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Fractal properties of financial markets

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HIGHLIGHTS

- Analysis of the S&P 500 index using a simple fractal function is proposed.
- The Besicovitch–Ursell function is fitted to the data for several financial growths.
- The fitting function reproduces complete financial growths in a natural way.
- Shortening the fitting interval causes deviations between the two curves.
- Fractal functions have great modeling abilities concerning market dynamics.

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ABSTRACT

We present an analysis of the USA stock market using a simple fractal function. Financial bubbles preceding the 1987, 2000 and 2007 crashes are investigated using the Besicovitch–Ursell fractal function. Fits show a good agreement with the S&P 500 data when a complete financial growth is considered, starting at the threshold of the abrupt growth and ending at the peak. Moving the final time of the fitting interval towards earlier dates causes growing discrepancy between two curves. On the basis of a detailed analysis of the financial index behavior we propose a method for identifying the stage of the current financial growth and estimating the time in which the index value is going to reach the maximum.

1. Introduction

Events such as earthquakes, financial crashes, material breaking, etc., are ubiquitous in nature. Describing them and finding tools for their prediction is a subject of great interest. Complex systems can generate large fluctuations and unpredictable outcomes, while at the same time they exhibit some general features that can be studied. One of the main tasks of statistical physics is finding the laws describing fluctuations. Recently, a lot of effort has been made in modeling and analyzing financial market dynamics using concepts and tools of statistical physics [1–9].

Scaling phenomena, characteristic of the systems that exhibit self-organized criticality, are observed also in financial markets. Analyzing the S&P 500 index, scaling behavior was observed for time intervals spanning three orders of magnitude—from 1 to 1000 min [4]. The probability distribution of the index variations was found to be non-Gaussian. One of the important characteristics of financial time series is a non-negligible probability of the occurrence of large market fluctuations—during the 1987 stock market crash the daily move of the S&P 500 index recorded a magnitude of about 20 standard deviations. Stock returns display tails that are much more pronounced than a simple Gaussian [10,11].

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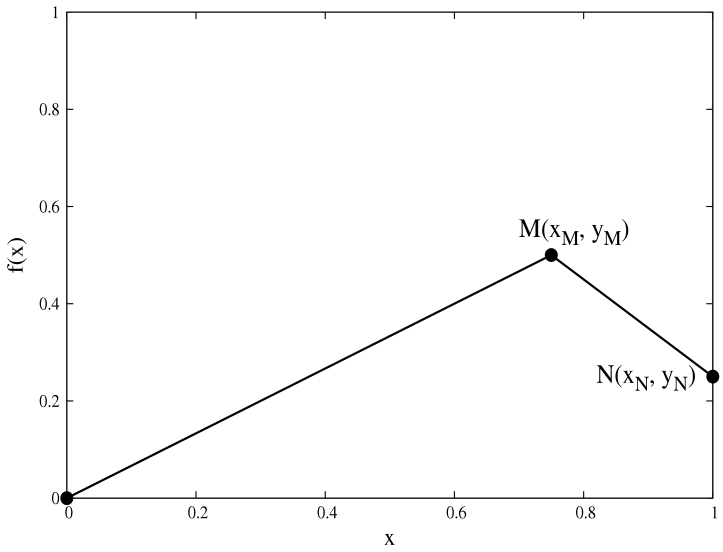


Fig. 1. Model function $f(x)$ (Eq. (1)) with $x_M = 0.75,\ y_M = 0.50,\ x_N = 1.0,\ y_N = 0.25$.

Table 1
Values of the fitting parameters obtained by fitting the B–U function to S&P 500 data for the financial growths ending in 1987, 2000 and 2007. The last column contains the r.m.s. errors of the fits.

Year	x_M	y_M	y_N	p_c	s_0	*r.m.s.*
1987	0.81528	0.12175	0.61857	1.9108	0.69802	0.0215
2000	0.91932	0.09926	0.32264	1.9045	0.48805	0.0400
2007	0.89584	0.30109	0.49857	1.8183	0.50153	0.0337

Possibility of identifying financial crises and predicting financial crashes has been analyzed in many papers [12–18]. Financial markets are nonlinear, complex systems described by enormous number of mostly unknown parameters. There are two main philosophies in applying tools of statistical physics to describe various financial events including crashes.

The first one is a global approach and its aim is to observe well defined structures in financial time series preceding the crash. One of the properties of a financial bubble is the faster-than-exponential growth of the price [16]. Each burst of super-exponential price growth is followed by a crash, i.e. the average exponential growth of the index consists of a succession of bubbles and crashes, which seem to be the norm rather than the exception. Price growth can be modeled by the so-called log-periodic power law (LPPL) [14,15]. Fitting the financial data by the log-periodic function can determine the critical time for a possible "phase transition" or crash. The end of a bubble is not necessarily accompanied by a crash, but the corresponding critical time is the time where the crash is the most probable [19]. Crashes can occur before, or a bubble can land smoothly, therefore, only probabilistic forecasts can be developed.

