867	0.98007	0.20271	1.22140	0.20134	1.02007
0.3	0.29552	0.95534	0.30934	1.34986	0.30452	1.04534
0.4	0.38942	0.92106	0.42279	1.49182	0.41075	1.08107
0.5	0.47943	0.87758	0.54630	1.64872	0.52110	1.12763
0.6	0.56464	0.82534	0.68414	1.82212	0.63665	1.18547
0.7	0.64422	0.76484	0.84229	2.01375	0.75858	1.25517
0.8	0.71736	0.69671	1.02964	2.22554	0.88811	1.33743
0.9	0.78333	0.62161	1.26016	2.45960	1.02652	1.43309
1.0	0.84147	0.54030	1.55741	2.71828	1.17520	1.54308
1.1	0.89121	0.45360	1.96476	3.00417	1.33565	1.66852
1.2	0.93204	0.36236	2.57215	3.32012	1.50946	1.81066
1.3	0.96356	0.26750	3.60210	3.66930	1.69838	1.97091
1.4	0.98545	0.16997	5.79788	4.05520	1.90430	2.15090
1.5	0.99750	0.07074	14.10142	4.48169	2.12928	2.35241
1.6	0.99957	−0.02920	−34.23253	4.95303	2.37557	2.57746
1.7	0.99166	−0.12884	−7.69660	5.47395	2.64563	2.82832
1.8	0.97385	−0.22720	−4.28626	6.04965	2.94217	3.10747
1.9	0.94630	−0.32329	−2.92710	6.68589	3.26816	3.41773
2.0	0.90930	−0.41615	−2.18504	7.38906	3.62686	3.76220

x	$\ln x$	x	$\ln x$	x	$\ln x$	x	$\ln x$
1.0	0.00000	2.0	0.69315	3.0	1.09861	5	1.60944
1.1	0.09531	2.1	0.74194	3.1	1.13140	7	1.94591
1.2	0.18232	2.2	0.78846	3.2	1.16315	11	2.39790
1.3	0.26236	2.3	0.83291	3.3	1.19392	13	2.56495
1.4	0.33647	2.4	0.87547	3.4	1.22378	17	2.83321
1.5	0.40547	2.5	0.91629	3.5	1.25276	19	2.94444
1.6	0.47000	2.6	0.95551	3.6	1.28093	23	3.13549
1.7	0.53063	2.7	0.99325	3.7	1.30833	29	3.36730
1.8	0.58779	2.8	1.02962	3.8	1.33500	31	3.43399
1.9	0.64185	2.9	1.06471	3.9	1.36098	37	3.61092

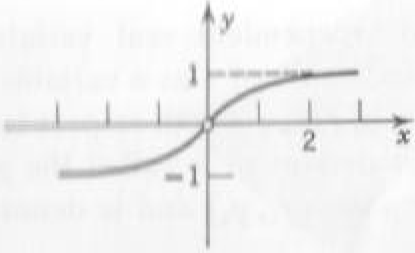


Fig. 11. tanh x.

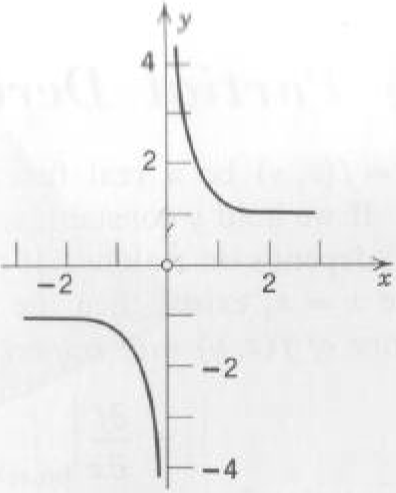


Fig. 12. coth x.

and furthermore the addition formulas

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

(19) $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

Problems

Are the following functions even, odd, or neither even nor odd?

1. $\sin(x^2)$ **2.** $\sin^2 x$ **3.** $\sin x + \cos x$ **4.** x, x^3, x^5

5. $\tan x$ **6.** $\tanh x$ **7.** x^2, x^4, x^6 **8.** $\ln(1 + e^x) - \dfrac{x}{2}$

9. Prove that the sum and the product of even functions are even functions.

10. Prove that the sum of odd functions is odd and the product of two odd functions is even.

11. Derive (9) from (7) and (8).

12. Derive (11) from (7) and (8).

13. Prove (18).

14. Prove the addition formulas of the hyperbolic sine and cosine.

Prove the following identities.

15. $\sinh x \sinh y = \frac{1}{2}[\cosh(x + y) - \cosh(x - y)]$

16. $\cosh x \cosh y = \frac{1}{2}[\cosh(x + y) + \cosh(x - y)]$

17. $\sinh x \cosh y = \frac{1}{2}[\sinh(x + y) + \sinh(x - y)]$

18. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

19. Using the differentiation formula of e^x, find the derivatives of $\sinh x$, $\cosh x$, and $\tanh x$.

20. Show that for large x, $\sinh x \approx e^x/2$, $\cosh x \approx e^x/2$.

0.2 *Partial Derivatives*

Let $z = f(x, y)$ be a real function of two independent real variables, x and y. If we hold y constant, say, $y = y_1$, and think of x as a variable, then $f(x, y_1)$ depends on x alone. If the derivative of $f(x, y_1)$ with respect to x for a value $x = x_1$ exists, then the value of this derivative is called the *partial derivative of $f(x, y)$ with respect to x at the point* (x_1, y_1) and is denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1)} \quad \text{or by} \quad \left.\frac{\partial z}{\partial x}\right|_{(x_1,y_1)}.$$

Other notations are

$$f_x(x_1, y_1) \quad \text{and} \quad z_x(x_1, y_1);$$

these may be used when subscripts are not used for another purpose and there is no danger of confusion.

We thus have, by the definition of the derivative,

$$\text{(1)} \qquad \left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1)} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x, y_1) - f(x_1, y_1)}{\Delta x}.$$

The partial derivative of $z = f(x, y)$ with respect to y is defined similarly; we now hold x constant, say, equal to x_1, and differentiate $f(x_1, y)$ with respect to y. Thus

$$\text{(2)} \qquad \left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1)} = \left.\frac{\partial z}{\partial y}\right|_{(x_1,y_1)} = \lim_{\Delta y \to 0} \frac{f(x_1, y_1 + \Delta y) - f(x_1, y_1)}{\Delta y}.$$

Other notations are $f_y(x_1, y_1)$ and $z_y(x_1, y_1)$.

It is clear that the values of those two partial derivatives will in general depend on the point (x_1, y_1), and so the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ at a variable point (x, y) are functions of x and y. The function $\partial z/\partial x$ is obtained as in ordinary calculus by differentiating $z = f(x, y)$ with respect to x, *treating y as a constant*, and $\partial z/\partial y$ is obtained by differentiating z with respect to y, *treating x as a constant.*

Example 1. Let $z = f(x, y) = x^2y + x \sin y$. Then

$$\frac{\partial f}{\partial x} = 2xy + \sin y, \qquad \frac{\partial f}{\partial y} = x^2 + x \cos y.$$

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of a function $z = f(x, y)$ have a very simple geometric interpretation. The function $z = f(x, y)$ can be represented by a surface in space. The equation $y = y_1$ then represents a vertical plane intersecting the surface in a curve, and the partial derivative $\partial z/\partial x$ at a point (x_1, y_1) is the slope of the tangent (i.e., $\tan \alpha$ where α is the angle shown in Fig. 13) to the curve. Similarly, the partial derivative $\partial z/\partial y$ at (x_1, y_1) is the slope of the tangent to the curve $x = x_1$ on the surface $z = f(x, y)$ at (x_1, y_1).

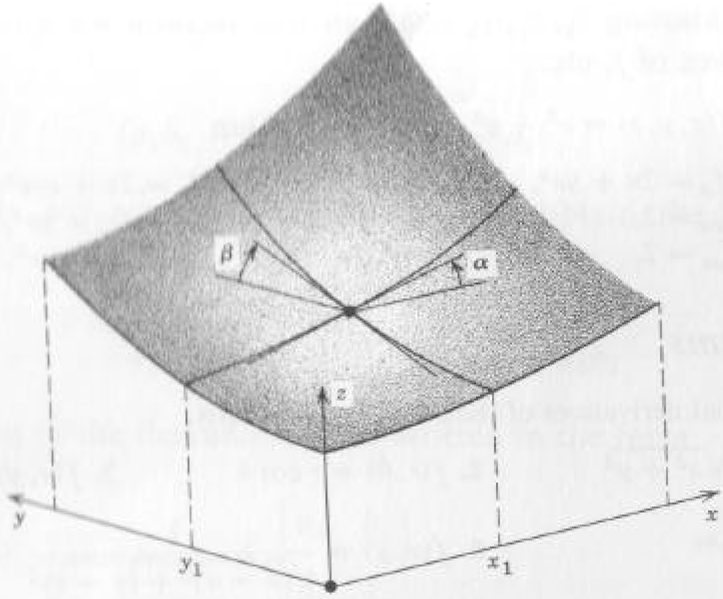


Fig. 13. Geometrical interpretation of first partial derivatives.

The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ are called *first partial derivatives* or *partial derivatives of the first order*. By differentiating these derivatives once more we obtain the four *second partial derivatives* (or *partial derivatives of the second order*):

$$(3) \quad \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x, y), \quad \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x, y), \quad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y).$$

Thus $\partial^2 z/\partial x^2$ is the partial derivative of $\partial z/\partial x$ with respect to x, $\partial^2 z/\partial x\,\partial y$ is the partial derivative of $\partial z/\partial x$ with respect to y, etc.

It can be shown[1] that if all the derivatives concerned are continuous, then

$$(4) \qquad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}\,;$$

i.e., the order of differentiation is then immaterial.

Example 2. For the function in Ex. 1,

$$f_{xx} = 2y, \qquad f_{xy} = 2x + \cos y = f_{yx}, \qquad f_{yy} = -x \sin y.$$

By differentiating the second partial derivatives again with respect to x and y, respectively, we obtain the *third partial derivatives* or *partial derivatives of the third order* of f, etc.

