etc. By differentiating f_x, f_y, f_z again in this fashion we obtain the second partial derivatives of f, etc.

Example 3. Let $f(x, y, z) = x^2 + y^2 + z^2 + xye^z$. Then

$$\begin{aligned} f_x &= 2x + ye^z, & f_y &= 2y + xe^z, & f_z &= 2z + xye^z, \\ f_{xx} &= 2, & f_{xy} &= f_{yx} = e^z, & f_{xz} &= f_{zx} = ye^z, \\ f_{yy} &= 2, & f_{yz} &= f_{zy} = xe^z, & f_{zz} &= 2 + xye^z. \end{aligned}$$

Problems

Find the first partial derivatives of the following functions.

1. $f(x, y) = \sqrt{x^2 + y^2}$ **2.** $f(r, \theta) = r \cos \theta$ **3.** $f(x, y) = \arctan \dfrac{y}{x}$

4. $f(x, y) = e^{xy}$ **5.** $f(x, y) = \dfrac{1}{\sqrt{(x - a)^2 + (y - b)^2}}$ (a, b constant)

6. $f(r, h) = \dfrac{\pi}{3} r^2 h$ **7.** $f(R_1, R_2) = \dfrac{R_1 + R_2}{R_1 R_2}$

8. Find $f_x + g_y$ where $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$.

9. Find $f_x + g_y$ where $f(x, y) = \ln (x^2 + y^2)$ and $g(x, y) = 2 \arctan \dfrac{y}{x}$.

Find $f_{xx} + f_{yy}$ where

10. $f(x, y) = x^2 - y^2$ **11.** $f(x, y) = e^x \cos y$ **12.** $f(x, y) = \sin x \cosh y$

Sketch the surfaces corresponding to the following functions.

13. $z = x^2 + y^2$ **14.** $z = \ln (x^2 + y^2)$ **15.** $z = e^{xy}$

16. The curves $z = f(x, y) = const$ are called **level curves** of $f(x, y)$. Draw the level curves of the functions in Probs. 13–15.

Find the first partial derivatives of the following functions at the given points.

17. $f(x, y) = \sqrt{1 - x^2 - y^2}$, at $(0, 0)$ **18.** $f(x, y) = (x^2 + y^2)^2$, at $(1, 2)$

Find the first and second partial derivatives of the following functions.

19. $f(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$ **20.** $f(x, y, z) = \dfrac{x + y + z}{xyz}$

0.3 Second and Third Order Determinants

Consider the system

$$(1) \qquad \begin{aligned} a_1 x + b_1 y &= k_1 \\ a_2 x + b_2 y &= k_2 \end{aligned}$$

consisting of two linear equations in the unknowns x and y. To solve this system we may multiply the first equation by b_2, the second by $-b_1$, and add, finding

$$(a_1 b_2 - a_2 b_1) x = k_1 b_2 - k_2 b_1.$$

Then we multiply the first equation of (1) by $-a_2$, the second by a_1, and add again, finding

$$(a_1b_2 - a_2b_1)y = a_1k_2 - a_2k_1.$$

If $a_1b_2 - a_2b_1$ is not zero, we may divide and obtain the desired result

(2) $$x = \frac{k_1b_2 - k_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1k_2 - a_2k_1}{a_1b_2 - a_2b_1}.$$

The expression in the denominators is written in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

and is called a **determinant of the second order**; thus

(3) $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The four numbers a_1, b_1, a_2, b_2 are called the **elements** of the determinant. The elements in a horizontal line are said to form a **row** and the elements in a vertical line are said to form a **column** of the determinant.

We may now write the solution (2) of the system (1) in the form

(4) $$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D} \qquad (D \neq 0)$$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}.$$

The formula (4) is called **Cramer's rule**.[2] Note that D_1 is obtained by replacing the first column of D by the column with elements k_1, k_2, and D_2 is obtained by replacing the last column of D by that column.

