Problems

Represent the following complex numbers in polar form.

1. $2 - 2i, i, 3 + 4i$ **2.** $5 + 5i, -5 + 5i, -5 - 5i$

3. Show that $\arg \bar{z} = -\arg z$ (up to multiples of 2π).

4. Show that $\arg (1/z) = -\arg z$ (up to multiples of 2π).

5. Prove that $|a - z| = |a - \bar{z}|$ where a is any real number. Interpret geometrically.

6. Prove that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 in the complex plane.

Find the absolute values of the following numbers.

7. $1 + i\sqrt{3}, -9i, 2 + i\sqrt{5}$ **8.** $2 - i\sqrt{5}, 2 + 3i, (4 + i)^3$

9. $\dfrac{(4 - 3i)(\frac{1}{2} + i)^4}{\left(1 - \dfrac{3i}{4}\right)^2(-3 + 4i)}$ **10.** $\left(\dfrac{1 + i}{1 - i}\right)^8, (3 + 4i)^3(-1 - i)^6$

What loci are represented by the following equations and inequalities? (Plot a graph in each case.)

11. $|z| = 1$ **12.** $|z - 1| = 1$ **13.** $\operatorname{Re}(z^2) = -1$ **14.** $\operatorname{Im}(2z) = -1$

15. $0 \leqq \arg z \leqq \pi/2$ **16.** $0 \leqq \arg (1/z) \leqq \pi/2$

17. Using (10), prove the identities

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

18. Find similar identities as in Prob. 17 for $\cos 2\theta$ and $\cos 4\theta$.

19. Find $(3 - i)(2 + i)$ by means of a graphical construction in the complex plane.

20. For given z find $1/z$ by a graphical construction.

0.6 *Some General Remarks About Numerical Computations*

The present section is devoted to a few simple but important general remarks about numerical computations.

It is clear that engineering mathematics ultimately comes down to numerical results, and the engineering student should, therefore, supplement his mathematical equipment with a definite knowledge of some fundamental numerical methods.

Various examples and problems in the text will help the student to learn how to arrange computations in a suitable form. However, numerical computation requires practical experience. It cannot be learned solely from books; practical training is needed, just as in swimming, driving a car, or playing the piano. Consequently, the student should not only work out book examples and problems, but also set up and calculate examples of his own. Active work is more important in numerical analysis than in many other branches of mathematics. Famous mathematicians like Gauss spent a considerable part of their time with numerical computations, and the engineering student will do well in developing a similar attitude.

In many cases formally complete answers to a problem obtained by theoretical considerations may be almost useless for numerical purposes. For example, in the case of systems of linear equations we obtain the solution in the form of quotients of determinants (Secs. 0.3 and 7.5), but if the number of equations and unknowns is large, the direct evaluation of the solution in this form is certainly not the *practical* answer to the problem of finding *numerical values* of the unknowns. In other cases, the theory may yield the solution of a problem in the form of an integral which cannot be evaluated by the usual methods of calculus. Then we need an approximation method for obtaining numerical values of that solution. Moreover, there are many practical problems which cannot be solved by exact methods at all, and in such cases, which arise quite frequently, we have to look for an appropriate approximation method which will yield numerical values of the solution of the problem.

Because we work with a finite number of digits and carry out a finite number of steps in computations, the methods of numerical analysis are *finite processes*, and a numerical result is an approximate value of the (unknown) exact result, except for the rare cases where the exact answer is a sufficiently simple rational number. The difference

$$E = a^* - a$$

between the exact value a and the approximate value a^* is called the *absolute error* of a^*, and the quotient

$$E_r = \frac{a^* - a}{a} \qquad (a \neq 0)$$

is called the *relative error* of a^*.

An error may arise by the process of *rounding off*, that is, by retaining only the more significant decimal digits of a number and rejecting the less significant beyond a certain point. This error is called a **rounding error**. Rounding errors can often be eliminated by carrying one, two, or even more extra figures, known as *guarding figures*, in the intermediate steps of calculation.

