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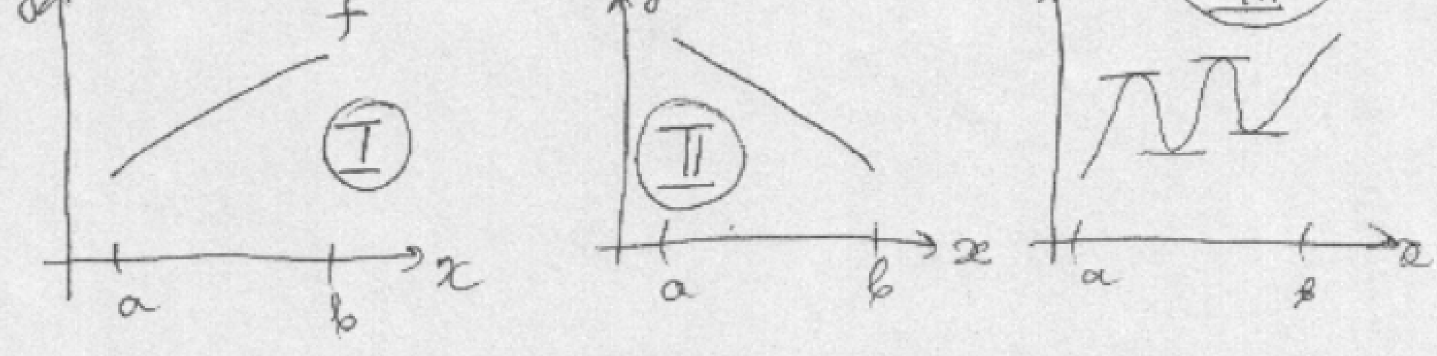
# Unconstrained Optimization with one Variable

$f: \mathbb{R} \to \mathbb{R}$ and we want to find

$\max_{x \in S} f(x)$, or $\min_{x \in S} f(x)$, where

$S \subseteq \mathbb{R}$, i.e. $S \subseteq D(f)$.

- We will assume that $f$ is differentiable, i.e. that $f'$ exist for $\forall x \in S$
- Let us also assume that $S$ is an interval, $S = [a, b]$ (or $S = (a, b)$, or any other combination of borders)
- We can have different possibilities:



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Case I: Function f is monotone increasing, so

$\min_{x \in [a,b]} f(x) = f(a)$, $x_{min} = a$

$\max_{x \in [a,b]} f(x) = f(b)$, $x_{max} = b$.

– This is equivalent to the condition

$$(\forall x \in (a,b))\ f'(x) > 0$$

(At the borders $f'$ can be zero, i.e. we can have $f'(a) = 0$ or $f'(b) = 0$, but $f'(a)$ and $f'(b)$ must be nonnegative)

Example: $f(x) = x^2$ $S = [0, 3]$

$f'(x) = 2x$, $\forall x \in (0,3)$ $f'(x) > 0 \Rightarrow$

$\max_{x \in [0,3]} x^2 = 3^2 = 9$ $\min_{x \in [0,3]} x^2 = 0^2 = 0$

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Example: $f(x) = x^2$, $S = [-1, 1]$

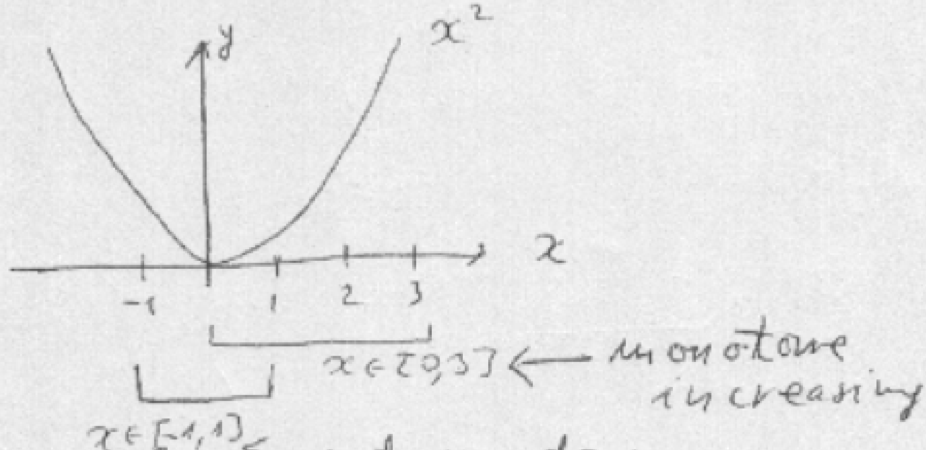
$f'(x) = 2x$, if $x \in (-1, 0)$, $f'(x) < 0$,

and if $x \in (0, 1)$, $f'(x) > 0$

So, this function on $[-1, 1]$ is

not monotone increasing.