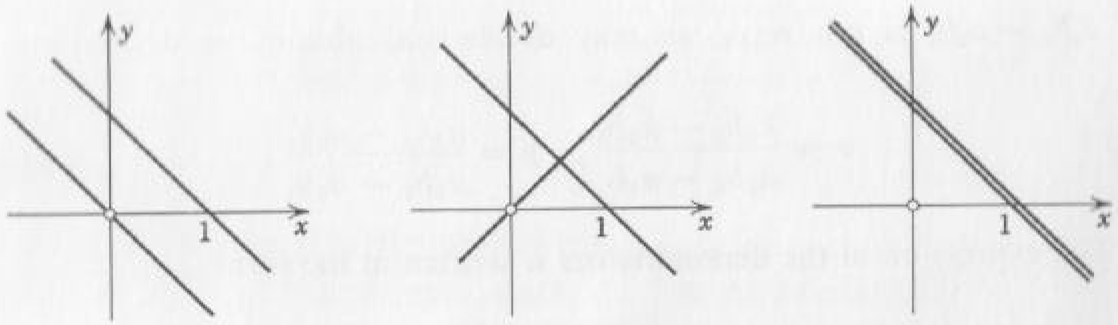
Each equation of the system (1) represents a straight line in the xy-plane, and a pair of numbers (x, y) is a solution of (1) if, and only if, the point P with coordinates x, y lies on both lines. Hence there are three possible cases:

(a) No solution if the lines are parallel.
(b) Precisely one solution if they intersect.
(c) Infinitely many solutions if they coincide.

[2] GABRIEL CRAMER (1704–1752), Italian mathematician, also known by his contributions to the theory of curves.

Example 1.

Case (a)	Case (b)	Case (c)
$x + y = 1$	$x + y = 1$	$x + y = 1$
$x + y = 0$	$x - y = 0$	$2x + 2y = 2$



If both k_1 and k_2 are zero, the system is said to be **homogeneous**; otherwise it is said to be **nonhomogeneous**.

If the system is homogeneous, Case (a) cannot occur, because then the lines represented by the equations pass through the origin, and the system has at least the *trivial solution* $x = 0$, $y = 0$.

The homogeneous system will have further solutions if, and only if, those two lines coincide, and then each point on the line is a solution. This happens if, and only if, $D = 0$, as the reader may show.

If the system is nonhomogeneous and $D \neq 0$, it has precisely one solution, which is obtained from (4).

A system of three linear equations in three unknowns x, y, z

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned} \tag{5}$$

may be considered in a similar fashion. To obtain an equation involving only x the equations are multiplied, respectively, by

$$b_2c_3 - b_3c_2, \qquad -(b_1c_3 - b_3c_1), \qquad b_1c_2 - b_2c_1.$$

We see that these expressions may be written as second-order determinants:

$$M_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \qquad -M_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad M_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Adding the resulting equations, we obtain

$$(a_1M_1 - a_2M_2 + a_3M_3)x = k_1M_1 - k_2M_2 + k_3M_3. \tag{6}$$

Two further equations containing only y and z, respectively, may be obtained in a similar manner.

To simplify our notation we now define a **determinant of the third order** by the equation

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \tag{7}$$

We see that

$$D = a_1M_1 - a_2M_2 + a_3M_3,$$

the coefficient of x in (6), and if we write the second-order determinants in (7) at length, we obtain

$$D = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

or

(8) $$D = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Obviously the determinant on the right-hand side of (7) which is multiplied by a_i, $i = 1, 2$, or 3, is obtained from D by omitting the first column and the ith row of D.

We see that (6) may now be written

$$Dx = D_1$$

where

$$D_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}.$$

The aforementioned equation containing only y may be written

$$Dy = D_2$$

where

$$D_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix},$$

and the equation containing only z may be written

$$Dz = D_3$$

where

$$D_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}.$$

Note that the elements of D are arranged in the same order as they occur as coefficients in the equations of (5), and D_j, $j = 1, 2$, or 3, is obtained from D by replacing the jth column by the column with elements k_1, k_2, k_3, the expressions on the right sides of the equations of (5).

It follows that if $D \neq 0$, then system (5) has the unique solution

(9) $$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$ **(Cramer's rule).**

Each equation of system (5) represents a plane in space, and a triple of numbers (x, y, z) is a solution of (5) if, and only if, the point P with coordinates x, y, z is a common point of the three planes. As before we have three possible cases:

(a) No solution if two (or all three) planes are parallel, if two of them coincide and the third is parallel, or if the intersections of each pair of them are three parallel lines.

(b) Precisely one solution, given by (9), if the planes have just one point in common.

(c) Infinitely many solutions if they have a line in common or coincide.