Another error may result from the fact that one or several of the formulas used in numerical work are obtained by replacing an infinite series by one of its partial sums. Such an error is called a **truncation error**. It occurs, for instance, in connection with numerical integration and differentiation.

Moreover, a computation may contain **mistakes** because of the fallibility of the human computer or the mechanical and electrical equipment used in the computation. While errors cannot be avoided, mistakes are avoidable in principle. **Checking** of numerical results is quite important and any numerical method should include checking procedures to confirm that the results contain no mistakes. This certainly applies to final results, but in a longer computation it is also advisable to check intermediate results.

Of course, *any computation should be arranged in a systematic tabular form and should contain all the intermediate results directly on the working sheet.* This will help to avoid mistakes and to locate and correct mistakes that have been made. *Numerical work should not be done on scraps of paper.* The numbers should be written in a neat and legible manner on paper that is full notebook page size.

Our last remark is concerned with approximation methods. If a problem cannot be solved by an exact method and we therefore use an approximation method, the immediate basic question is how much at most can the approximate result deviate from the (unknown) exact result. This requires the *estimation of error* involved in approximation methods. In many cases formulas for estimating the maximum possible error are known, and they should be used whenever such an approximation formula is applied. This is the only way to obtain a clear picture of the quality of an approximation and to eliminate a source of stupid mistakes which otherwise may, and actually do, occur in the mathematical part of engineering work. The determination of new and more effective methods of estimating errors is one of the important and interesting problems in modern applied mathematics. The most recent developments in this direction are influenced considerably by the increasing use of electronic computers. Older methods were brought into forms suitable for electronic computation, and new methods have been added; this development is still in progress.

0.7 *Solution of Equations*

A frequent task in engineering mathematics is the determination of the roots of an equation of the form

$$f(x) = 0 \qquad (x \text{ real}). \tag{1}$$

If $f(x)$ is a simple function, then there may be a formula which yields the exact values of the roots. For example, if

$$f(x) = ax^2 + bx + c \qquad (a \neq 0)$$

we have to solve a *quadratic equation*, the solutions being

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}). \tag{2}$$

However, if $f(x)$ is more complicated, there may not be a formula for the exact values of the roots and in such a case we may use an approximation method which yields approximate values of the roots. Let us explain an important method of this type.

Suppose we want to determine the real roots of the equation $f(x) = 0$ where f is a differentiable function. Then we may plot a graph of $f(x)$ in the usual way. Assuming that the equation has a real root, we may obtain from the graph a first rough approximation of this root; let x_1 be this approximation. We may then determine a second approximation x_2, namely, the point of intersection of the tangent to the curve of $f(x)$ at x_1 and the x-axis (Fig. 22). Since

$$\tan \alpha = f'(x_1) = \frac{f(x_1)}{x_1 - x_2}$$

by solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In the next step we compute

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

etc., and in general

$$(3) \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (n = 1, 2, \cdots)$$

This method of successive approximations is called **Newton's[8] method.**

Example 1. Given the equation

$$f(x) = x^3 + x - 1 = 0$$

we see that it has a root near $x_1 = 1$ (Fig. 23). From (3) we obtain

$$x_{n+1} = x_n - \frac{{x_n}^3 + x_n - 1}{3{x_n}^2 + 1} = \frac{2{x_n}^3 + 1}{3{x_n}^2 + 1} = \frac{N_n}{D_n}.$$