If we consider a function $f(x, y, z)$ of three independent variables, then we have the three first partial derivatives $f_x(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$. Here f_x is obtained by differentiating f with respect to x, *treating both y and z as constants*. Thus, in analogy to (1), we now have

$$\left.\frac{\partial f}{\partial x}\right|_{(x_1, y_1, z_1)} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x, y_1, z_1) - f(x_1, y_1, z_1)}{\Delta x},$$

[1] Cf. Ref. [A1] in Appendix 1.

etc. By differentiating f_x, f_y, f_z again in this fashion we obtain the second partial derivatives of f, etc.

Example 3. Let $f(x, y, z) = x^2 + y^2 + z^2 + xye^z$. Then

$$\begin{aligned} &f_x = 2x + ye^z, \quad && f_y = 2y + xe^z, \quad && f_z = 2z + xye^z, \\ &f_{xx} = 2, && f_{xy} = f_{yx} = e^z, && f_{xz} = f_{zx} = ye^z, \\ &f_{yy} = 2, && f_{yz} = f_{zy} = xe^z, && f_{zz} = 2 + xye^z. \end{aligned}$$

Problems

Find the first partial derivatives of the following functions.

1. $f(x, y) = \sqrt{x^2 + y^2}$ 2. $f(r, \theta) = r \cos \theta$ 3. $f(x, y) = \arctan \dfrac{y}{x}$

4. $f(x, y) = e^{xy}$ 5. $f(x, y) = \dfrac{1}{\sqrt{(x - a)^2 + (y - b)^2}}$ (a, b constant)

6. $f(r, h) = \dfrac{\pi}{3} r^2 h$ 7. $f(R_1, R_2) = \dfrac{R_1 + R_2}{R_1 R_2}$

8. Find $f_x + g_y$ where $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$.

9. Find $f_x + g_y$ where $f(x, y) = \ln (x^2 + y^2)$ and $g(x, y) = 2 \arctan \dfrac{y}{x}$.

Find $f_{xx} + f_{yy}$ where

10. $f(x, y) = x^2 - y^2$ 11. $f(x, y) = e^x \cos y$ 12. $f(x, y) = \sin x \cosh y$

Sketch the surfaces corresponding to the following functions.

13. $z = x^2 + y^2$ 14. $z = \ln (x^2 + y^2)$ 15. $z = e^{xy}$

16. The curves $z = f(x, y) = const$ are called **level curves** of $f(x, y)$. Draw the level curves of the functions in Probs. 13–15.

Find the first partial derivatives of the following functions at the given points.

17. $f(x, y) = \sqrt{1 - x^2 - y^2}$, at $(0, 0)$ 18. $f(x, y) = (x^2 + y^2)^2$, at $(1, 2)$

Find the first and second partial derivatives of the following functions.

19. $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ 20. $f(x, y, z) = \dfrac{x + y + z}{xyz}$

0.3 *Second and Third Order Determinants*

Consider the system

$$(1) \qquad \begin{aligned} a_1 x + b_1 y &= k_1 \\ a_2 x + b_2 y &= k_2 \end{aligned}$$

consisting of two linear equations in the unknowns x and y. To solve this system we may multiply the first equation by b_2, the second by $-b_1$, and add, finding

$$(a_1 b_2 - a_2 b_1) x = k_1 b_2 - k_2 b_1.$$

Then we multiply the first equation of (1) by $-a_2$, the second by a_1, and add again, finding

$$(a_1b_2 - a_2b_1)y = a_1k_2 - a_2k_1.$$

If $a_1b_2 - a_2b_1$ is not zero, we may divide and obtain the desired result

$$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}. \tag{2}$$

The expression in the denominators is written in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and is called a **determinant of the second order**; thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1. \tag{3}$$

The four numbers a_1, b_1, a_2, b_2 are called the **elements** of the determinant. The elements in a horizontal line are said to form a **row** and the elements in a vertical line are said to form a **column** of the determinant.

We may now write the solution (2) of the system (1) in the form

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D} \qquad (D \neq 0) \tag{4}$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad D_1 = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}.$$

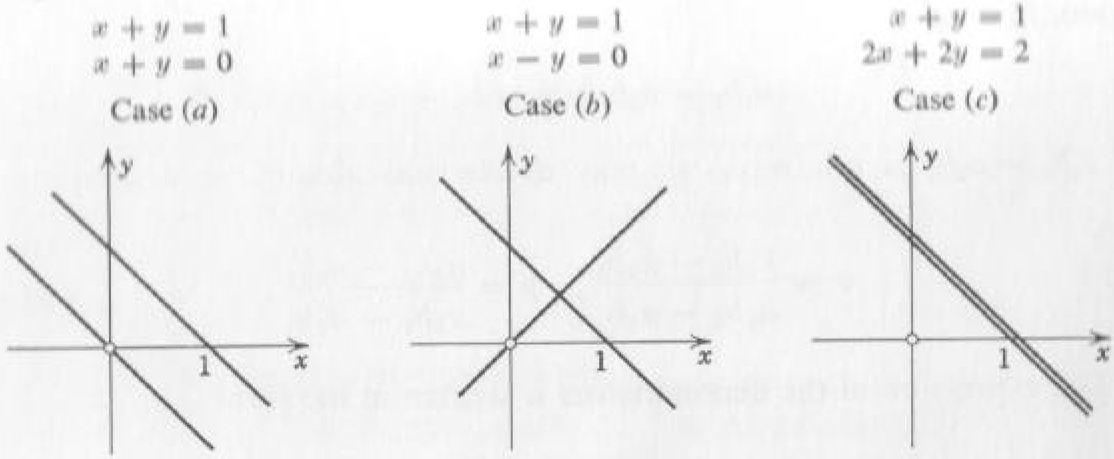
The formula (4) is called **Cramer's rule**.[2] Note that D_1 is obtained by replacing the first column of D by the column with elements k_1, k_2, and D_2 is obtained by replacing the last column of D by that column.

Each equation of the system (1) represents a straight line in the xy-plane, and a pair of numbers (x, y) is a solution of (1) if, and only if, the point P with coordinates x, y lies on both lines. Hence there are three possible cases:

(a) No solution if the lines are parallel.
(b) Precisely one solution if they intersect.
(c) Infinitely many solutions if they coincide.

[2] GABRIEL CRAMER (1704–1752), Italian mathematician, also known by his contributions to the theory of curves.

Example 1.



If both k_1 and k_2 are zero, the system is said to be **homogeneous**; otherwise it is said to be **nonhomogeneous**.

If the system is homogeneous, Case (*a*) cannot occur, because then the lines represented by the equations pass through the origin, and the system has at least the *trivial solution* $x = 0$, $y = 0$.

The homogeneous system will have further solutions if, and only if, those two lines coincide, and then each point on the line is a solution. This happens if, and only if, $D = 0$, as the reader may show.

If the system is nonhomogeneous and $D \neq 0$, it has precisely one solution, which is obtained from (4).

A system of three linear equations in three unknowns x, y, z

$$(5)\qquad \begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

may be considered in a similar fashion. To obtain an equation involving only x the equations are multiplied, respectively, by

$$b_2c_3 - b_3c_2, \qquad -(b_1c_3 - b_3c_1), \qquad b_1c_2 - b_2c_1.$$

We see that these expressions may be written as second-order determinants:

$$M_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad -M_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad M_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Adding the resulting equations, we obtain

$$(6)\qquad (a_1M_1 - a_2M_2 + a_3M_3)x = k_1M_1 - k_2M_2 + k_3M_3.$$

Two further equations containing only y and z, respectively, may be obtained in a similar manner.

To simplify our notation we now define a **determinant of the third order** by the equation

$$(7)\quad D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

We see that

$$D = a_1M_1 - a_2M_2 + a_3M_3,$$

the coefficient of x in (6), and if we write the second-order determinants in (7) at length, we obtain

$$D = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

or

(8) $$D = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Obviously the determinant on the right-hand side of (7) which is multiplied by a_i, $i = 1, 2,$ or 3, is obtained from D by omitting the first column and the ith row of D.

We see that (6) may now be written

$$Dx = D_1$$

where

$$D_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}.$$

The aforementioned equation containing only y may be written

$$Dy = D_2$$

where

$$D_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix},$$

and the equation containing only z may be written

$$Dz = D_3$$

where

$$D_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}.$$

Note that the elements of D are arranged in the same order as they occur as coefficients in the equations of (5), and D_j, $j = 1, 2,$ or 3, is obtained from D by replacing the jth column by the column with elements k_1, k_2, k_3, the expressions on the right sides of the equations of (5).

It follows that if $D \neq 0$, then system (5) has the unique solution

(9) $$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$ **(Cramer's rule).**

Each equation of system (5) represents a plane in space, and a triple of numbers (x, y, z) is a solution of (5) if, and only if, the point P with coordinates x, y, z is a common point of the three planes. As before we have three possible cases:

(a) No solution if two (or all three) planes are parallel, if two of them coincide and the third is parallel, or if the intersections of each pair of them are three parallel lines.

(b) Precisely one solution, given by (9), if the planes have just one point in common.

(c) Infinitely many solutions if they have a line in common or coincide.