The other approach is to study the local scaling properties of financial time series and investigate the long-memory correlations [20–22]. It is known that the Hurst exponent measures the level of persistency in the given signal. The value $H = 0.5$ corresponds to the "Markovian" behavior and the absence of long-term correlations. For $H > 0.5$ there is a persistence, and for $H < 0.5$ antipersistence in the time series. When the trend in the market is strong and well determined one should observe some long-range correlations in returns and consequently higher values of the Hurst exponent. Contrary, at the index maximum the increasing trend is broken and the decreasing one is set and before the crash the signal of anti-correlation and the drop of the Hurst exponent appear.

Fractal market hypothesis is tightly connected to multifractality and long-range dependence in financial time series. Financial markets are considered as complex systems consisting of many heterogeneous agents which are distinguishable mainly with respect to their investment horizons. These horizons range from seconds (market makers) up to several years (pension funds). Each of the investors group has its own trading rules and strategies. It was found [22,23] that the turbulent times are characterized by the dominance of short investment horizons.

Self-similarity in financial price records manifests itself in the virtual impossibility to distinguish the price records on different time scales, when the axes are not labeled [24]. This leads to the idea to describe the oscillations in financial markets by fractal functions [25]. Here we demonstrate that financial growth can be successfully modeled by a simple fractal function. We provide the fits to real financial data. On the basis of the well known S&P 500 index we analyze financial growths preceding the 1987, 2000 and 2007 crashes. We also analyze the current financial growth, starting from the minimal value reached during the last economic crisis.

Although having different triggers, the mentioned crises display similar behavior. Some economists think that a crash may be a result of social interactions and that the crashes have fundamentally similar origins that should be found in

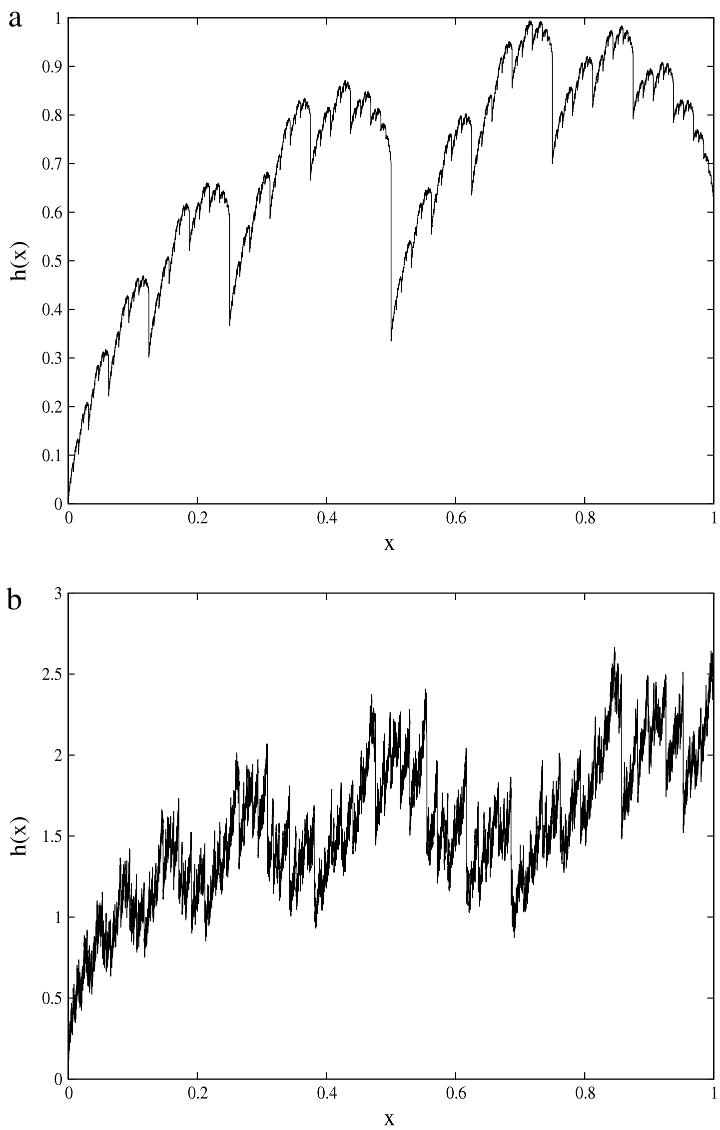


Fig. 2. Besicovitch–Ursell function $h(x)$ (Eq. (3)) obtained for: (a) $x_M = 0.75,\ y_M = 0.50,\ y_N = 0.25,\ p_c = 2.0,\ s_0 = 0.75$; (b) $x_M = 0.70,\ y_M = 0.40,\ y_N = 0.90,\ p_c = 1.8,\ s_0 = 0.40$.

the collective organization of the market traders. Finding a way to describe the financial data resulting from extremely complicated market dynamics would enhance the possibility of predictions.