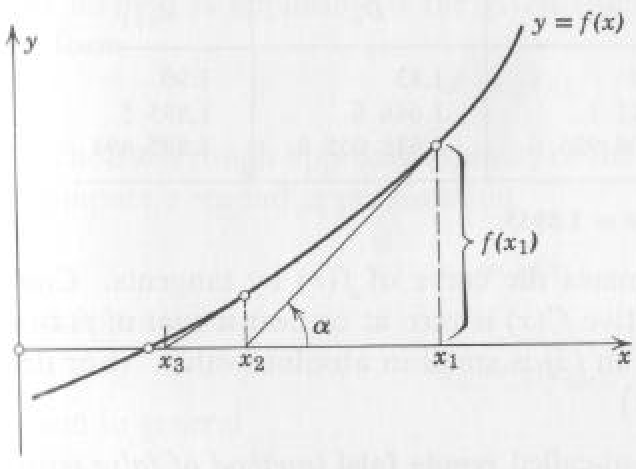

Fig. 22. Newton's method.

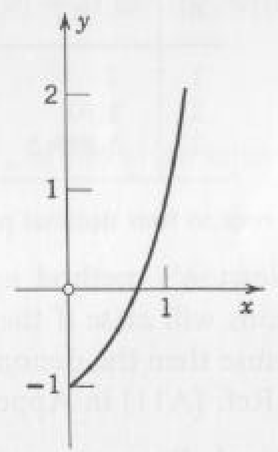

Fig. 23. Example 1.

[8] Sir ISAAC NEWTON (1642–1727), English physicist and mathematician. Newton and the German mathematician and philosopher, GOTTFRIED WILHELM LEIBNIZ (1646–1716), invented (independently) the differential and integral calculus. Newton discovered many basic physical laws and introduced the method of investigating physical problems by means of calculus. His work is of greatest importance to both mathematics and physics.

The computation may be arranged in tabular form:

n	x_n	x_n^2	x_n^3	N_n	D_n	x_{n+1}
1	1	1	1	3	4	0.75
2	0.75	0.562 5	0.421 875	1.843 750	2.687 500	0.686 047
3	0.686 047	0.470 660	0.322 895	1.645 790	2.411 980	0.682 340
4	0.682 340	0.465 588	0.317 689	1.635 378	2.396 764	0.682 328

The result seems to indicate that the desired root to four decimal places is $x = 0.6823$. (We mention that even all six decimals of x_5 are the correct first decimals of the root.)

Sometimes work may be saved by starting with a small number of digits and increasing it gradually. Our present computation may then become:

n	x_n	x_n^2	x_n^3	N_n	D_n	x_{n+1}
1	1	1	1	3	4	0.75
2	0.75	0.562 5	0.421 9	1.843 8	2.687 5	0.686
3	0.686	0.470 60	0.322 83	1.645 66	2.411 80	0.682 3
4	0.682 3	0.465 533	0.317 633	1.635 266	2.396 599	0.682 33

Example 2. Find the positive root of the equation

$$2 \sin x = x.$$

Here $f(x) = x - 2 \sin x = 0$, and from (3),

$$x_{n+1} = x_n - \frac{x_n - 2 \sin x_n}{1 - 2 \cos x_n} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2 \cos x_n} = \frac{N_n}{D_n}.$$

From the graph we conclude that the root is near $x_1 = 2$. Using tables of the sine and cosine functions, the computation yields

n	x_n	N_n	D_n	x_{n+1}
1	2	3.48	1.83	1.90
2	1.90	3.121 1	1.646 6	1.895 5
3	1.895 5	3.104 925 6	1.638 055 9	1.895 494

The root to four decimal places is $x = 1.8955$.

Newton's method approximates the curve of $f(x)$ by tangents. Complications will arise if the derivative $f'(x)$ is zero at or near a root of $f(x) = 0$, because then the denominator in (3) is small in absolute value. (For details see Ref. [A11] in Appendix 1.)

We shall now consider the so-called **regula falsi** (*method of false position*) which approximates that curve by a chord, as shown in Fig. 25. The chord intersects the x-axis at the point

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)} \tag{4}$$

and this is an approximation for the root X_0 of the equation $f(x) = 0$.