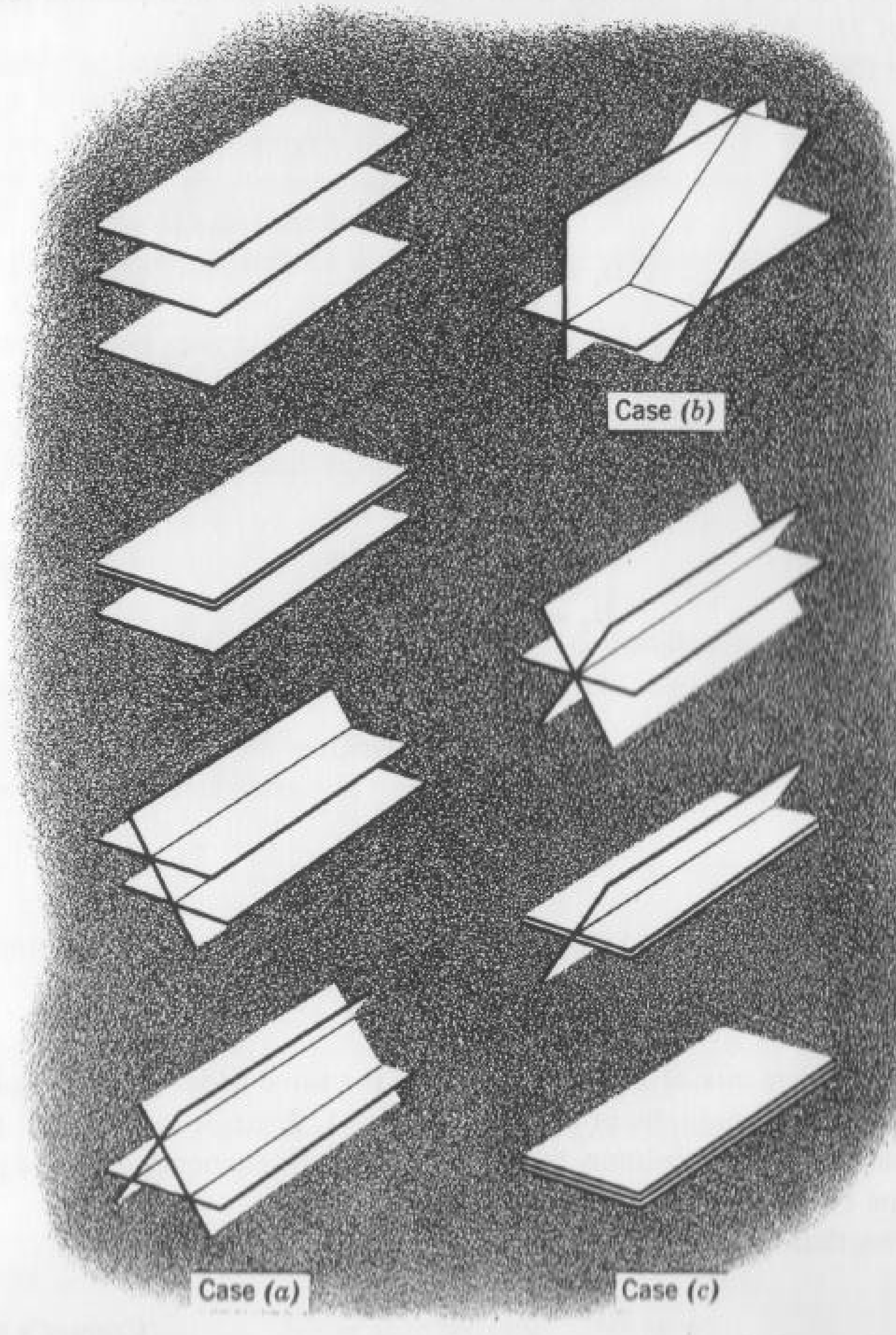


Fig. 14. Geometrical interpretation of three linear equations in three unknowns.

If (5) *is homogeneous, i.e.,* $k_1 = k_2 = k_3 = 0$, *it has at least the trivial solution* $x = y = z = 0$, *and nontrivial solutions exist if, and only if,* $D = 0$. *If* (5) *is nonhomogeneous and* $D \neq 0$, *it has precisely one solution which is obtained from* (9).

We shall now list the most important properties of our determinants; the corresponding proofs follow from (7) by direct calculation.[3]

(A) *The value of a determinant is not altered if its rows are written as columns in the same order,*

$$(10) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(B) *If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by* -1. Example:

$$(11) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The second-order determinant obtained from

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by deleting one row and one column is called the **minor** of the element which belongs to the deleted row and column. Example: The minors of a_2 and b_2 in D are

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix},$$

respectively, etc.

The **cofactor** of the element of D in the ith row and the kth column is defined as $(-1)^{i+k}$ times the minor of that element. Example: The cofactors of a_2 and b_2 are

$$-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix},$$

respectively. Furthermore we see that we may write (7) in the form

$$D = a_1C_1 + a_2C_2 + a_3C_3,$$

[3] Determinants of arbitrary order n will be defined in Sec. 7.3, and we shall see that they have quite similar properties.

where C_i is the cofactor of a_i in D. From this and the properties **(A)** and **(B)** we obtain the following property.

(C) *The determinant D may be developed by any row or column, that is, it may be written as the sum of the three elements of any row (or column) each multiplied by its cofactor.* For example, the development of D by its second row is

$$D = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

From **(C)** we obtain

(D) *A factor of the elements of any row (or column) can be placed before the determinant.* Example:

$$\begin{vmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 2\cdot 3 & 1 \\ 3 & -3\cdot 3 & 2 \\ -1 & 4\cdot 3 & 5 \end{vmatrix} = 3 \begin{vmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 5 \end{vmatrix}.$$

From the properties **(B)** and **(D)** we may draw the following conclusion.

(E) *If corresponding elements of two rows (or columns) of a determinant are proportional, the value of the determinant is zero.*

The following property is basic for simplifying determinants to be evaluated.

(F) *The value of a determinant remains unaltered if the elements of one row (or column) are altered by adding to them any constant multiple of the corresponding elements in any other row (or column).* Example:

$$\begin{vmatrix} -6 & 21 & -30 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix} = \begin{vmatrix} -6+1\cdot 7 & 21-3\cdot 7 & -30+5\cdot 7 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 5 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 5 \\ 0 & -3 & 0 \\ 2 & 7 & -4 \end{vmatrix} = -3 \begin{vmatrix} 1 & 5 \\ 2 & -4 \end{vmatrix} = 42.$$

(G) *If each element of a row (or column) of a determinant is expressed as a binomial, the determinant can be written as the sum of two determinants.* Example:

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

By applying the product rule of differentiation we obtain the following property.

(11) ***If the elements of a determinant are differentiable functions of a variable, the derivative of the determinant may be written as a sum of three determinants,***

$$\frac{d}{dx}\begin{vmatrix} f & g & h \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} f' & g' & h' \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p' & q' & r' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p & q & r \\ u' & v' & w' \end{vmatrix},$$

where primes denote derivatives with respect to x.

Determinants of higher order and more general systems of linear equations will be considered in Chap. 7.

Problems

Evaluate

1. $\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 3 \\ 5 & 7 \end{vmatrix}$ **3.** $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

4. $\begin{vmatrix} 4.6 & -4.1 \\ -2.0 & 0.2 \end{vmatrix}$ **5.** $\begin{vmatrix} 4 & 12 \\ 40 & 20 \end{vmatrix}$ **6.** $\begin{vmatrix} \sqrt{3} & -2 \\ 0.5 & \sqrt{27} \end{vmatrix}$

7. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ **8.** $\begin{vmatrix} -4 & 18 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{vmatrix}$ **9.** $\begin{vmatrix} 9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

10. $\begin{vmatrix} 6 & 13 & -2 \\ 5 & 37 & -5 \\ 1 & 13 & -2 \end{vmatrix}$ **11.** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ **12.** $\begin{vmatrix} 17 & 4 & 6 \\ 8 & 0 & 5 \\ 4 & 0 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}$ **14.** $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ **15.** $\begin{vmatrix} a+b & b & 0 \\ b & b+c & c \\ 0 & c & c+d \end{vmatrix}$

Solve the following systems of equations.

16. $17x + 4y = -24$, $x - 3y = 18$ **17.** $3x - y = 1$, $x + 3y = 7$ **18.** $3x - 4y = 14$, $-x + 3y = -8$

19. Plot the straight lines represented by the equations in Prob. 17 and determine their point of intersection.

20. Show that the equation

$$\begin{vmatrix} y & x^2 & x \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

represents a parabola passing through the points (0, 0), (−1, 1), and (1, 2).

0.4 Complex Numbers

The fact that there are equations such as

$$x^2 + 3 = 0, \qquad x^2 - 10x + 40 = 0,$$

which are not satisfied by any real number, leads to the introduction of complex numbers.[4] These are of the form $a + ib$ (or $a + bi$) where a and b are real numbers. The symbol i is called the **imaginary unit;** the number a is called the **real part** and b the **imaginary part** of $a + ib$. For example, the real part of $4 - 3i$ is 4, and the imaginary part is -3. To each ordered pair of real numbers a, b there corresponds one complex number $a + ib$, and conversely.

Complex numbers can be represented as points in the plane. For this purpose we choose two perpendicular coordinate axes, the horizontal x-axis and the vertical y-axis and on both axes the same unit of length (Fig. 15).

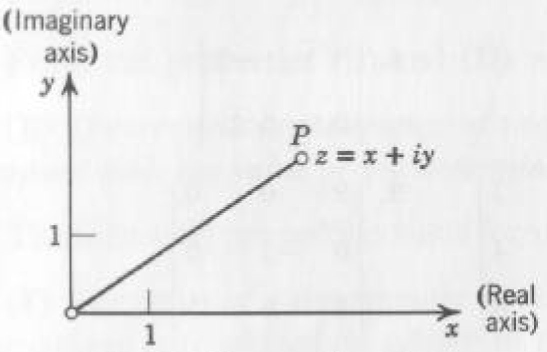


Fig. 15. The complex plane.

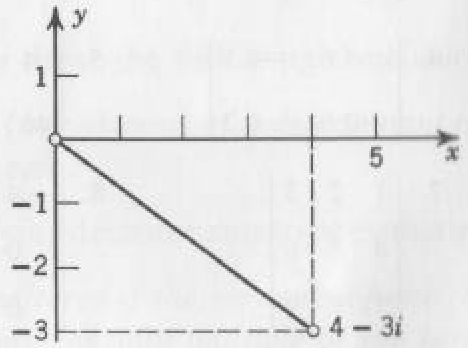


Fig. 16. The number $4 - 3i$ in the complex plane.