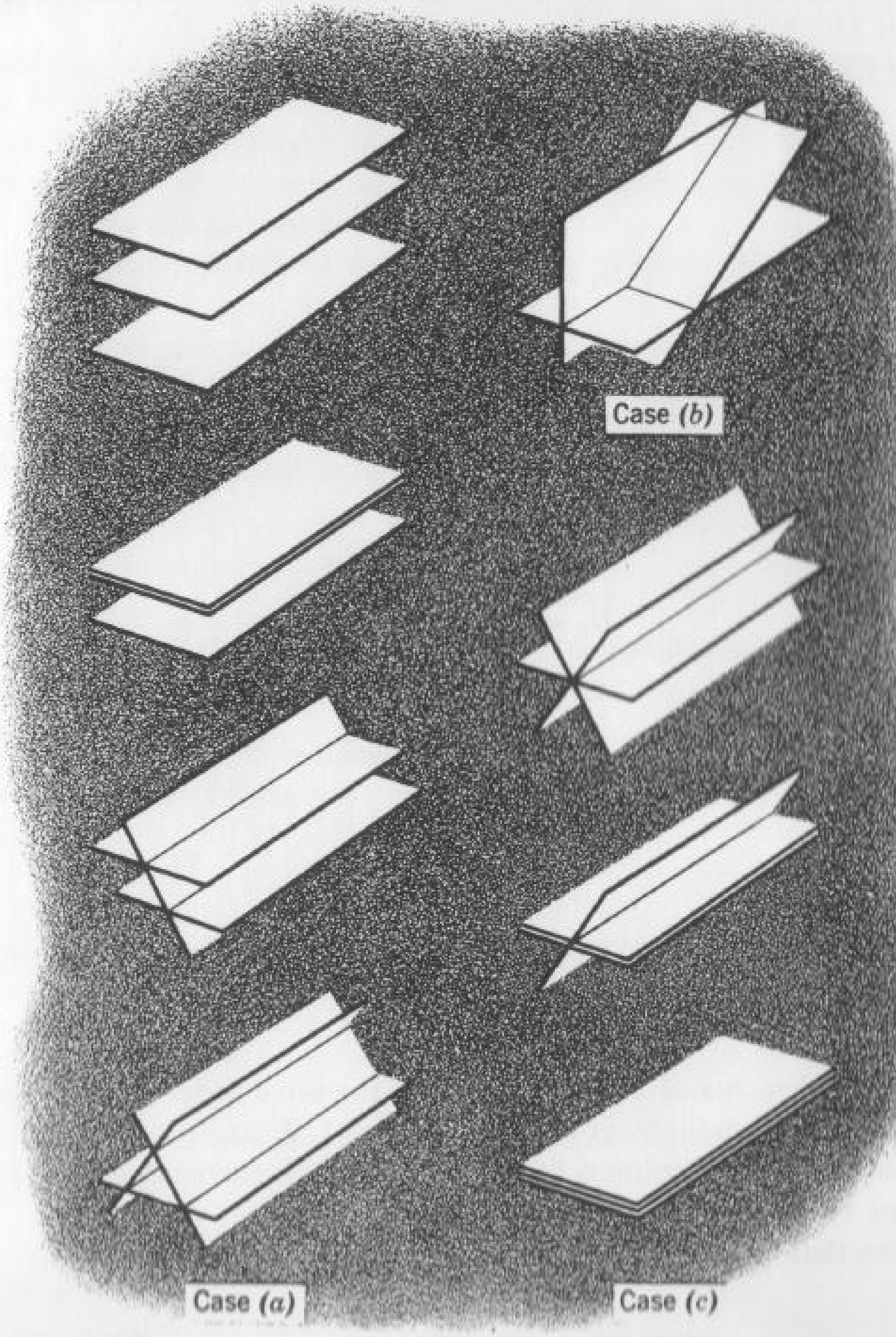


Fig. 14. Geometrical interpretation of three linear equations in three unknowns.

If (5) *is homogeneous, i.e.,* $k_1 = k_2 = k_3 = 0$, *it has at least the trivial solution* $x = y = z = 0$, *and nontrivial solutions exist if, and only if,* $D = 0$. *If* (5) *is nonhomogeneous and* $D \neq 0$, *it has precisely one solution which is obtained from* (9).

We shall now list the most important properties of our determinants; the corresponding proofs follow from (7) by direct calculation.[3]

(A) *The value of a determinant is not altered if its rows are written as columns in the same order,*

$$(10)\qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(B) *If any two rows (or two columns) of a determinant are interchanged, the value of the determinant is multiplied by* -1. Example:

$$(11)\qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The second-order determinant obtained from

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by deleting one row and one column is called the **minor** of the element which belongs to the deleted row and column. Example: The minors of a_2 and b_2 in D are

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix},$$

respectively, etc.