2. Model function

It is well known that time-varying quantities often turn out to have fractal graphs. Examples include physiological signals, multiscale fluctuations of temperature during long-term climate change and fluctuations of TCP flow dynamics in communication networks. Actually, many phenomena display fractal features when plotted as a function of time, at least when recorded over fairly long time spans.

Here, we define a simple function for modeling financial data, so it is suitable for both theoretical analysis and numerical experiments. At first, we define a "saw-tooth" function $f : [0, 1] \to \mathcal{R}$ by

$$f(x) = \begin{cases} \dfrac{y_M}{x_M}, & 0 \le x \le x_M; \\ \dfrac{y_M - y_N}{x_M - 1}\, x + \dfrac{x_M y_N - y_M}{x_M - 1}, & x_M < x \le 1 \end{cases} \tag{1}$$

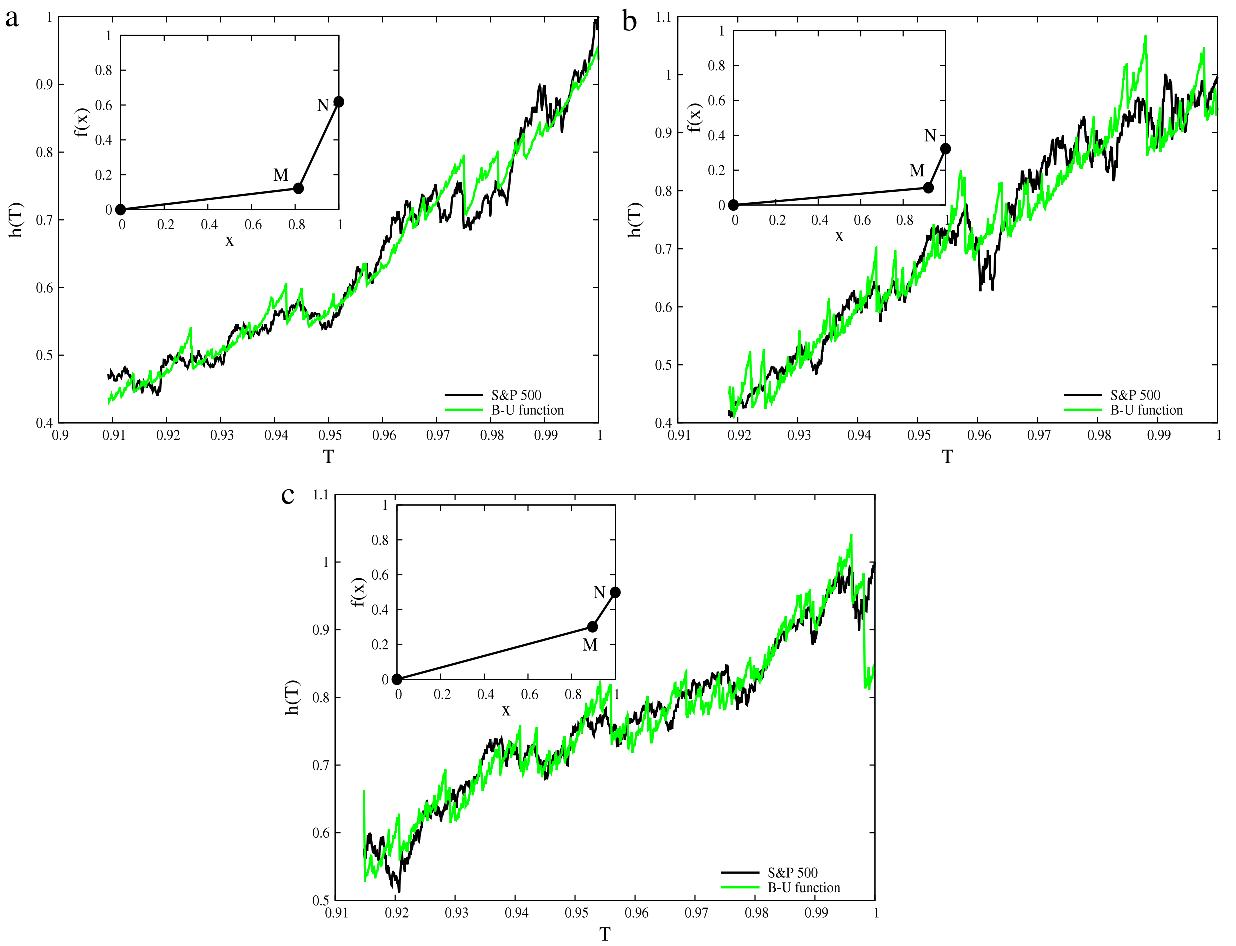


Fig. 3. Fit of the fractal Besicovitch–Ursell function (green, gray line) to the S&P 500 data (black line) for the financial growth preceding: (a) the 1987 crash; (b) the 2000 crash; and (c) the 2007 crash. Inset: model function $f(x)$ (Eq. (1)). The final time and the maximal index value are normalized to unity. The time is counted in days from 3 January 1950. Values of the r.m.s. errors of the fits are given in Table 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $x_M, y_M, y_N \in (0, 1)$. Function f can be seen graphically in Fig. 1. Here, $0 < x_M < 1$ is the asymmetry parameter that determines the relative position of the maximum within one period of the saw-tooth function $f(x)$. Let $g : \mathcal{R} \to \mathcal{R}$ be a periodic function of period 1, defined on $\mathcal{R}$ by