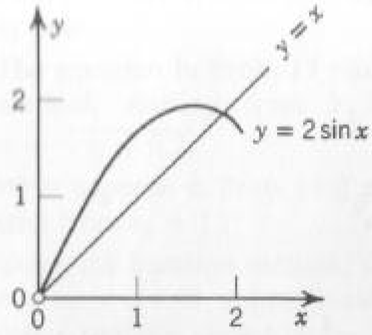

Fig. 24. Example 2.

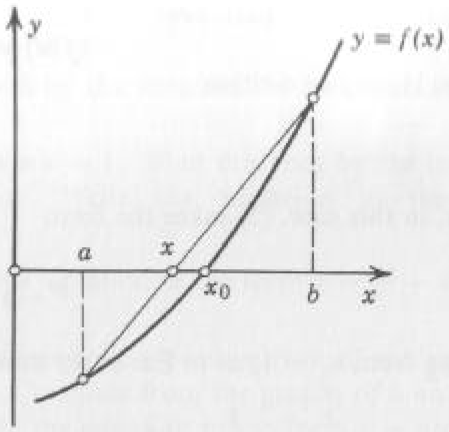

Fig. 25. Regula falsi.

Example 3. Find an approximate value of the root of the equation

$$f(x) = x^3 + x - 1 = 0$$

near $x = 1$ (cf. Ex. 1). We find that

$$f(0.5) = -0.375 \quad \text{and} \quad f(1) = 1.$$

Choosing $a = 0.5$ and $b = 1$, formula (4) yields

$$x_1 = \frac{0.5 \cdot 1 - 1 \cdot (-0.375)}{1 - (-0.375)} = 0.64.$$

Since $f(0.64) = -0.0979 < 0$, in the next step we may choose $a = 0.64$ and $b = 1$, and obtain from (4) the more accurate approximation $x_2 = 0.672$, and so on.

A third important method for determining the roots of an equation $f(x) = 0$ is the so-called **iteration method**.

This method is applicable if the given equation $f(x) = 0$ can be written in the form

$$x = g(x).$$

We choose a rough approximation x_1 of the root X_0 of the given equation and compute a second approximation

$$x_2 = g(x_1),$$

a third approximation

$$x_3 = g(x_2),$$

etc., and in general

$$x_{n+1} = g(x_n) \qquad (n = 1, 2, \cdots). \tag{5}$$

It can be shown that if

$$|g'(X_0)| < 1$$

and x_1 is sufficiently accurate (sufficiently close to X_0), then $x_n \to X_0$ as $n \to \infty$, that is, (5) will yield better and better approximations as n increases.

Example 4. The equation

$$f(x) = x^3 + x - 1 = 0$$

(cf. Ex. 1) may be written

$$x = \frac{1}{1 + x^2} \tag{6}$$

Hence, in this case, (5) takes the form

$$x_{n+1} = \frac{1}{1 + x_n^2}.$$

Starting from $x_1 = 1$, as in Ex. 1, we thus obtain

$$x_2 = \frac{1}{1 + 1^2} = \frac{1}{2} = 0.5, \qquad x_3 = \frac{1}{1 + (1/2)^2} = \frac{4}{5} = 0.8, \qquad x_4 = 0.61, \quad \text{etc.}$$

Note that the right-hand side of (6) is

$$g(x) = \frac{1}{1 + x^2}, \qquad \text{and} \qquad g'(x) = \frac{-2x}{(1 + x^2)^2}.$$

Now $X_0 \approx 0.7$, and $|g'(0.7)| = 1.4/1.49^2 < 1$, which shows that for x_1 sufficiently close to X_0 we have $x_n \to X_0$ as $n \to \infty$. Our numerical results seem to confirm this fact.

It is clear that a given equation $f(x) = 0$ may be written in the form $x = g(x)$ in many ways. In the present example we may write

$$x = 1 - x^3$$

but then

$$g(x) = 1 - x^3, \qquad |g'(x)| = 3x^2 > 1 \text{ when } x = X_0 \approx 0.7,$$

and the iteration will not work. The reader may try to start from $x_1 = 1$, $x_1 = 0.7$, $x_1 = 2$, and see what happens.