The xy-coordinate system thus obtained is called a **Cartesian coordinate system.**[5] To the complex number $z = x + iy$ there corresponds the point P with Cartesian coordinates (x, y). The xy-plane in which the complex numbers are represented geometrically in this fashion is called the **complex plane** or *Argand diagram.*[6] The x-axis is called the *real axis*, and the y-axis the *imaginary axis.*

Two complex numbers

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2$$

[4] First to use complex numbers for this purpose was the Italian mathematician GIROLAMO CARDANO (1501–1576), who found the formula for solving cubic equations. The term "complex number" was introduced by the great German mathematician CARL FRIEDRICH GAUSS (1777–1855) whose work was of basic importance in algebra, number theory, differential equations, differential geometry, non-Euclidean geometry, complex analysis, numerical analysis, and theoretical mechanics. He also paved the way for a general and systematic use of complex numbers.

[5] Named after the French philosopher and mathematician RENATUS CARTESIUS (latinized for RENÉ DESCARTES (1596–1650)), who invented analytic geometry.

[6] JEAN ROBERT ARGAND (1768–1822), French mathematician. His paper on the complex plane appeared in 1806, nine years after a similar memoir by the Norwegian mathematician CASPAR WESSEL (1745–1818).

are defined to be **equal** if, and only if, their real parts are equal and their imaginary parts are equal; that is,

$$z_1 = z_2 \quad \textit{if, and only if,} \quad x_1 = x_2 \text{ and } y_1 = y_2.$$

If z_1 and z_2 are different (not equal), then we may write $z_1 \neq z_2$.

Inequalities between complex numbers, such as $z_1 < z_2$ or $z_1 \geqq z_2$, have no meaning. (Inequalities may hold between the *absolute values* of complex numbers; see Sec. 0.5.)

Addition. The sum $z_1 + z_2$ of z_1 and z_2 is defined as the complex number obtained by adding the real parts and the imaginary parts of z_1 and z_2, that is,

(1) $$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

We see that addition of complex numbers is in accordance with the "*parallelogram law*" by which forces are added in mechanics (Fig. 17).

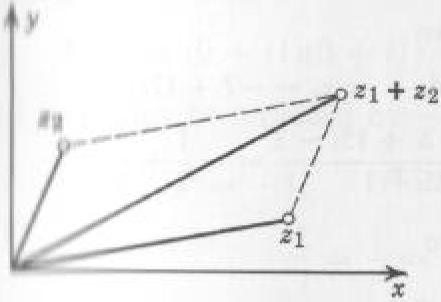


Fig. 17. Addition of complex numbers.

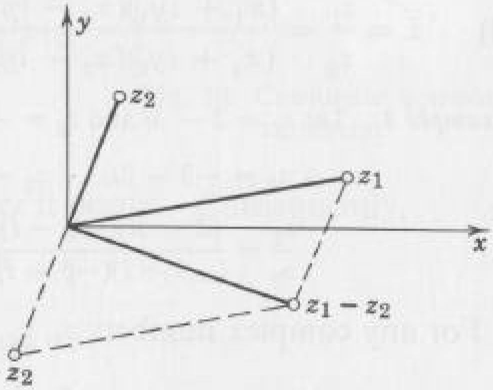


Fig. 18. Subtraction of complex numbers.

Subtraction. This operation is defined as the inverse operation of addition; that is, the difference $z_1 - z_2$ is the complex number z for which $z_1 = z + z_2$. Obviously,

(2) $$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

Multiplication. The product z_1z_2 is defined as the complex number

(3) $$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

which is obtained formally by applying the ordinary rules of arithmetic for real numbers, treating the symbol i as a number, and replacing $i^2 = ii$ by -1.

Division. This operation is defined as the inverse operation of multiplication; that is, the quotient $z = z_1/z_2$ is the complex number $z = x + iy$ for which

$$z_1 = zz_2 = (x + iy)(x_2 + iy_2).$$

By using (3) we see that this can be written

$$x_1 + iy_1 = (x_2x - y_2y) + i(y_2x + x_2y).$$

By definition of equality the real parts and the imaginary parts on both sides must be equal:

$$x_1 = x_2 x - y_2 y$$
$$y_1 = y_2 x + x_2 y.$$

This is a system of two linear equations in the unknowns x and y. The determinant of the coefficients has the value $x_2^2 + y_2^2$. Assuming that $z_2 = x_2 + iy_2 \neq 0$ that determinant is not zero, and Cramer's rule (Sec. 0.3) yields the unique solution

$$x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \quad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0).$$

From this we see that the number $z = x + iy$ can be obtained formally by multiplying both the numerator and the denominator of the quotient z_1/z_2 by $x_2 - iy_2$; thus

(4) $$z = \frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}.$$

Example 1. Let $z_1 = 2 - 3i$ and $z_2 = -5 + i$. Then

$$z_1 + z_2 = -3 - 2i, \quad z_1 - z_2 = 7 - 4i, \quad z_1 z_2 = -7 + 17i,$$
$$\frac{z_1}{z_2} = \frac{(2 - 3i)(-5 - i)}{(-5 + i)(-5 - i)} = \frac{-10 - 2i + 15i - 3}{25 + 1} = -\frac{1}{2} + \frac{i}{2}.$$

For any complex numbers z_1, z_2, z_3 we have

(5)
$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \qquad (\textit{Commutative laws})$$
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3) \qquad (\textit{Associative laws})$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \qquad (\textit{Distributive law}).$$

These laws follow immediately from the corresponding laws for real numbers and the preceding definitions of the algebraic operations for complex numbers.

Let $z = x + iy$ be any complex number; then $x - iy$ is called the **conjugate** of z and is denoted by $\bar{z}$ (Fig. 19). Thus

$$z = x + iy, \qquad \bar{z} = x - iy.$$

For example, the conjugate of $4 + 7i$ is $4 - 7i$. Now

$$z + \bar{z} = 2x, \qquad z - \bar{z} = 2iy$$

and therefore

(6) $$\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

where the symbol Re denotes the real part and Im denotes the imaginary part. Obviously,

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

In the fundamental operations of arithmetic the complex number $x + i0$ behaves like the real number x. Indeed, equality, sum, and product as defined for two such numbers $x_1 + i0$ and $x_2 + i0$ carry over, respectively, into equality, sum, and product as defined for two real numbers x_1 and x_2. For this reason we may say that a complex number whose imaginary part is zero is a real number, and write simply x instead of $x + i0$. The complex number $0 + iy$ is written iy and is called a **pure imaginary number**. Defining i to mean $0 + i1$ we have

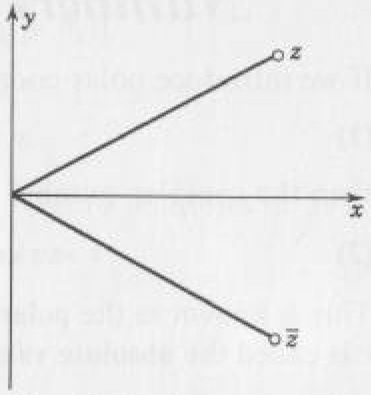


Fig. 19. Conjugate complex numbers.

(8) $$i^2 = (0 + i1)(0 + i1) = -1$$

and so i^2 may be replaced by -1 whenever it occurs. Consequently,

(9) $$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \cdots$$
$$\frac{1}{i} = \frac{i}{i^2} = -i, \quad \frac{1}{i^2} = -1, \cdots.$$

Problems

Let $z_1 = 3 - 2i$ and $z_2 = 2 + 4i$. Reduce each of the following expressions to the form $a + ib$.

1. $z_1 z_2, z_1/z_2, z_1^2, z_1^2/z_2, (1 + i)z_1$

2. $z_1 + z_2, z_1 - z_2, z_2 - z_1, z_2^3, (z_1/z_2)^2, z_2/(1 - 2i)$

Reduce each of the following expressions to the form $a + ib$.

3. $(1 + i)^2, \left(\frac{1 + i}{1 - i}\right)^2$ **4.** $\left(\frac{1 + i}{1 - i}\right)^2 - \left(\frac{1 - i}{1 + i}\right)^2$

Solve the following equations.

5. $z^2 + 9 = 0$ **6.** $z^2 - 2z + 2 = 0$ **7.** $z^2 + 2z + 5 = 0$ **8.** $z^2 + z + 9 = 0$

9. Prove the first two formulas of (7).

10. Prove the last two formulas of (7).

11. If $z = \bar{z}$, show that z is real. If $z = -\bar{z}$, show that z is pure imaginary or zero.

12. If $z^2 = \bar{z}^2$, show that z is either real or pure imaginary.

13. Prove that any number is equal to the conjugate of its conjugate.

14. Plot the numbers $2 + 4i, 2 - 4i, -2 + 4i, -2i$, and $1 - i$ as points in the complex plane.

15. If $z_1 z_2 = 0$, show that at least $z_1 = 0$ or $z_2 = 0$.

Let $z = x + iy$. Find

16. $\mathrm{Re}\,(1/z)$, $\mathrm{Im}\,(1/z)$
17. $\mathrm{Im}\,(z^3)$
18. $\mathrm{Re}\,(1/z^2)$, $\mathrm{Re}\,(z^2 + z)$
19. $\mathrm{Re}\,(-iz^2)$, $\mathrm{Im}\,(4iz^2 - 6z + 8i)$
20. $\mathrm{Re}\,[1/(z - i)]$

0.5 *Polar Form of Complex Numbers*

If we introduce polar coordinates r, θ in the complex plane by setting

(1) $$x = r\cos\theta, \qquad y = r\sin\theta,$$

then the complex number $z = x + iy$ may be written

(2) $$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

This is known as the **polar form** or *trigonometric form* of a complex number. r is called the **absolute value** or *modulus* of z and is denoted by $|z|$. Thus

(3) $$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \qquad (\geqq 0).$$

Obviously, r is the distance of the point P corresponding to z from the origin O in the complex plane (Fig. 20). The directed angle measured from the positive x-axis to OP is called the **argument** of z and is denoted by $\arg z$; *angles will be measured positive in the counterclockwise direction and in terms of radians.* We have, then,

(4) $$\arg z = \theta = \arcsin\frac{y}{r} = \arccos\frac{x}{r} = \arctan\frac{y}{x},$$

as in trigonometry. Note that for given z, *the argument θ is determined only up to multiples of 2π.*

Example 1. Let $z = 1 + i$. Then $|z| = \sqrt{2}$ and $\arg z = \dfrac{\pi}{4} \pm 2n\pi$ where $n = 0, 1, 2, \cdots$

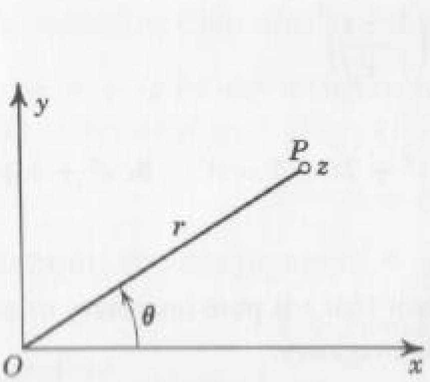


Fig. 20. Trigonometric form of complex numbers.