$$g(x) = f(\mathrm{rem}(|x|, 1)), \quad x \in \mathcal{R}. \tag{2}$$

Here, the function $R = \mathrm{rem}(x, y)$ returns the remainder after division of x by y. Let $p_c > 1$ and choose a parameter $0 < s_0 < 1$. Then the Besicovitch–Ursell (B–U) function is defined as follows [26]:

$$h(x) = \sum_{k=0}^{\infty} \frac{g(p_c^k x)}{p_c^{s_0 k}} = f(\mathrm{rem}(|x|, 1)) + \sum_{k=1}^{\infty} \frac{f(\mathrm{rem}(|p_c^k x|, 1))}{p_c^{s_0 k}}, \quad x \in \mathcal{R}. \tag{3}$$

The Besicovitch–Ursell function is continuous and nowhere differentiable on $\mathcal{R}$ for $p_c > 1$ and $s_0 \in (0, 1)$ [27]. The scaling parameter $0 < s_0 < 1$ (Hurst exponent) is related to the box-counting dimension $\dim_B$ of the graph of the B–U function (3) according to the following formula [28]:

$$\dim_B \mathrm{graph}(h) = 2 - s_0. \tag{4}$$

It is well known that the Hausdorff dimension $\dim_H$ of the graph of the B–U function (3) is at most $2 - s_0$ ($\dim_H \mathrm{graph}(h) \le 2 - s_0$) [28]. The fractal dimension $2 - s_0$ measures the global irregularity of the curve. Two different cases of the B–U function are shown in Fig. 2.

The analysis of the behavior of the S&P 500 index is carried out using nonlinear fitting routine FMINSEARCH in MATLAB® (MathWorks, Natick, MA). This is an implementation of the Nelder–Mead simplex algorithm [29] which minimizes a nonlinear function of several variables. This is a direct search method that does not use numerical or analytic gradients.