Problems

Find a root of the following equations; carry out three steps of Newton's method, starting from the given x_1.

1. $x^3 - 1.2x^2 + 2x - 2.4 = 0, x_1 = 1$
2. $x^3 - 1.2x^2 + 2x - 2.4 = 0, x_1 = 2$
3. $x^3 - 3.9x^2 + 0.9x + 5.8 = 0, x_1 = 3$
4. $x^3 - 3.9x^2 + 0.9x + 5.8 = 0, x_1 = 1$
5. $x^3 - 3.9x^2 + 4.79x - 1.881 = 0, x_1 = 1$
6. The roots of the equation in Prob. 5 are 0.9, 1.1, and 1.9. Although $x_1 = 1$ lies close to 0.9 and 1.1, Newton's method does not yield one of these roots. Why? Choose another x_1 such that the method yields approximations for the root 1.1.

Find all real roots of the following equations by Newton's method.

7. $x^4 - 0.1x^3 - 0.82x^2 - 0.1x - 1.82 = 0$
8. $x^3 - 4x^2 + 2x - 8 = 0$
9. $x^3 - 5x^2 + 6.64x - 1.92 = 0$ **10.** $\cos x = x$
11. $x + \ln x = 2$ **12.** $2x + \ln x = 1$

Find the real roots of the following equations by means of the regula falsi.

13. $x^4 = 2$ **14.** $x^4 = 2x$ **15.** $3 \sin x = 2x$

16. In Prob. 13 the approximate values of the positive root will always be somewhat smaller than the exact values of the root. Why?

17. Find the root of $x^5 = x + 0.2$ near $x = 0$ by the iteration method, starting from $x_1 = 0$.

18. The equation in Prob. 17 has a root near $x = 1$. Find this root by the iteration method, starting from $x_1 = 1$. *Hint:* Write the equation in the form $x = \sqrt[5]{x + 0.2}$.

19. What happens in Prob. 18 if you write the equation in the form $x = x^5 - 0.2$ and start from $x_1 = 1$?

20. Using the iteration method, show that the smallest positive root of the equation $x = \tan x$ is 4.49, approximately. *Hint:* Conclude from the graphs of x and $\tan x$ that a root lies close to $x_1 = 3\pi/2$; write the equation in the form $x = \arctan x$. (Why?)

0.8 *Approximate Integration*

If $f(x)$ is a given function and we can find a function $F(x)$ whose derivative is $f(x)$, then we may evaluate definite integrals of $f(x)$ by using the familiar formula

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad [F'(x) = f(x)]. \tag{1}$$

Tables of integrals[9] may be helpful for this purpose. However, in engineering applications there frequently occur integrals which cannot be evaluated by familiar methods known from calculus. These may be integrals of two types, namely, integrals such as

$$\int e^{-x^2}\,dx, \qquad \int \frac{\sin x}{x}\,dx, \qquad \int \cos(x^2)\,dx,$$

which cannot be represented in terms of finitely many elementary functions, or integrals whose integrands are empirical functions, given by a table of numerical values which are obtained from a physical experiment or in some other way. In many cases, integrals of the first type may be evaluated by one of the advanced methods (complex integration, use of power series, or asymptotic expansions), and in the case of an integral of the second type we may try to approximate the empirical function by a polynomial or some other elementary function.

Nevertheless, in various situations it will be preferable to apply one of the standard methods of numerical integration which we shall consider in the present section. These methods use the fact that the definite integral in (1) equals the area A of the shaded region R in Fig. 26 on p. 32.

To determine A we may cut R from cardboard, weight the piece, and divide the result by the weight of a square of this cardboard whose side is 1.

Another simple method is to draw R on a graph paper and count squares.

[9] Cf. for example, Ref. [A3] in Appendix 1.