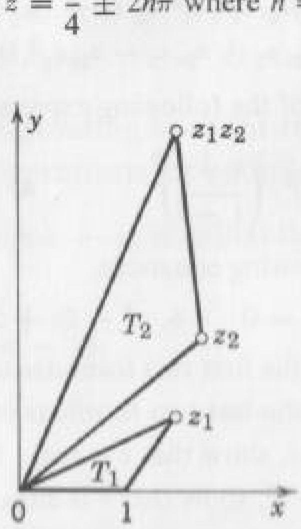


Fig. 21. Multiplication of complex numbers.

The polar form of complex numbers is particularly useful in analyzing multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then the product is

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

If we apply the familiar addition theorems of the sine and the cosine, this assumes the simple form

(5) $$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

We thus obtain the important rules

(6) $$|z_1 z_2| = |z_1|\,|z_2|$$

and

(7) $$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad \text{(up to multiples of } 2\pi).$$

Similarly, from the definition of division it follows that

(8) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

and

(9) $$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \quad \text{(up to multiples of } 2\pi).$$

Table 2. arc tan $\frac{y}{x}$

$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$	$\frac{y}{x}$	arc tan $\frac{y}{x}$
0	0.00000	1.0	0.78540	2.0	1.10715	4.0	1.32582
0.1	0.09967	1.1	0.83298	2.2	1.14417	4.5	1.35213
0.2	0.19740	1.2	0.87606	2.4	1.17601	5.0	1.37340
0.3	0.29146	1.3	0.91510	2.6	1.20362	5.5	1.39094
0.4	0.38051	1.4	0.95055	2.8	1.22777	6.0	1.40565
0.5	0.46365	1.5	0.98279	3.0	1.24905	7.0	1.42890
0.6	0.54042	1.6	1.01220	3.2	1.26791	8.0	1.44644
0.7	0.61073	1.7	1.03907	3.4	1.28474	9.0	1.46014
0.8	0.67474	1.8	1.06370	3.6	1.29985	10.0	1.47113
0.9	0.73282	1.9	1.08632	3.8	1.31347	11.0	1.48014

Example 2. From (6) and (7) we obtain

$$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$$

and from this so-called **formula of De Moivre**[7]

(10) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

From (6) and (7) it follows that the triangles T_1 and T_2 in Fig. 21 are similar. This can be used as the basis for a graphical construction of the product of complex numbers.

[7] ABRAHAM DE MOIVRE (1667–1754), French mathematician, who introduced imaginary quantities in trigonometry and contributed to the theory of mathematical probability.

Problems

Represent the following complex numbers in polar form.

1. $2 - 2i, i, 3 + 4i$ 2. $5 + 5i, -5 + 5i, -5 - 5i$

3. Show that $\arg \bar{z} = -\arg z$ (up to multiples of 2π).

4. Show that $\arg (1/z) = -\arg z$ (up to multiples of 2π).

5. Prove that $|a - z| = |a - \bar{z}|$ where a is any real number. Interpret geometrically.

6. Prove that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane.

Find the absolute values of the following numbers.

7. $1 + i\sqrt{3}, -9i, 2 + i\sqrt{5}$ 8. $2 - i\sqrt{5}, 2 + 3i, (4 + i)^3$

9. $\dfrac{(4 - 3i)(\frac{1}{2} + i)^4}{\left(1 - \frac{3i}{4}\right)^2(-3 + 4i)}$ 10. $\left(\dfrac{1 + i}{1 - i}\right)^8, (3 + 4i)^3(-1 - i)^6$

What loci are represented by the following equations and inequalities? (Plot a graph in each case.)

11. $|z| = 1$ 12. $|z - 1| = 1$ 13. $\mathrm{Re}\,(z^2) = -1$ 14. $\mathrm{Im}\,(2z) = -1$

15. $0 \leqq \arg z \leqq \pi/2$ 16. $0 \leqq \arg (1/z) \leqq \pi/2$

17. Using (10), prove the identities

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta, \qquad \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$

18. Find similar identities as in Prob. 17 for $\cos 2\theta$ and $\cos 4\theta$.

19. Find $(3 - i)(2 + i)$ by means of a graphical construction in the complex plane.

20. For given z find $1/z$ by a graphical construction.

0.6 *Some General Remarks About Numerical Computations*

The present section is devoted to a few simple but important general remarks about numerical computations.

It is clear that engineering mathematics ultimately comes down to numerical results, and the engineering student should, therefore, supplement his mathematical equipment with a definite knowledge of some fundamental numerical methods.

Various examples and problems in the text will help the student to learn how to arrange computations in a suitable form. However, numerical computation requires practical experience. It cannot be learned solely from books; practical training is needed, just as in swimming, driving a car, or playing the piano. Consequently, the student should not only work out book examples and problems, but also set up and calculate examples of his own. Active work is more important in numerical analysis than in many other branches of mathematics. Famous mathematicians like Gauss spent a considerable part of their time with numerical computations, and the engineering student will do well in developing a similar attitude.

In many cases formally complete answers to a problem obtained by theoretical considerations may be almost useless for numerical purposes. For example, in the case of systems of linear equations we obtain the solution in the form of quotients of determinants (Secs. 0.3 and 7.5), but if the number of equations and unknowns is large, the direct evaluation of the solution in this form is certainly not the *practical* answer to the problem of finding *numerical values* of the unknowns. In other cases, the theory may yield the solution of a problem in the form of an integral which cannot be evaluated by the usual methods of calculus. Then we need an approximation method for obtaining numerical values of that solution. Moreover, there are many practical problems which cannot be solved by exact methods at all, and in such cases, which arise quite frequently, we have to look for an appropriate approximation method which will yield numerical values of the solution of the problem.

Because we work with a finite number of digits and carry out a finite number of steps in computations, the methods of numerical analysis are *finite processes*, and a numerical result is an approximate value of the (unknown) exact result, except for the rare cases where the exact answer is a sufficiently simple rational number. The difference

$$E = a^* - a$$

between the exact value a and the approximate value a^* is called the *absolute error* of a^*, and the quotient

$$E_r = \frac{a^* - a}{a} \qquad (a \neq 0)$$

is called the *relative error* of a^*.

An error may arise by the process of *rounding off*, that is, by retaining only the more significant decimal digits of a number and rejecting the less significant beyond a certain point. This error is called a **rounding error**. Rounding errors can often be eliminated by carrying one, two, or even more extra figures, known as *guarding figures*, in the intermediate steps of calculation.

Another error may result from the fact that one or several of the formulas used in numerical work are obtained by replacing an infinite series by one of its partial sums. Such an error is called a **truncation error**. It occurs, for instance, in connection with numerical integration and differentiation.

Moreover, a computation may contain **mistakes** because of the fallibility of the human computer or the mechanical and electrical equipment used in the computation. While errors cannot be avoided, mistakes are avoidable in principle. **Checking** of numerical results is quite important and any numerical method should include checking procedures to confirm that the results contain no mistakes. This certainly applies to final results, but in a longer computation it is also advisable to check intermediate results.

Of course, *any computation should be arranged in a systematic tabular form and should contain all the intermediate results directly on the working sheet.* This will help to avoid mistakes and to locate and correct mistakes that have been made. *Numerical work should not be done on scraps of paper.* The numbers should be written in a neat and legible manner on paper that is full notebook page size.

Our last remark is concerned with approximation methods. If a problem cannot be solved by an exact method and we therefore use an approximation method, the immediate basic question is how much at most can the approximate result deviate from the (unknown) exact result. This requires the *estimation of error* involved in approximation methods. In many cases formulas for estimating the maximum possible error are known, and they should be used whenever such an approximation formula is applied. This is the only way to obtain a clear picture of the quality of an approximation and to eliminate a source of stupid mistakes which otherwise may, and actually do, occur in the mathematical part of engineering work. The determination of new and more effective methods of estimating errors is one of the important and interesting problems in modern applied mathematics. The most recent developments in this direction are influenced considerably by the increasing use of electronic computers. Older methods were brought into forms suitable for electronic computation, and new methods have been added; this development is still in progress.

0.7 *Solution of Equations*

A frequent task in engineering mathematics is the determination of the roots of an equation of the form

$$f(x) = 0 \qquad (x \text{ real}). \tag{1}$$

If $f(x)$ is a simple function, then there may be a formula which yields the exact values of the roots. For example, if

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

we have to solve a *quadratic equation*, the solutions being

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}). \tag{2}$$

However, if $f(x)$ is more complicated, there may not be a formula for the exact values of the roots and in such a case we may use an approximation method which yields approximate values of the roots. Let us explain an important method of this type.

