Second Edition

ADVANCED ENGINEERING MATHEMATICS

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INTRODUCTION

Review of Some Topics from Algebra and Calculus

This introductory chapter includes some topics which are usually covered in elementary algebra and calculus. We shall refer to sections of this chapter whenever we need some of these topics as a prerequisite for our further consideration.

References: Appendix 1, Ref. [A13].
Answers to problems: Appendix 2.

0.1 *Elementary Functions*

In this section we shall present a collection of some basic formulas for reference.

Figure 1 shows the graph of the **exponential function** e^x, where

$$e = 2.7\,1828\,1828\,45\,90\,45\,2\cdots.$$

Basic identities are

$$\text{(1)}\qquad e^x e^y = e^{x+y}, \qquad e^x/e^y = e^{x-y}, \qquad (e^x)^y = e^{xy}.$$

The inverse of e^x is the **natural logarithm** $\ln x$ (Fig. 2). It satisfies the identities

$$\text{(2)}\qquad \ln(xy) = \ln x + \ln y, \quad \ln\frac{x}{y} = \ln x - \ln y, \qquad \ln(x^a) = a\ln x.$$

Furthermore,

$$e^{\ln x} = x, \qquad e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}.$$

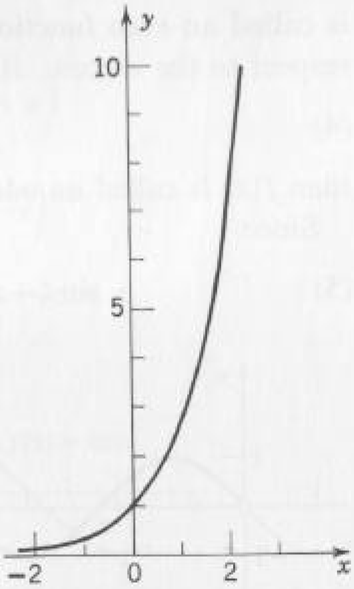

Fig. 1. Exponential function e^x.

The inverse of the exponential function 10^x is the *logarithm of base* 10 which is denoted by

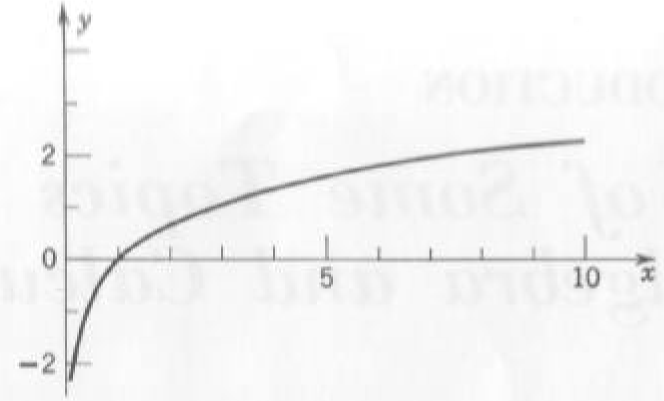

Fig. 2. Natural logarithm $\ln x$.

$\log_{10} x$ or simply by $\log x$. We have

$$\log x = M \ln x$$

where

$$M = \log e = 0.43429\ 44819\ 03251\ 82765$$

and conversely

$$\ln x = \frac{1}{M} \log x$$

where

$$\frac{1}{M} = \ln 10 = 2.30258\ 50929\ 94045\ 68402.$$

The **sine** and **cosine functions** $\sin x$ and $\cos x$ are defined in trigonometry for all values of x. *Throughout calculus, angles are measured in radians* so that both functions have the period 2π.

A function $w = f(x)$ which is defined for all x and has the property

(3) $$f(-x) = f(x) \qquad \text{for all } x$$

is called an **even function**. The graph of such a function is symmetric with respect to the w-axis. If $f(x)$ is defined for all x and

(4) $$f(-x) = -f(x) \qquad \text{for all } x,$$

then $f(x)$ is called an **odd function**. These are two quite important concepts.

Since

(5) $$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x$$

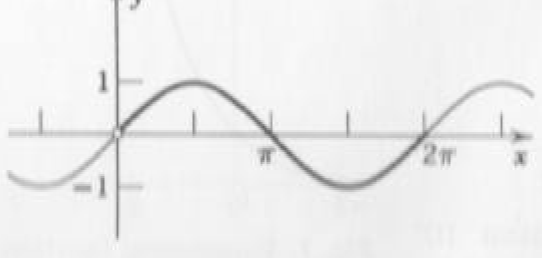

Fig. 3. $\sin x$.