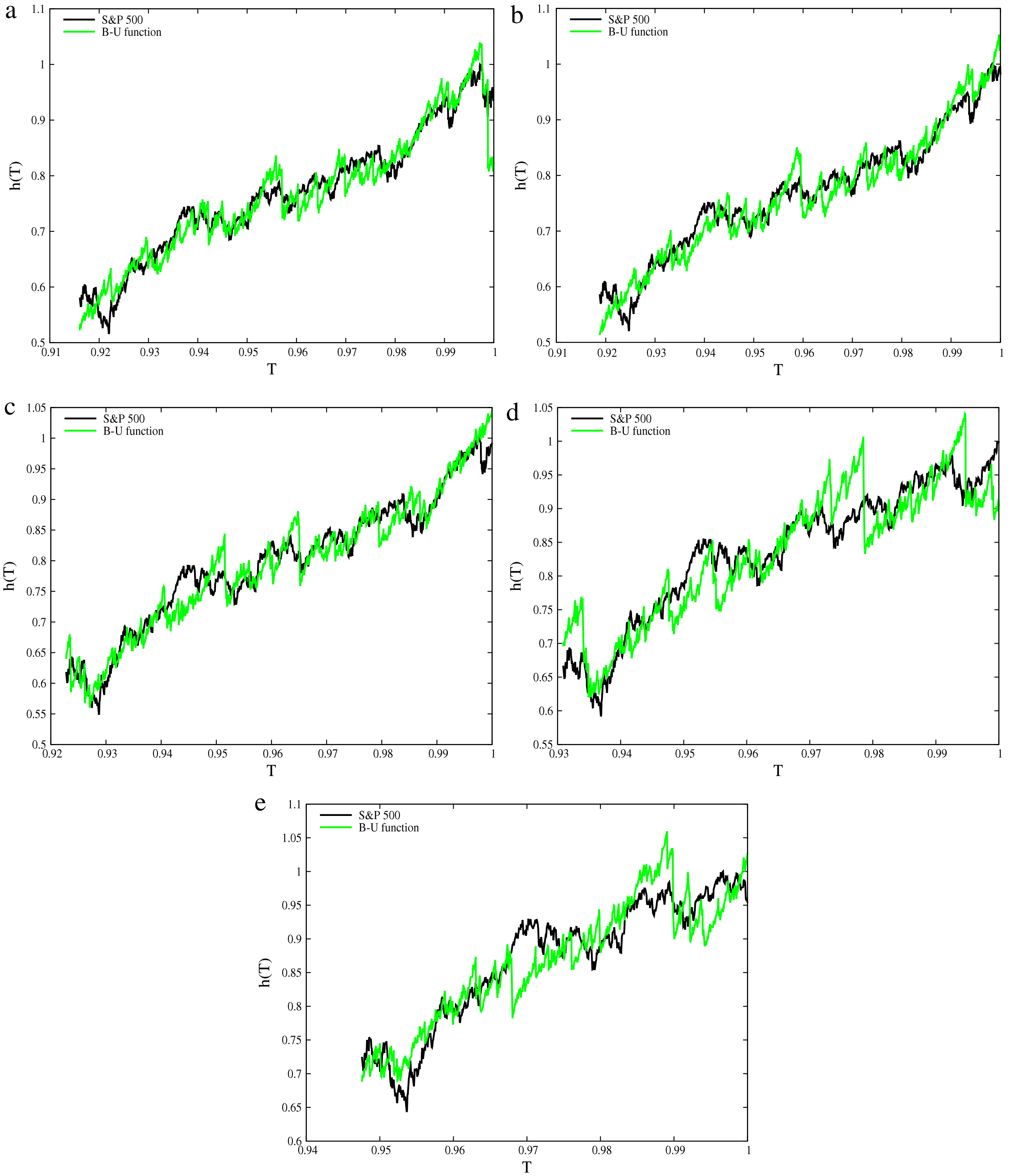


Fig. 4. Fit of the fractal Besicovitch–Ursell function (green, gray line) to the S&P 500 data (black line) for the financial growth preceding the 2007 crash with initial date fixed at 6 November 2002 and the final date: (a) one month before the peak; (b) three months before the peak; (c) six months before the peak; (d) one year before the peak; and (e) two years before the peak. The final time and the maximal index value are normalized to unity. The time is counted in days from 3 January 1950. The r.m.s. error values are: 0.0300, 0.0288, 0.0278, 0.0411 and 0.0392 for (a), (b), (c), (d) and (e), respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Therefore, this algorithm is suitable for fitting with functions that are continuous and nowhere differentiable. Fitting parameters $x_M, y_M, y_N \in (0, 1), p_c > 1$ and $s_0 \in (0, 1)$ in the fitting function (3) are obtained for various intervals of the S&P 500 data corresponding to financial growths.

Table 2
Values of the fitting parameters obtained by fitting the B–U function to the S&P 500 data for the growth ending in 2007, for various time intervals ending before the peak. End of the time window used for the fitting is given in the first column, and the normalizing factor K is given in the second column of the table. The last column contains the r.m.s. errors of the fits.

T_f	K	x_M	y_M	y_N	p_c	s_0	r.m.s.
$21\,100 \times 0.995 = 20\,994$	0.995	0.92173	0.29982	0.43282	1.8184	0.49287	0.0250
$21\,100 \times 0.99 = 20\,889$	0.99	0.92161	0.30979	0.44285	1.8183	0.49017	0.0271
$21\,100 \times 0.98 = 20\,678$	0.98	0.92148	0.30985	0.45311	1.8184	0.49002	0.0314
$21\,100 \times \mathbf{0.98} = 20\,678$	**0.97**	0.89022	0.30734	0.48888	1.8290	0.49015	0.0424