Suppose we want to determine the real roots of the equation $f(x) = 0$ where f is a differentiable function. Then we may plot a graph of $f(x)$ in the usual way. Assuming that the equation has a real root, we may obtain from the graph a first rough approximation of this root; let x_1 be this approximation. We may then determine a second approximation x_2, namely, the point of intersection of the tangent to the curve of $f(x)$ at x_1 and the x-axis (Fig. 22). Since

$$\tan \alpha = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

by solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In the next step we compute

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

etc., and in general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 1, 2, \cdots) \tag{3}$$

This method of successive approximations is called **Newton's[8] method.**

Example 1. Given the equation

$$f(x) = x^3 + x - 1 = 0$$

we see that it has a root near $x_1 = 1$ (Fig. 23). From (3) we obtain

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} = \frac{2x_n^3 + 1}{3x_n^2 + 1} = \frac{N_n}{D_n}.$$

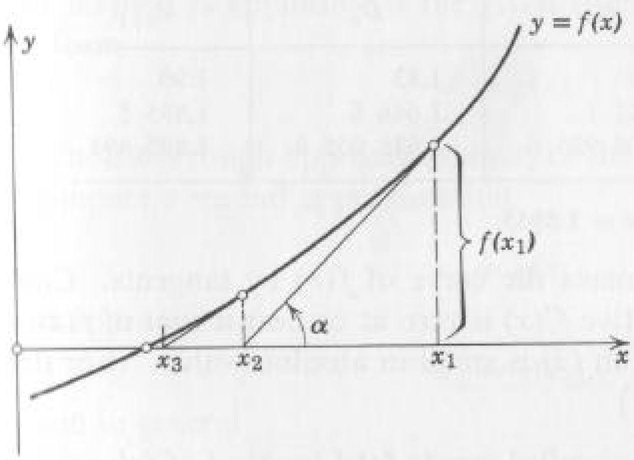


Fig. 22. Newton's method.

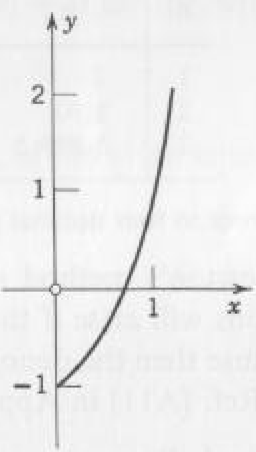


Fig. 23. Example 1.

[8] Sir ISAAC NEWTON (1642–1727), English physicist and mathematician. Newton and the German mathematician and philosopher, GOTTFRIED WILHELM LEIBNIZ (1646–1716), invented (independently) the differential and integral calculus. Newton discovered many basic physical laws and introduced the method of investigating physical problems by means of calculus. His work is of greatest importance to both mathematics and physics.

The computation may be arranged in tabular form:

n	x_n	x_n^2	x_n^3	N_n	D_n	x_{n+1}
1	1	1	1	3	4	0.75
2	0.75	0.562 5	0.421 875	1.843 750	2.687 500	0.686 047
3	0.686 047	0.470 660	0.322 895	1.645 790	2.411 980	0.682 340
4	0.682 340	0.465 588	0.317 689	1.635 378	2.396 764	0.682 328

The result seems to indicate that the desired root to four decimal places is $x = 0.6823$. (We mention that even all six decimals of x_5 are the correct first decimals of the root.)

Sometimes work may be saved by starting with a small number of digits and increasing it gradually. Our present computation may then become:

n	x_n	x_n^2	x_n^3	N_n	D_n	x_{n+1}
1	1	1	1	3	4	0.75
2	0.75	0.562 5	0.421 9	1.843 8	2.687 5	0.686
3	0.686	0.470 60	0.322 83	1.645 66	2.411 80	0.682 3
4	0.682 3	0.465 533	0.317 633	1.635 266	2.396 599	0.682 33

Example 2. Find the positive root of the equation

$$2 \sin x = x.$$

Here $f(x) = x - 2 \sin x = 0$, and from (3),

$$x_{n+1} = x_n - \frac{x_n - 2 \sin x_n}{1 - 2 \cos x_n} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2 \cos x_n} = \frac{N_n}{D_n}.$$

From the graph we conclude that the root is near $x_1 = 2$. Using tables of the sine and cosine functions, the computation yields

n	x_n	N_n	D_n	x_{n+1}
1	2	3.48	1.83	1.90
2	1.90	3.121 1	1.646 6	1.895 5
3	1.895 5	3.104 925 6	1.638 055 9	1.895 494

The root to four decimal places is $x = 1.8955$.

Newton's method approximates the curve of $f(x)$ by tangents. Complications will arise if the derivative $f'(x)$ is zero at or near a root of $f(x) = 0$, because then the denominator in (3) is small in absolute value. (For details see Ref. [A11] in Appendix 1.)

We shall now consider the so-called **regula falsi** (*method of false position*) which approximates that curve by a chord, as shown in Fig. 25. The chord intersects the x-axis at the point

(4) $$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

and this is an approximation for the root X_0 of the equation $f(x) = 0$.

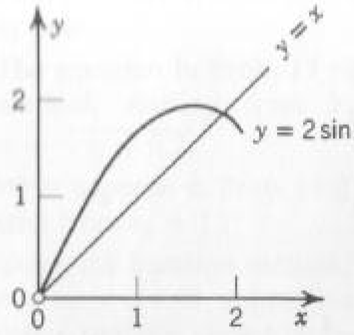


Fig. 24. Example 2.

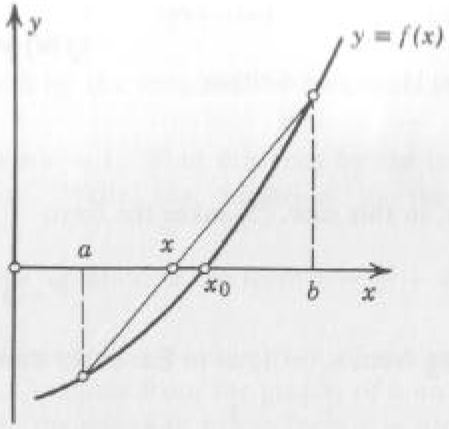


Fig. 25. Regula falsi.

Example 3. Find an approximate value of the root of the equation

$$f(x) = x^3 + x - 1 = 0$$

near $x = 1$ (cf. Ex. 1). We find that

$$f(0.5) = -0.375 \qquad \text{and} \qquad f(1) = 1.$$

Choosing $a = 0.5$ and $b = 1$, formula (4) yields

$$x_1 = \frac{0.5 \cdot 1 - 1 \cdot (-0.375)}{1 - (-0.375)} = 0.64.$$

Since $f(0.64) = -0.0979 < 0$, in the next step we may choose $a = 0.64$ and $b = 1$, and obtain from (4) the more accurate approximation $x_2 = 0.672$, and so on.

A third important method for determining the roots of an equation $f(x) = 0$ is the so-called **iteration method**.

This method is applicable if the given equation $f(x) = 0$ can be written in the form

$$x = g(x).$$

We choose a rough approximation x_1 of the root X_0 of the given equation and compute a second approximation

$$x_2 = g(x_1),$$

a third approximation

$$x_3 = g(x_2),$$

etc., and in general

$$x_{n+1} = g(x_n) \qquad (n = 1, 2, \cdots). \tag{5}$$

It can be shown that if

$$|g'(X_0)| < 1$$

and x_1 is sufficiently accurate (sufficiently close to X_0), then $x_n \to X_0$ as $n \to \infty$, that is, (5) will yield better and better approximations as n increases.

Example 4. The equation

$$f(x) = x^3 + x - 1 = 0$$

(cf. Ex. 1) may be written

$$x = \frac{1}{1 + x^2} \tag{6}$$

Hence, in this case, (5) takes the form

$$x_{n+1} = \frac{1}{1 + x_n^2}.$$

Starting from $x_1 = 1$, as in Ex. 1, we thus obtain

$$x_2 = \frac{1}{1 + 1^2} = \frac{1}{2} = 0.5, \qquad x_3 = \frac{1}{1 + (1/2)^2} = \frac{4}{5} = 0.8, \qquad x_4 = 0.61, \quad \text{etc.}$$

Note that the right-hand side of (6) is

$$g(x) = \frac{1}{1 + x^2}, \quad \text{and} \quad g'(x) = \frac{-2x}{(1 + x^2)^2}.$$

Now $X_0 \approx 0.7$, and $|g'(0.7)| = 1.4/1.49^2 < 1$, which shows that for x_1 sufficiently close to X_0 we have $x_n \to X_0$ as $n \to \infty$. Our numerical results seem to confirm this fact.

It is clear that a given equation $f(x) = 0$ may be written in the form $x = g(x)$ in many ways. In the present example we may write

$$x = 1 - x^3$$

but then

$$g(x) = 1 - x^3, \qquad |g'(x)| = 3x^2 > 1 \text{ when } x = X_0 \approx 0.7,$$

and the iteration will not work. The reader may try to start from $x_1 = 1$, $x_1 = 0.7$, $x_1 = 2$, and see what happens.

Problems

Find a root of the following equations; carry out three steps of Newton's method, starting from the given x_1.

1. $x^3 - 1.2x^2 + 2x - 2.4 = 0, x_1 = 1$
2. $x^3 - 1.2x^2 + 2x - 2.4 = 0, x_1 = 2$
3. $x^3 - 3.9x^2 + 0.9x + 5.8 = 0, x_1 = 3$
4. $x^3 - 3.9x^2 + 0.9x + 5.8 = 0, x_1 = 1$
5. $x^3 - 3.9x^2 + 4.79x - 1.881 = 0, x_1 = 1$
6. The roots of the equation in Prob. 5 are 0.9, 1.1, and 1.9. Although $x_1 = 1$ lies close to 0.9 and 1.1, Newton's method does not yield one of these roots. Why? Choose another x_1 such that the method yields approximations for the root 1.1.

Find all real roots of the following equations by Newton's method.

7. $x^4 - 0.1x^3 - 0.82x^2 - 0.1x - 1.82 = 0$
8. $x^3 - 4x^2 + 2x - 8 = 0$
9. $x^3 - 5x^2 + 6.64x - 1.92 = 0$
10. $\cos x = x$
11. $x + \ln x = 2$
12. $2x + \ln x = 1$

Find the real roots of the following equations by means of the regula falsi.

13. $x^4 = 2$
14. $x^4 = 2x$
15. $3 \sin x = 2x$

16. In Prob. 13 the approximate values of the positive root will always be somewhat smaller than the exact values of the root. Why?