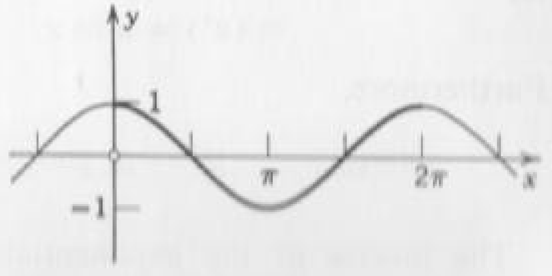

Fig. 4. $\cos x$.

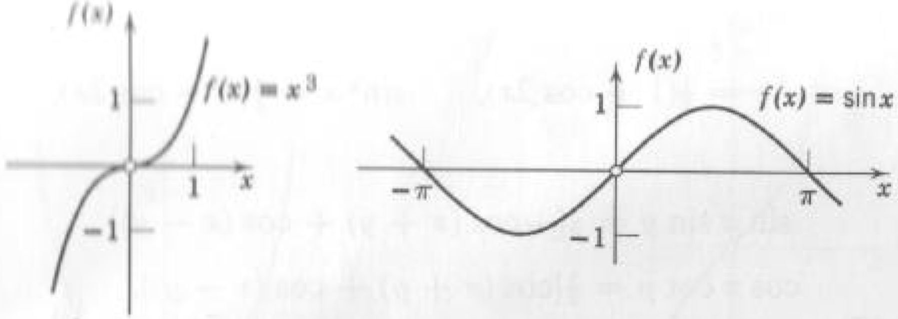

Fig. 5. Odd functions.

sin x is odd while cos x is even. The exponential function e^x is neither odd nor even.

The functions sin x and cos x are related by the identity

(6) $$\sin^2 x + \cos^2 x = 1.$$

The addition formulas of the sine function are

(7) $$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y. \end{aligned}$$

In particular,

(7*) $$\sin 2x = 2 \sin x \cos x.$$

The addition formulas of the cosine function are

(8) $$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y. \end{aligned}$$

In particular,

(8*) $$\cos 2x = \cos^2 x - \sin^2 x.$$

From (7) and (8),

(9) $$\begin{aligned} \sin x &= \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) \\ \cos x &= \sin\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right) \end{aligned}$$

$$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos x.$$

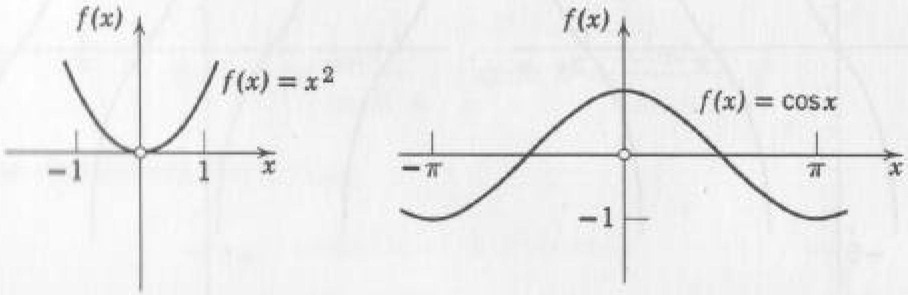

Fig. 6. Even functions.

From (8*) and (6),

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x). \tag{10}$$

From (8),

$$\begin{aligned} \sin x \sin y &= \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)] \\ \cos x \cos y &= \tfrac{1}{2}[\cos(x+y) + \cos(x-y)], \end{aligned} \tag{11a}$$

and from (7),

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]. \tag{11b}$$

From this, by setting $x + y = u$ and $x - y = v$,

$$\begin{aligned} \sin u + \sin v &= 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2} \\ \cos u + \cos v &= 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2} \\ \cos v - \cos u &= 2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}. \end{aligned} \tag{12}$$

The other trigonometric functions are defined by the identities

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}. \tag{13}$$

The addition formulas for the tangent are

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \quad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \tag{14}$$

In applications it is sometimes required to write $A \cos x + B \sin x$ where A and B are given constants, in the form $C \cos(x - \delta)$ where C and δ are

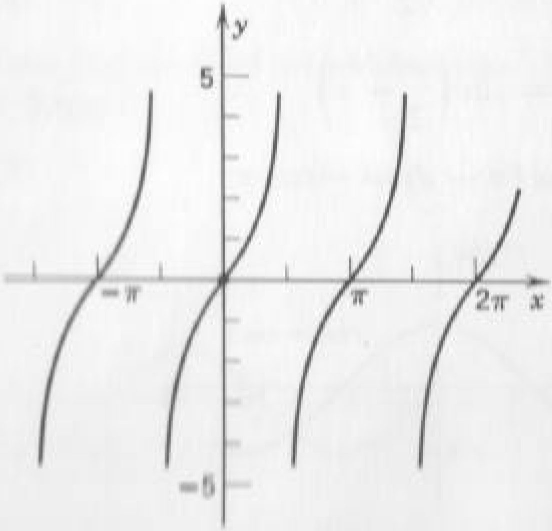

Fig. 7. tan x.