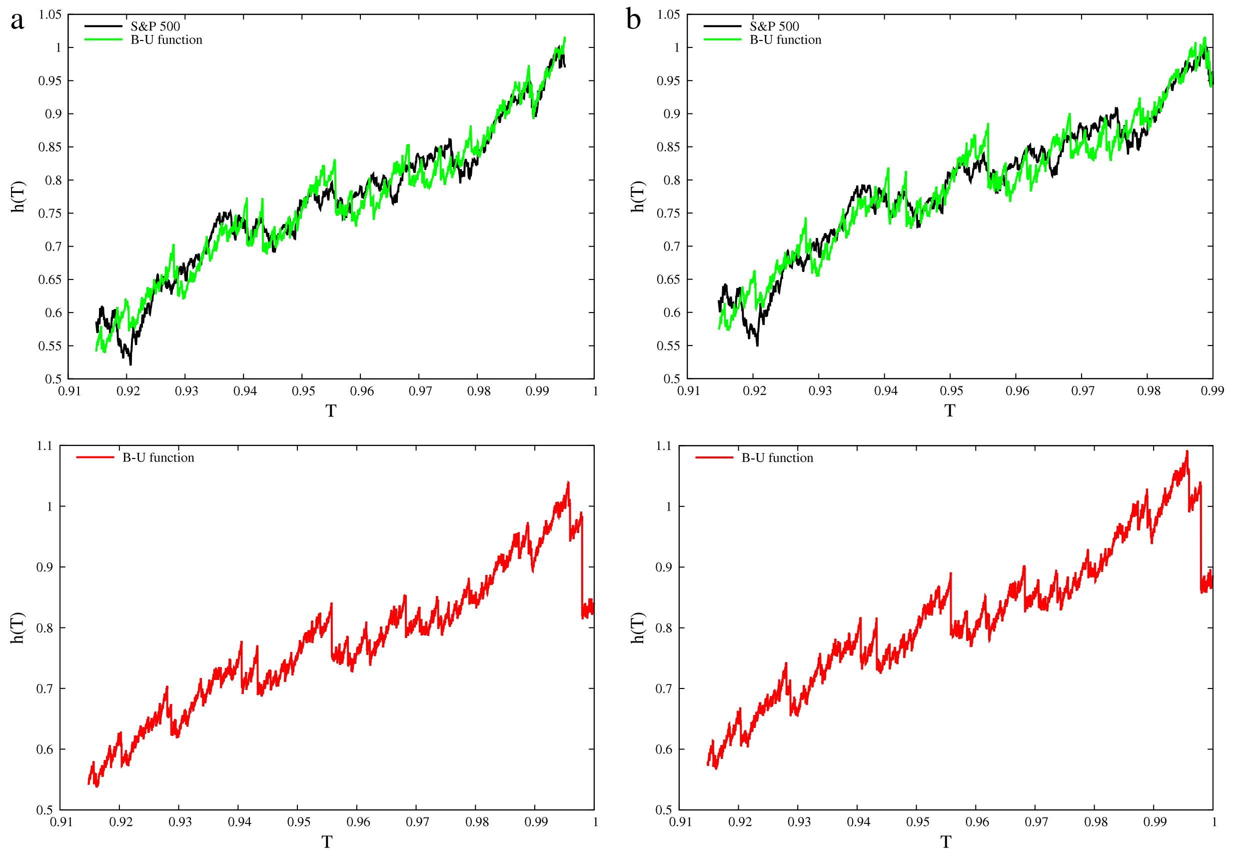
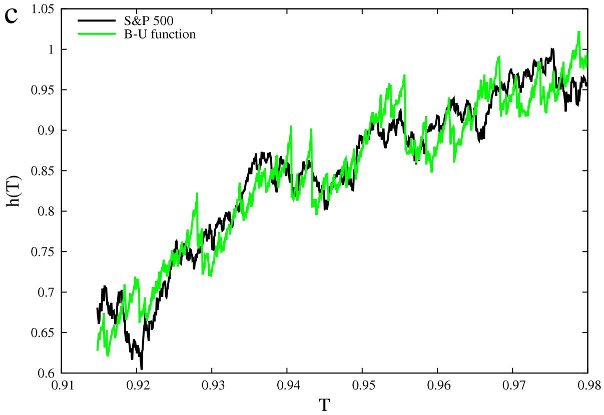


Fig. 5. Fit of the fractal Besicovitch–Ursell function (green, gray line) to the S&P 500 data (black line) for the financial growth preceding the 2007 crash (upper graphs) with the initial date fixed at 6 November 2002 and final dates taken to be at: (a) $\Delta T = 106$ days before the peak, normalized to $K = 0.995$; (b) $\Delta T = 211$ days before the peak, normalized to $K = 0.99$; and (c) $\Delta T = 422$ days before the peak, normalized to $K = 0.98$. Here the values of $(1 - K) \times 21\,100$ precisely match the number of days before the peak. Lower graphs are showing the fractal function extended to the time $T = 1$ (red line), giving a prediction for the days remaining to the known peak. Values of the r.m.s. errors of the fits are given in Table 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3. Results and discussion