17. Find the root of $x^5 = x + 0.2$ near $x = 0$ by the iteration method, starting from $x_1 = 0$.

18. The equation in Prob. 17 has a root near $x = 1$. Find this root by the iteration method, starting from $x_1 = 1$. *Hint:* Write the equation in the form $x = \sqrt[5]{x + 0.2}$.

19. What happens in Prob. 18 if you write the equation in the form $x = x^5 - 0.2$ and start from $x_1 = 1$?

20. Using the iteration method, show that the smallest positive root of the equation $x = \tan x$ is 4.49, approximately. *Hint:* Conclude from the graphs of x and $\tan x$ that a root lies close to $x_1 = 3\pi/2$; write the equation in the form $x = \text{arc tan } x$. (Why?)

0.8 *Approximate Integration*

If $f(x)$ is a given function and we can find a function $F(x)$ whose derivative is $f(x)$, then we may evaluate definite integrals of $f(x)$ by using the familiar formula

$$(1) \qquad \int_a^b f(x)\,dx = F(b) - F(a) \qquad [F'(x) = f(x)].$$

Tables of integrals[9] may be helpful for this purpose. However, in engineering applications there frequently occur integrals which cannot be evaluated by familiar methods known from calculus. These may be integrals of two types, namely, integrals such as

$$\int e^{-x^2}\,dx, \qquad \int \frac{\sin x}{x}\,dx, \qquad \int \cos(x^2)\,dx,$$

which cannot be represented in terms of finitely many elementary functions, or integrals whose integrands are empirical functions, given by a table of numerical values which are obtained from a physical experiment or in some other way. In many cases, integrals of the first type may be evaluated by one of the advanced methods (complex integration, use of power series, or asymptotic expansions), and in the case of an integral of the second type we may try to approximate the empirical function by a polynomial or some other elementary function.

Nevertheless, in various situations it will be preferable to apply one of the standard methods of numerical integration which we shall consider in the present section. These methods use the fact that the definite integral in (1) equals the area A of the shaded region R in Fig. 26 on p. 32.

To determine A we may cut R from cardboard, weight the piece, and divide the result by the weight of a square of this cardboard whose side is 1.

Another simple method is to draw R on a graph paper and count squares.

[9] Cf. for example, Ref. [A3] in Appendix 1.

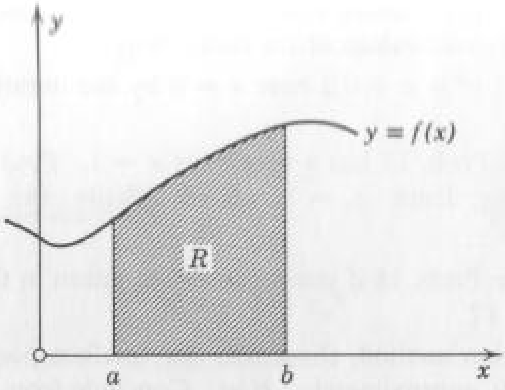


Fig. 26. Geometrical interpretation of a definite integral.

More accurate results are obtained by using a **planimeter**. So let us briefly discuss this instrumental method. Planimeters are inexpensive precision instruments which measure the area of any region R bounded by a closed curve C. There are several types of planimeters.[10] Figure 27 shows a so-called *polar planimeter*. This instrument has a rod, the tracing arm (I), on one end of which is mounted the tracing pin (II) which follows the curve C. The other end of the tracing arm is moved on a circular path by means of a second rod, the pole arm (III), which can be turned about the end point 0 (the pole) and is joined to the other end of the tracing arm by means of a hinge. An integrating wheel (IV), mounted on the tracing arm, measures the area of the region R when the pin traces once around the entire boundary curve C, starting from any point of C and returning to that point; the area is obtained as the difference of the initial and final reading on the scale of the integrating wheel. The accuracy can be increased by repeating this procedure m times, adding the m results, and dividing their sum by m.

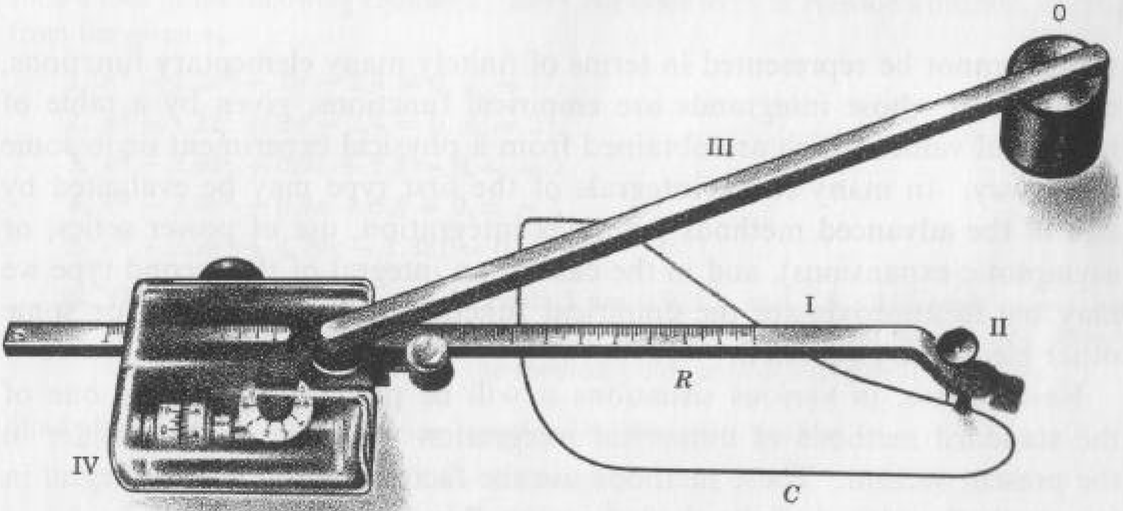


Fig. 27. Polar planimeter, made by A. Ott, Kempten, Germany, manufacturers of high precision instruments.

[10] These types and the corresponding mathematical theories are considered in Ref. [A15] in Appendix 1.

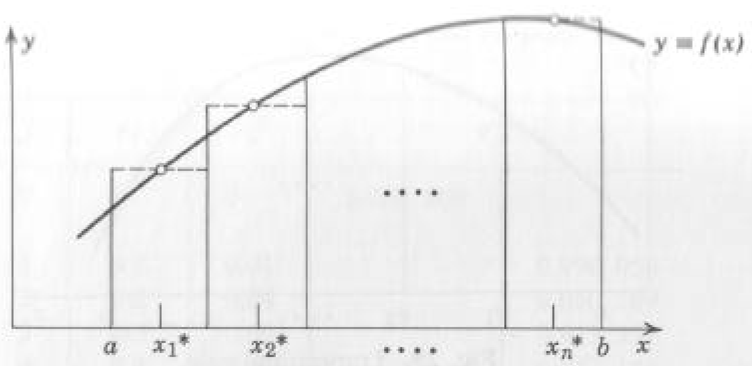


Fig. 28. Rectangular rule.

We shall now consider the simplest formulas of numerical integration, assuming that $b > a$ in (1).

We subdivide the interval of integration, whose length is $b - a$, into n equal parts of length

$$\Delta x = \frac{b-a}{n}.$$

Let $x_1^*, \cdots, x_n^*$ be the midpoints of these n intervals. Then the n rectangles in Fig. 28 have the areas $f(x_1^*)\,\Delta x, \cdots, f(x_n^*)\,\Delta x$. Therefore,

$$\textbf{(2)} \qquad \int_a^b f(x)\,dx \approx \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)].$$

This simple formula is called the **rectangular rule**.

Example 1. Use (2) to find an approximate value of ln 2 (= 0.693 15). We have

$$\ln 2 = \int_1^2 \frac{dx}{x}.$$

We choose $n = 5$ and $n = 10$ and arrange the work in tabular form.

j	x_j^*	$1/x_j^*$	j	x_j^*	$1/x_j^*$	j	x_j^*	$1/x_j^*$
1	1.1	0.909 091	1	1.05	0.952 381	6	1.55	0.645 161
2	1.3	0.769 231	2	1.15	0.869 565	7	1.65	0.606 061
3	1.5	0.666 667	3	1.25	0.800 000	8	1.75	0.571 429
4	1.7	0.588 235	4	1.35	0.740 741	9	1.85	0.540 541
5	1.9	0.526 316	5	1.45	0.689 655	10	1.95	0.512 821
Sum $S = 3.459\ 540$			Sum $S_1 = 4.052\ 342$			Sum $S_2 = 2.876\ 013$		
$\ln 2 \approx \frac{1}{5}S = 0.691\ 91$ Error: 0.001 24 (0.2%)			$\ln 2 \approx \frac{1}{10}(S_1 + S_2) = 0.692\ 84$ Error: 0.000 31 (0.05%)					
Computation for $n = 5$			Computation for $n = 10$					

We see that the result for $n = 10$ is more accurate than that for $n = 5$.

We shall now derive another integration formula. We subdivide the interval $a \leqq x \leqq b$ into n equal parts of length $\Delta x = (b - a)/n$, as before, and denote the end points of these subintervals by $a, x_1, x_2, \cdots x_{n-1}, b$.

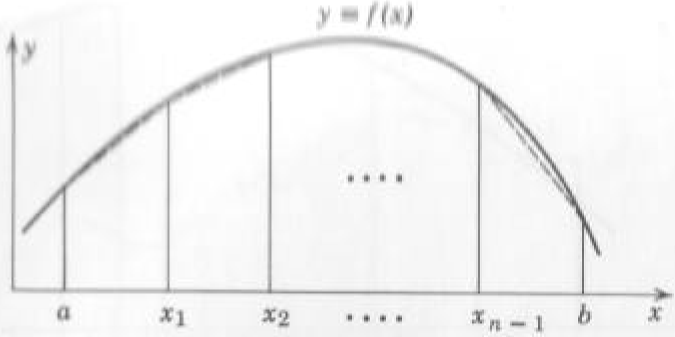


Fig. 29. Trapezoidal rule.