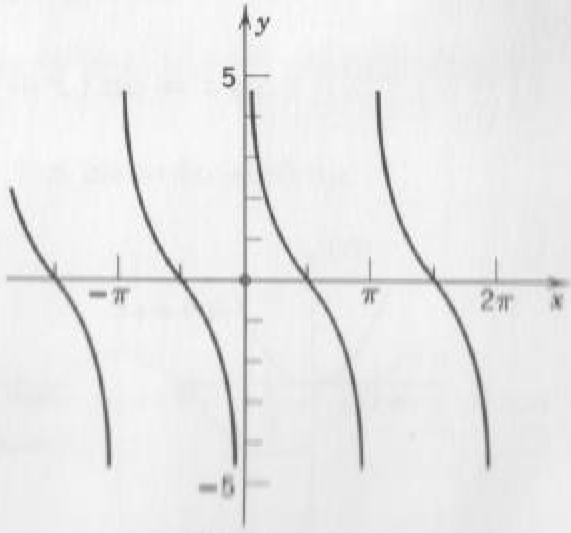

Fig. 8. cot x.

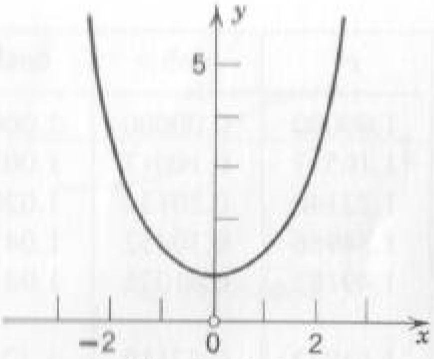

Fig. 9. Cosh x.

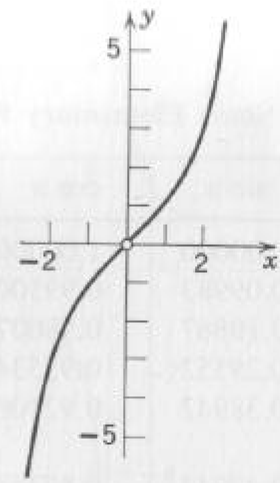

Fig. 10. Sinh x.

constants. From (8) we obtain

$$C\cos(x-\delta) = C\cos\delta\cos x + C\sin\delta\sin x$$

and this is equal to $A\cos x + B\sin x$, if $C\cos\delta = A$ and $C\sin\delta = B$. Using (6), we thus obtain

(15a) $$A\cos x + B\sin x = \sqrt{A^2+B^2}\cos(x-\delta)$$

where

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{B}{A}.$$

Similarly,

(15b) $$A\cos x + B\sin x = \sqrt{A^2+B^2}\sin(x\pm\delta)$$

where

$$\tan\delta = \frac{\sin\delta}{\cos\delta} = \pm\frac{A}{B}.$$

The **hyperbolic cosine** and **sine functions** are defined by the identities

(16) $$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

$\cosh x$ is even, while $\sinh x$ is odd. The other hyperbolic functions are defined by the identities