The next step is the fitting of the Besicovitch–Ursell fractal function to the real financial data. As mentioned in the introduction, we have investigated the growths of the S&P 500 index having peaks in 1987, 2000 and 2007. The best fits of the B–U fractal function to these data are shown in Fig. 3, with fitting parameters given in Table 1. The time scale is normalized in such a way that the final time corresponds to unity. (The days are counted from 3 January 1950, but it should be mentioned that the shape of the fitting function and the results are not affected by the starting point from which the data are available.) Despite the complexity of the financial data, there is a remarkable agreement between this function and the real data. The fitting function exhibits a great flexibility in following the fluctuations, reaching maxima and minima at the same times as the S&P 500 data. Nevertheless, the model function (Eq. (1)) is always of the same shape, shown in the insets of figures, and the fitting parameters have very similar values for all investigated crisis.

It should be stressed that the graphs given in Fig. 3 comprise the complete financial growths starting at the threshold of the abrupt growth and ending at the peak. In the whole range selected this way, the B–U fractal function reconstructs the



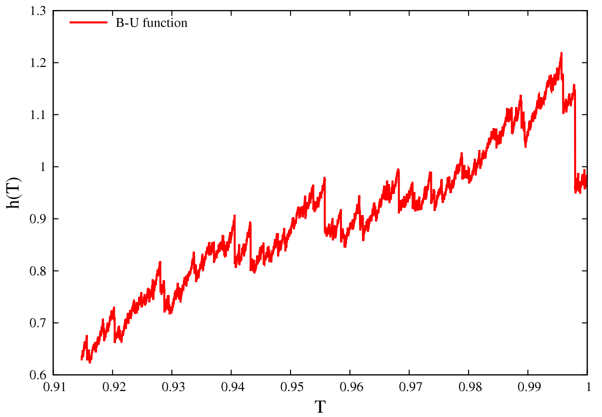
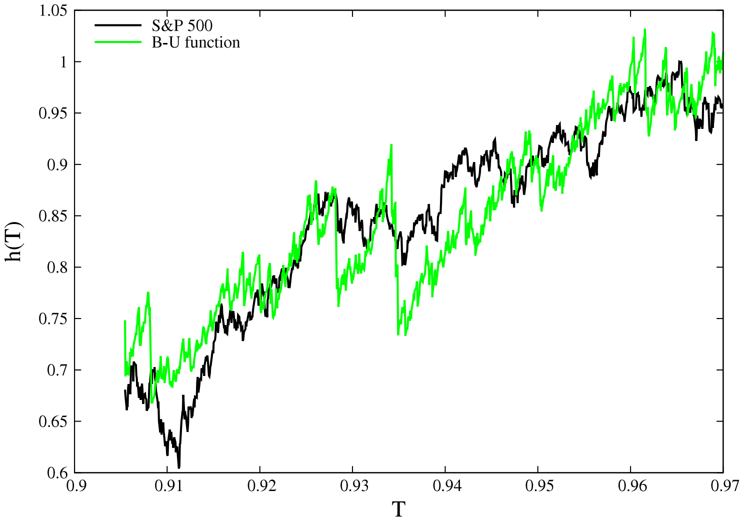


Fig. 5. (*continued*)

Table 3
Values of the parameters of the B–U fitting function for the current financial growth in the interval 21 622–23 271 (16 March 2009–20 September 2013) for various values of the normalizing factor K. Fitting parameters are here limited to the intervals obtained for previous crisis: $x_M \in (0.80, 0.94)$, $y_M \in (0.09, 0.31)$, $y_N \in (0.28, 0.62)$, $p_c \in (1.80, 2.00)$, $s_0 \in (0.49, 0.70)$. The last column contains the r.m.s. errors of the fits.

K	x_M	y_M	y_N	p_c	s_0	r.m.s.
1.0	0.92615	0.27397	0.52983	1.8162	0.49003	0.0346
0.997	0.91429	0.25837	0.53143	1.8170	0.49043	0.0352
0.995	0.92457	0.27134	0.54375	1.8176	0.49000	0.0333
0.993	0.92085	0.26470	0.54792	1.8180	0.49000	0.0328
0.99	0.92191	0.27346	0.54233	1.8194	0.49015	0.0337
0.985	0.91706	0.25424	0.53481	1.8207	0.49000	0.0349
0.98	0.91791	0.28022	0.53272	1.8220	0.49005	0.0387
0.97	0.90584	0.28059	0.61766	1.8253	0.50406	0.0363
0.96	0.89547	0.29289	0.56215	1.8361	0.49002	0.0487
0.95	0.88831	0.27674	0.55898	1.8481	0.49012	0.0555
0.94	0.88926	0.30255	0.49531	1.8549	0.49001	0.0683
0.93	0.89617	0.30400	0.58494	1.8004	0.49010	0.0790