Then the n trapezoids in Fig. 29 have the areas

$$\tfrac{1}{2}[f(a) + f(x_1)]\,\Delta x, \qquad \tfrac{1}{2}[f(x_1) + f(x_2)]\,\Delta x, \cdots, \qquad \tfrac{1}{2}[f(x_{n-1}) + f(b)]\,\Delta x.$$

Thus

$$\text{(3)} \quad \int_a^b f(x)\,dx \approx \Delta x\left[\frac{f(a)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(b)}{2}\right]$$

where $\Delta x = (b - a)/n$. This formula is called the **trapezoidal rule.**

Let A be the exact value of the integral under consideration, and let A^* be the approximate value obtained by the trapezoidal rule (3). Then the difference

$$E_T = A^* - A$$

is called the *error* of A^*, and

$$\text{(4)} \qquad A = \int_a^b f(x)\,dx = A^* - E_T.$$

Clearly, if $f(x)$ is a *linear* function, its graph is a straight line, and the trapezoidal rule will yield the exact value of the integral, that is, $E_T = 0$. Now in this case, f' is constant and f'' is zero for all x. Hence the error results from the fact that in general $f'' \neq 0$, and its seems to be plausible that its magnitude will depend on the values of f'' in the interval of integration and also on the number n of subintervals in (3). In fact, it can be shown that E_T always lies in the interval

$$\text{(5)} \qquad KM^* \leqq E_T \leqq KM$$

where M^* and M are the smallest and largest values of the second derivative of f in the interval of integration, and

$$K = \frac{(b - a)^3}{12n^2}.$$

Example 2. Compute the integral

$$I = \int_0^1 e^{-x^2}\,dx$$

by means of the trapezoidal rule (3), taking $n = 10$, and estimate the error.

j	x_j	x_j^2	$e^{-x_j^2}$	
0	0	0	1.000 000	
1	0.1	0.01		0.990 050
2	0.2	0.04		0.960 789
3	0.3	0.09		0.913 931
4	0.4	0.16		0.852 144
5	0.5	0.25		0.778 801
6	0.6	0.36		0.697 676
7	0.7	0.49		0.612 626
8	0.8	0.64		0.527 292
9	0.9	0.81		0.444 858
10	1.0	1.00	0.367 879	
		Sums	1.367 879	6.778 167

We have $\Delta x = 0.1$, and

$$I \approx 0.1\left[\frac{1.363\ 879}{2} + 6.778\ 176\right] = 0.746\ 211.$$

Estimate of error.

$$f''(x) = 2(2x^2 - 1)e^{-x^2}, \qquad M^* = f''(0) = -2, \qquad M = f''(1) = 0.735\ 758.$$

From this and (5),

$$-2K \leqq E_T \leqq 0.735\ 758\ K$$

where $K = 1/1200$. Thus

$$-0.001\ 667 \leqq E_T \leqq 0.000\ 614.$$

From this and (4) it follows that the exact value of I must lie between

$$0.746\ 211 - 0.000\ 614 = 0.745\ 597$$

and

$$0.746\ 211 + 0.001\ 667 = 0.747\ 878.$$

(The exact value is $I = 0.746\ 824 \cdots$.)

The rectangular rule is obtained by approximating the integrand $f(x)$ by a step function (piecewise constant function). The trapezoidal rule results by approximating $f(x)$ by linear functions. We may expect to obtain a more accurate integration formula by approximating the curve of $f(x)$ by portions of parabolas.

For this purpose we subdivide the interval of integration $a \leqq x \leqq b$ into an *even* number of equal subintervals, say, into $2n$ subintervals of length $\Delta x = (b - a)/2n$, with end points $x_0(= a), x_1, \cdots, x_{2n-1}, x_{2n}(= b)$. In the first two intervals we approximate the curve of $f(x)$ by the parabola of the form $\alpha x^2 + \beta x + \gamma$ passing through the points A_0, A_1, A_2 of that curve

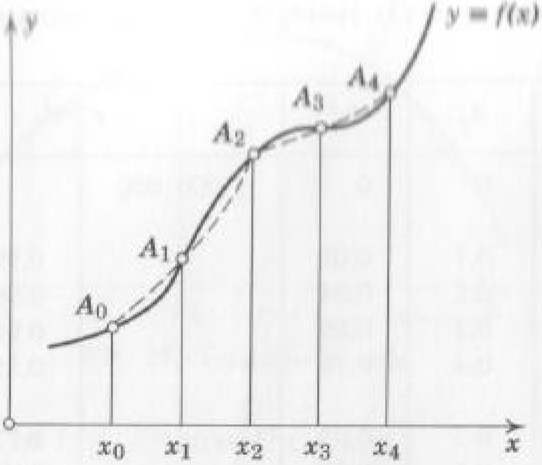


Fig. 30. Simpson's rule.

(Fig. 30). In the next two intervals we approximate that curve by another such parabola through A_2, A_3, A_4, and so on. Proceeding in this fashion, we obtain a curve consisting of n portions of parabolas, and the area under that curve is an approximation for the area under the curve of $f(x)$ between a and b. It can be shown that the integration formula thus obtained is

$$\textbf{(6)} \quad \int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})]$$

where $\Delta x = (b - a)/2n$ (cf. Ref. [A13] in Appendix 1). This important formula for approximate integration is called **Simpson's rule.**

If we write (6) in the form

$$(7) \qquad \int_a^b f(x)\,dx = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + \cdots + f(x_{2n})] - E_S,$$

then E_S is the error of the approximation. It can be shown that E_S lies in the interval

$$(8) \qquad CM_4^* \leqq E_S \leqq CM_4$$

where M_4^* and M_4 are the smallest and the largest value of the fourth derivative of f in the interval of integration, and

$$C = \frac{(b-a)^5}{180\,(2n)^4}\,.$$

From (8) we see that Simpson's rule yields the exact result not only for polynomials of the second degree but even for polynomials of the third degree.

Example 3. Evaluate $I = \int_0^1 e^{-x^2}\,dx$ by Simpson's rule with $2n = 10$ and estimate the error.

j	x_j	x_j^2	$e^{-x_j^2}$		
0	0	0	1.000 000		
1	0.1	0.01		0.990 050	
2	0.2	0.04			0.960 789
3	0.3	0.09		0.913 931	
4	0.4	0.16			0.852 144
5	0.5	0.25		0.778 801	
6	0.6	0.36			0.697 676
7	0.7	0.49		0.612 626	
8	0.8	0.64			0.527 292
9	0.9	0.81		0.444 858	
10	1.0	1.00	0.367 879		
Sums			1.367 879	3.740 266	3.037 901

We find, since $\Delta x = 0.1$,

$$I \approx \frac{0.1}{3}(1.367\ 879 + 4 \cdot 3.740\ 266 + 2 \cdot 3.037\ 901) = 0.746\ 825.$$

Estimate of error. The fourth derivative of the integrand is

$$f^{\text{IV}}(x) = 4(4x^4 - 12x^2 + 3)e^{-x^2}.$$

By considering the derivative of f^{IV} we find that the smallest value of f^{IV} in the interval of integration occurs at $x = x^* = 2.5 + 0.5\sqrt{10}$ and the largest value occurs at $x = 0$. Computing the corresponding values of f^{IV} we obtain in (8)

$$M_4^* = f^{\text{IV}}(x^*) = -7.359 \cdots \quad \text{and} \quad M_4 = f^{\text{IV}}(0) = 12.$$

Furthermore, since $2n = 10$ and $b - a = 1$, in (8),

$$C = 1/1\ 800\ 000 = 0.000\ 000\ 55 \cdots.$$

Therefore

$$-0.000\ 004 \cdots \leqq E_S \leqq 0.000\ 006 \cdots.$$

This shows that the first four decimals of the above approximation are correct and, by (7),

$$0.746\ 818 < I < 0.746\ 830.$$

The exact value to six decimal places is $I = 0.746\ 824$, and we see that even five decimals or our result are correct. Our present result is much better than that obtained in Ex. 2 by the trapezoidal rule, while the amount of work is almost the same in both cases.

Problems

Review some integration formulas and methods by integrating

1. $\int \frac{dx}{a^2 + x^2}$
2. $\int \frac{dx}{\sqrt{a^2 - x^2}}$
3. $\int e^{ax} \cos bx\,dx$
4. $\int e^{ax} \sin bx\,dx$
5. $\int \ln x\,dx$
6. $\int \cos^2 x\,dx$

7. $\int \sin^2 \omega x\, dx$ 8. $\int \tan x\, dx$ 9. $\int \frac{dx}{x^2(x^2+1)^2}$

10. Compute $\int_0^1 x^3\, dx$ by the rectangular rule (2) with $n = 5$. What is the error?

11. Compute the integral in Prob. 10 by the trapezoidal rule (3) with $n = 5$. What error bounds are obtained from (5)? What is the actual error of the result? Why is this result larger than the exact value?

12. Compute the integral in Ex. 2 by using (2) with $n = 5$.

Using the column of $\sin x$ in Table 1 in Sec. 0.1 evaluate $\int_0^1 \frac{\sin x}{x}\, dx$:

13. By the rectangular rule (2) with $n = 5$.
14. By the trapezoidal rule (3) with $n = 5$.
15. By (3) with $n = 10$.
16. By Simpson's rule with $2n = 2$.
17. By Simpson's rule with $2n = 10$.

18. Evaluate $\int_0^1 x^5\, dx$ by Simpson's rule with $2n = 10$. What error bounds are obtained from (8)? What is the actual error of the result?

19. Find an approximate value of $\ln 2 = \int_1^2 \frac{dx}{x}$ by Simpson's rule with $2n = 4$. Estimate the error by (8).

20. If subintervals of different lengths $\Delta_1 x, \Delta_2 x, \cdots, \Delta_n x$ are chosen, show that the trapezoidal rule assumes the form

$$\int_a^b f(x)\, dx \approx \tfrac{1}{2}[f(a) + f(x_1)]\, \Delta_1 x + \cdots + \tfrac{1}{2}[f(x_{n-1}) + f(b)]\, \Delta_n x.$$