We will use other methods to find min and max of this function.




Example: $f(x) = e^x$, $S = \mathbb{R}$

$f'(x) = e^x$, $(\forall x \in \mathbb{R})$ $f'(x) > 0$, monotone increasing

But, S does not have borders, so f does not have min and max!

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Case II: Function is monotone decreasing, so

$$\min_{x\in[a,b]} f(x) = f(b), \quad x_{min} = b$$

$$\max_{x\in[a,b]} f(x) = f(a), \quad x_{max} = a$$

This is equivalent to the condition

$$(\forall x \in (a,b)) \; f'(x) < 0$$

(At the borders we can have $f'(a)=0$ or $f'(b)=0$, but they must be nonnegative)

Example: $f(x) = x^4$, $S = [-4, -2]$

$f'(x) = 4x^3$, $(\forall x \in [-4,-2]) \; f'(x) < 0 \Rightarrow$

$\max_{x\in[-4,-2]} x^4 = (-4)^4 = 256$, $\min_{x\in[-4,-2]} x^4 = (-2)^4 = 16$

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- If $f$ is monotone increasing at some interval, then $-f$ is monotone decreasing on the same interval and vice versa;
that exchanges min's and max's

Case III : Function is not monotone and have max or min at some point $x^* \in (a, b)$ (interior of $S$)

- At min or max tangent is parallel to $x$-axis and have slope $= \emptyset$, i.e.

$$\boxed{f'(x^*) = 0}$$

- This is neccessary condition!
- It is not sufficient to tell if $x^*$ is min or max

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– Sufficient conditions:

If $f''(x^*) > 0$, then $x^*$ is min

If $f''(x^*) < 0$, then $x^*$ is max

– The algorithm:

1) Find all zeros of the $f'(x)$ on $x \in (a, b)$, i.e. all $x^*$'s so that $f'(x^*) = 0$

2) Check the sign of $f''(x^*)$ for all $x^*$'s; those with $f''(x^*) > 0$ are minima, and those with $f''(x^*) < 0$ are maxima

3) If there are no zeros of $f'$ on $(a, b)$, then $f'$ is either positive (Case I) or negative (Case II), and maximum and minimum are on the borders

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- Example: $f(x) = x^2$, $S = \mathbb{R}$

Step 1: $f'(x) = 2x$, $f'(x^*) = 2x^* = 0 \Rightarrow$

$x^* = 0$ unique solution!

step 2: $f''(x) = 2 > 0 \Rightarrow f''(x^*) = 2 > 0$

$\Rightarrow x^* = 0$ is minimum

Since there are no other zeros of $f'$, this is the only extremal point.

- Example: $f(x) = x^2 \sin x - \frac{x^3}{3} \cos x$, $S = \mathbb{R}$

step 1: $f'(x) = 2x \sin x + \cancel{x^2 \cos x} -$

$- \cancel{\frac{3x^2}{3} \cos x} + \frac{x^3}{3} \sin x =$

$= x\left(2 + \frac{x^2}{3}\right) \sin x$

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since $2+\frac{x^2}{3}>0$, and

$\sin x=0$ for $x_k=k\pi$, $k\in\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$

We have $f'(x^*)=0$ for

$x^*=0$ and $x_k^*=k\pi$, $k\in\mathbb{Z}$

(for $k=0$ we get $x_0^*=x^*$, so $x_k^*=k\pi$, $k\in\mathbb{Z}$ contains all solutions)

<u>Step 2</u>: $f''(x)=\left(2+\frac{x^2}{3}\right)\sin x + x\cdot\frac{2x}{3}\sin x + x\left(2+\frac{x^2}{3}\right)\cos x$

$$f''(x_k^*)=\left(2+\frac{k^2\pi^2}{3}\right)\sin k\pi+\frac{2}{3}k^2\pi^2\sin k\pi + k\pi\left(2+\frac{k^2\pi^2}{3}\right)\cos k\pi$$