(17) $$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}.$$

From the definitions we obtain

(18) $$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x}$$

Table 1. Some Elementary Functions

x	$\sin x$	$\cos x$	$\tan x$	e^x	$\sinh x$	$\cosh x$
0	0.00000	1.00000	0.00000	1.00000	0.00000	1.00000
0.1	0.09983	0.99500	0.10033	1.10517	0.10017	1.00500
0.2	0.19867	0.98007	0.20271	1.22140	0.20134	1.02007
0.3	0.29552	0.95534	0.30934	1.34986	0.30452	1.04534
0.4	0.38942	0.92106	0.42279	1.49182	0.41075	1.08107
0.5	0.47943	0.87758	0.54630	1.64872	0.52110	1.12763
0.6	0.56464	0.82534	0.68414	1.82212	0.63665	1.18547
0.7	0.64422	0.76484	0.84229	2.01375	0.75858	1.25517
0.8	0.71736	0.69671	1.02964	2.22554	0.88811	1.33743
0.9	0.78333	0.62161	1.26016	2.45960	1.02652	1.43309
1.0	0.84147	0.54030	1.55741	2.71828	1.17520	1.54308
1.1	0.89121	0.45360	1.96476	3.00417	1.33565	1.66852
1.2	0.93204	0.36236	2.57215	3.32012	1.50946	1.81066
1.3	0.96356	0.26750	3.60210	3.66930	1.69838	1.97091
1.4	0.98545	0.16997	5.79788	4.05520	1.90430	2.15090
1.5	0.99750	0.07074	14.10142	4.48169	2.12928	2.35241
1.6	0.99957	−0.02920	−34.23253	4.95303	2.37557	2.57746
1.7	0.99166	−0.12884	−7.69660	5.47395	2.64563	2.82832
1.8	0.97385	−0.22720	−4.28626	6.04965	2.94217	3.10747
1.9	0.94630	−0.32329	−2.92710	6.68589	3.26816	3.41773
2.0	0.90930	−0.41615	−2.18504	7.38906	3.62686	3.76220

x	$\ln x$	x	$\ln x$	x	$\ln x$	x	$\ln x$
1.0	0.00000	2.0	0.69315	3.0	1.09861	5	1.60944
1.1	0.09531	2.1	0.74194	3.1	1.13140	7	1.94591
1.2	0.18232	2.2	0.78846	3.2	1.16315	11	2.39790
1.3	0.26236	2.3	0.83291	3.3	1.19392	13	2.56495
1.4	0.33647	2.4	0.87547	3.4	1.22378	17	2.83321
1.5	0.40547	2.5	0.91629	3.5	1.25276	19	2.94444
1.6	0.47000	2.6	0.95551	3.6	1.28093	23	3.13549
1.7	0.53063	2.7	0.99325	3.7	1.30833	29	3.36730
1.8	0.58779	2.8	1.02962	3.8	1.33500	31	3.43399
1.9	0.64185	2.9	1.06471	3.9	1.36098	37	3.61092

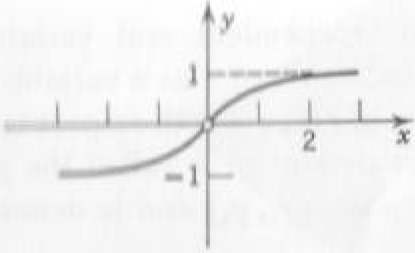

Fig. 11. tanh x.

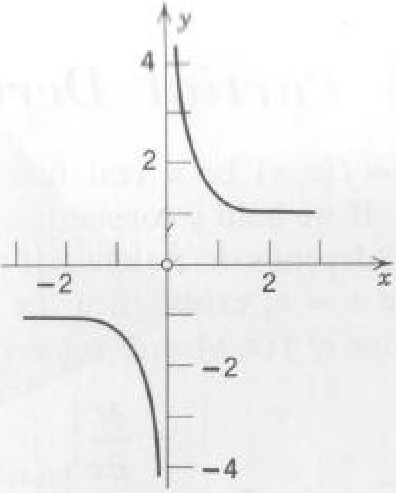

Fig. 12. coth x.

and furthermore the addition formulas

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

(19) $$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

Problems

Are the following functions even, odd, or neither even nor odd?

1. $\sin(x^2)$ **2.** $\sin^2 x$ **3.** $\sin x + \cos x$ **4.** x, x^3, x^5

5. $\tan x$ **6.** $\tanh x$ **7.** x^2, x^4, x^6 **8.** $\ln(1 + e^x) - \frac{x}{2}$

9. Prove that the sum and the product of even functions are even functions.

10. Prove that the sum of odd functions is odd and the product of two odd functions is even.

11. Derive (9) from (7) and (8).

12. Derive (11) from (7) and (8).

13. Prove (18).

14. Prove the addition formulas of the hyperbolic sine and cosine.

Prove the following identities.

15. $\sinh x \sinh y = \frac{1}{2}[\cosh(x + y) - \cosh(x - y)]$

16. $\cosh x \cosh y = \frac{1}{2}[\cosh(x + y) + \cosh(x - y)]$

17. $\sinh x \cosh y = \frac{1}{2}[\sinh(x + y) + \sinh(x - y)]$

18. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1)$, $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

19. Using the differentiation formula of e^x, find the derivatives of $\sinh x$, $\cosh x$, and $\tanh x$.

20. Show that for large x, $\sinh x \approx e^x/2$, $\cosh x \approx e^x/2$.