The **cofactor** of the element of D in the ith row and the kth column is defined as $(-1)^{i+k}$ times the minor of that element. Example: The cofactors of a_2 and b_2 are

$$-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix},$$

respectively. Furthermore we see that we may write (7) in the form

$$D = a_1C_1 + a_2C_2 + a_3C_3,$$

[3] Determinants of arbitrary order n will be defined in Sec. 7.3, and we shall see that they have quite similar properties.

where C_i is the cofactor of a_i in D. From this and the properties **(A)** and **(B)** we obtain the following property.

(C) *The determinant D may be developed by any row or column, that is, it may be written as the sum of the three elements of any row (or column) each multiplied by its cofactor.* For example, the development of D by its second row is

$$D = -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

From **(C)** we obtain

(D) *A factor of the elements of any row (or column) can be placed before the determinant.* Example:

$$\begin{vmatrix} 4 & 6 & 1 \\ 3 & -9 & 2 \\ -1 & 12 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 2\cdot 3 & 1 \\ 3 & -3\cdot 3 & 2 \\ -1 & 4\cdot 3 & 5 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 & 1 \\ 3 & -3 & 2 \\ -1 & 4 & 5 \end{vmatrix}.$$

From the properties **(B)** and **(D)** we may draw the following conclusion.

(E) *If corresponding elements of two rows (or columns) of a determinant are proportional, the value of the determinant is zero.*

The following property is basic for simplifying determinants to be evaluated.

(F) *The value of a determinant remains unaltered if the elements of one row (or column) are altered by adding to them any constant multiple of the corresponding elements in any other row (or column).* Example:

$$\begin{vmatrix} -6 & 21 & -30 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix} = \begin{vmatrix} -6+1\cdot 7 & 21-3\cdot 7 & -30+5\cdot 7 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 5 \\ 1 & -3 & 5 \\ 2 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 5 \\ 0 & -3 & 0 \\ 2 & 7 & -4 \end{vmatrix} = -3\begin{vmatrix} 1 & 5 \\ 2 & -4 \end{vmatrix} = 42.$$

(G) *If each element of a row (or column) of a determinant is expressed as a binomial, the determinant can be written as the sum of two determinants.* Example:

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

By applying the product rule of differentiation we obtain the following property.

(11) ***If the elements of a determinant are differentiable functions of a variable, the derivative of the determinant may be written as a sum of three determinants,***

$$\frac{d}{dx}\begin{vmatrix} f & g & h \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} f' & g' & h' \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p' & q' & r' \\ u & v & w \end{vmatrix} + \begin{vmatrix} f & g & h \\ p & q & r \\ u' & v' & w' \end{vmatrix},$$

where primes denote derivatives with respect to x.

Determinants of higher order and more general systems of linear equations will be considered in Chap. 7.

Problems

Evaluate

1. $\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 3 \\ 5 & 7 \end{vmatrix}$ **3.** $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

4. $\begin{vmatrix} 4.6 & -4.1 \\ -2.0 & 0.2 \end{vmatrix}$ **5.** $\begin{vmatrix} 4 & 12 \\ 40 & 20 \end{vmatrix}$ **6.** $\begin{vmatrix} \sqrt{3} & -2 \\ 0.5 & \sqrt{27} \end{vmatrix}$

7. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ **8.** $\begin{vmatrix} -4 & 18 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{vmatrix}$ **9.** $\begin{vmatrix} 9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

10. $\begin{vmatrix} 6 & 13 & -2 \\ 5 & 37 & -5 \\ 1 & 13 & -2 \end{vmatrix}$ **11.** $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ **12.** $\begin{vmatrix} 17 & 4 & 6 \\ 8 & 0 & 5 \\ 4 & 0 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}$ **14.** $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ **15.** $\begin{vmatrix} a+b & b & 0 \\ b & b+c & c \\ 0 & c & c+d \end{vmatrix}$

Solve the following systems of equations.

16. $17x + 4y = -24$, $x - 3y = 18$ **17.** $3x - y = 1$, $x + 3y = 7$ **18.** $3x - 4y = 14$, $-x + 3y = -8$

19. Plot the straight lines represented by the equations in Prob. 17 and determine their point of intersection.

20. Show that the equation

$$\begin{vmatrix} y & x^2 & x \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

represents a parabola passing through the points (0, 0), (−1, 1), and (1, 2).