Since $\sin k\pi=0$, $\cos k\pi=(-1)^k$, we have

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$$f''(x^*_{k\pi}) = (-1)^k k\pi\left(2 + \frac{k^2\pi^2}{3}\right)$$

$2 + \frac{k^2\pi^2}{3} > 0$ for all $k \in \mathbb{Z}$

We can consider three cases:

$1^\circ$ $k = n \in \mathbb{N} = \{1, 2, 3, \dots\}$

$2^\circ$ $k = -n$, $n \in \mathbb{N}$

$3^\circ$ $k = 0$

For $k = n \in \mathbb{N}$, we have

$k\pi = n\pi > 0$, and $(-1)^k = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

$$\Rightarrow f''(x^*_k) = f''(x^*_n) \begin{cases} < 0, & n \text{ odd} \Rightarrow \max \\ > 0, & n \text{ even} \Rightarrow \min \end{cases}$$

For $k = -n$, $n \in \mathbb{N}$, we have

$k\pi = -n\pi < 0$, and $(-1)^k = (-1)^n = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$

$$\Rightarrow f''(x^*_k) = f''(x^*_{-n}) \begin{cases} > 0, & n \text{ odd} \Rightarrow \min \\ < 0, & n \text{ even} \Rightarrow \max \end{cases}$$

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- For $k=0$ we have $f''(x_0^*) = f''(0) = 0$.
- If this is the case, we need to calculate further derivatives and find first one different ~~from~~ zero at $x_0^*=0$ (say it is $f^{(n)}(x_0^*) = = f^{(n)}(0) \neq 0$). That means

$f^{(m)}(0) = 0$ for $m = 1, \dots, n-1$, and

$f^{(n)}(0) \neq 0$

- If $n$ is even, then

if $f^{(n)}(0) > 0 \Rightarrow x_0^* = 0$ is min

if $f^{(n)}(0) < 0 \Rightarrow x_0^* = 0$ is max

- If $n$ is odd, $x_0^* = 0$ is not min nor max! (it is saddle point of $f$)

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Unconstrained Optimization
with Multiple Variables

$f: \mathbb{R}^n \to \mathbb{R}$, i.e. we have $f(x, y, z, \dots)$

Set $S$ is some subset of $\mathbb{R}^n$

We want to find $\max\limits_{x \in S} f$ or $\min\limits_{x \in S} f$.

- Note that is $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ satisfies $f(x^*) = \min\limits_{x \in S} f(x)$, then

$-f(x^*) = \max\limits_{x \in S} (-f(x)) \implies$ we can exchange min for max or vice versa!

- We will here consider only minima and maxima at the interior of $S$.
- If there are no extremal points at the interior, we will have to look at the borders of $S$, but that

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can easily be done.

- The Algorithm:

1) Find all first derivatives of $f$ and solve the system

$$\frac{\partial f}{\partial x_1}=0, \dots, \frac{\partial f}{\partial x_n}=0$$

The solution is denoted by $x^*=(x_1^*, \dots, x_n^*)$

2) For every solution of 1) calculate Hessian matrix (at $x^*$)

$$H=\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$H$ is symmetric matrix!

If the corresponding quadratic form $x^T H x$ is:

positively defined, $x^*$ is min
negatively defined, $x^*$ is max

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- If $H$ is semidefinite, we need to look closely at the function – we will not cover this case.

- Example: $f(x,y) = x^2+y^2$

step 1:
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x = 0 \\ \frac{\partial f}{\partial y} &= 2y = 0 \end{aligned}\right\} \Rightarrow x^* = y^* = 0$$

step 2:
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\det H_{11} > 0$, $\det H > 0$ $\Rightarrow$

$(0,0)$ is minimum

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- Example: $f(x,y) = x^4 + x^2y^2 + y^4 - 4x^2 - 4y^2$

$$\frac{\partial f}{\partial x} = 4x^3 + 2xy^2 - 8x = 0$$

$$\frac{\partial f}{\partial y} = 2x^2y + 4y^3 - 8y = 0$$

If $x=0$ then $\frac{\partial f}{\partial x} = 0$ and

$\frac{\partial f}{\partial y} = 4y(y^2-2) = 0$, so we have

solutions $\boxed{(0,0),\ (0, \pm\sqrt{2})}$

If $x \neq 0$, we will divide $\frac{\partial f}{\partial x}$ with $2x$, and obtain

$$2x^2 + y^2 = 4$$

$$\frac{\partial f}{\partial y} = 2y(x^2 + 2y^2 - 4) = 0$$

If $y=0$, $\frac{\partial f}{\partial y} = 0$, and $2x^2 + 0^2 = 4 \Rightarrow x = \pm\sqrt{2}$,
so the solutions are $\boxed{(\pm\sqrt{2}, 0)}$

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If $y \neq 0$, then we have a system

$$2x^2 + y^2 = 4$$
$$x^2 + 2y^2 = 4$$

which has a unique solution $x^2 = y^2 = \frac{4}{3}$, so the solutions to the original system are all combination of signs

$$\left(\pm\sqrt{\tfrac{4}{3}}, \pm\sqrt{\tfrac{4}{3}}\right), \left(\pm\sqrt{\tfrac{4}{3}}, \mp\sqrt{\tfrac{4}{3}}\right).$$

$$H = \begin{bmatrix} 12x^2 + 2y^2 - 8 & 4xy \\ 4xy & 2x^2 + 12y^2 - 8 \end{bmatrix}$$

For $(0,0)$ we have $H = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$

$\det H_{11} = -8 < 0$, $\det H = 64 > 0 \Rightarrow$

$H$ is negatively defined $\Rightarrow$

$(0,0)$ is maximum of $f$

Check other solutions for HW!

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- Economic Example: Suppose a single firm faces two markets, A and B. Each market has its own demand function $d_i(q_i)$ ($i = A, B$), where $q_A$ and $q_B$ is the number of products offered at the markets. Prices at the markets are different, say $p_A$ and $p_B$ per single product. The cost function is $C(q_A + q_B)$. How to maximize the profit

$$\pi(q_A, q_B) = p_A d_A(q_A) + p_B d_B(q_B) - C(q_A + q_B)$$

- We are searching for $\max\limits_{q_A, q_B} \pi(q_A, q_B)$

So

$$\frac{\partial \pi}{\partial q_A} = p_A d_A'(q_A) - \frac{\partial C}{\partial q_A} = 0$$

$$\frac{\partial \pi}{\partial q_B} = p_B d_B'(q_B) - \frac{\partial C}{\partial q_B} = 0$$

These are conditions for extremal point

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Hessian matrix is:

$$H = \begin{bmatrix} p_A d_A''(q_A) - \frac{\partial^2 C}{\partial q_A^2} & -\frac{\partial^2 C}{\partial q_A \partial q_B} \\ -\frac{\partial^2 C}{\partial q_A \partial q_B} & p_B d_B''(q_B) - \frac{\partial^2 C}{\partial q_B^2} \end{bmatrix},$$

and for a found solutions of

$\frac{\partial \Pi}{\partial q_A} = 0$, $\frac{\partial \Pi}{\partial q_B} = 0$ the neccessary conditions are (it has to be maximum)

$$H_{11} < 0 \Leftrightarrow p_A d_A''(q_A^*) - \frac{\partial^2 C}{\partial q_A^2}(q_A^*, q_B^*) < 0$$

$$\det H > 0 \Leftrightarrow \left( p_A d_A''(q_A^*) - \frac{\partial^2 C}{\partial q_A^2}(q_A^*, q_B^*) \right) \cdot$$

$$\cdot \left( p_B d_B''(q_B^*) - \frac{\partial^2 C}{\partial q_B^2}(q_A^*, q_B^*) \right) - \left( \frac{\partial^2 C}{\partial q_A \partial q_B}(q_A^*, q_B^*) \right)^2 > 0$$

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Homework Problems: Find minima and maxima of the following functions:

1) $f(x) = x^2 + x^3$

2) $f(x) = e^x \sin x$

3) $f(x,y) = x^4y^2 + x^2y^4 - x^2y^2$

4) $f(x,y,z) = 2x^2 + 2xy + 2xz + 2y^2 + 2yz + 2z^2$

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