market trends in a natural way. It is interesting to see how would shortening of the intervals affect the fits. Fig. 4 shows the fits of the B–U fractal function to the S&P 500 data for the financial growth preceding the 2007 crash, but with the final date taken to be (a) one month, (b) three months, (c) six months, (d) one year and (e) two years before the peak, while the initial date is fixed. We can see (Fig. 4(a)) that when the time window ends one month before the peak, the fitting function follows the financial data in a realistic way, very likely to the case of the complete growth. When the final time for the fitting is moving towards earlier dates, the maxima and minima of the fitting function are more and more shifted with regard to the maxima and minima of the real data. When the final data for fitting is taken one and two years before the peak, as in Fig. 4(d) and (e) respectively, there is an obvious discrepancy between the two curves. The root-mean-square (r.m.s.) error values of these fits are given in the figure caption. Smaller values are obtained for the fits that comprise nearly the complete financial growth, while for the time-windows ending a year or more before the peak there is a noticeable increase of the r.m.s. errors.



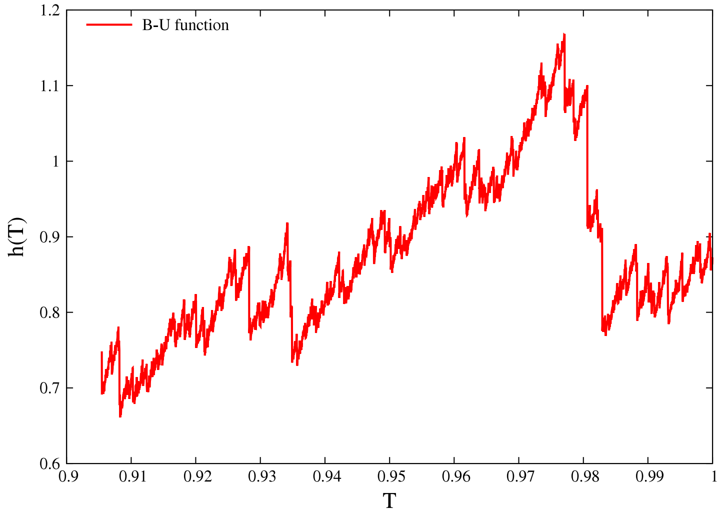


Fig. 6. Fit of the fractal Besicovitch–Ursell function (green, gray line) to the S&P 500 data (black line) for the financial growth preceding the 2007 crash (upper graphs) with the initial date fixed at 6 November 2002. The final time is taken to be $\Delta T = 422$ before the peak, corresponding to $K = 0.98$, but it is normalized to 0.97. Lower graph is showing the fractal function extended to the time $T = 1$ (red line). Value of the r.m.s. error of the fit is given in Table 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The same analysis has been carried out for the other two financial growths, ending in 1987 and 2000, and similar results have been obtained. Best fits of the B–U function are obtained for complete financial growths. Moving the final time towards earlier times causes growing deviations between the two curves.

In order to investigate the possibility of making predictions on the basis of fits of the B–U fractal functions to the financial data, we present some further analysis of the growth preceding the 2007 crash. Namely, speaking in days, the complete growth was fitted from the initial date $T_i = 19\,300$ (6 November 2002) till the final date $T_f = 21\,100$ (11 October 2007) and the time scale was normalized in such a way that the final time corresponds to unity. Now we are going to fit this financial growth from the same initial time, to the final time $T_f < 21\,100$. We can imagine that T_f is the last point of the S&P 500 curve that is known and we want to extend it to the future. The final date is now determined as $T_f = K * 21\,100$, with $0 < K < 1$. In the fitting procedure the final time is normalized to the value of constant K. Let us first consider three cases with different T_f obtained by taking $K = 0.995, 0.99$, and 0.98. These final times correspond to the dates $\Delta T = 106$, 211, and 422 days before the peak. Fits obtained this way are illustrated in Fig. 5(a)–(c). Upper graphs show the fits to the data, and the lower graphs give the fitting function extended to unity, i.e. extended for the corresponding number of days. Extension of the fitting function enables determining the time when the peak appears. It can be seen that all the fitting functions from Fig. 5 have the same shape. Moreover, they have the same shape as the fitting function from Fig. 3(c), showing the fit of the complete growth. They are predicting the end of the growth at the same time and this prediction is in a good agreement with the known behavior of the S&P 500 index. This is the case only if the value of the factor K accurately determines the time distance to the end of the growth. On the other hand, if the value assigned to the normalizing factor is not correct, it

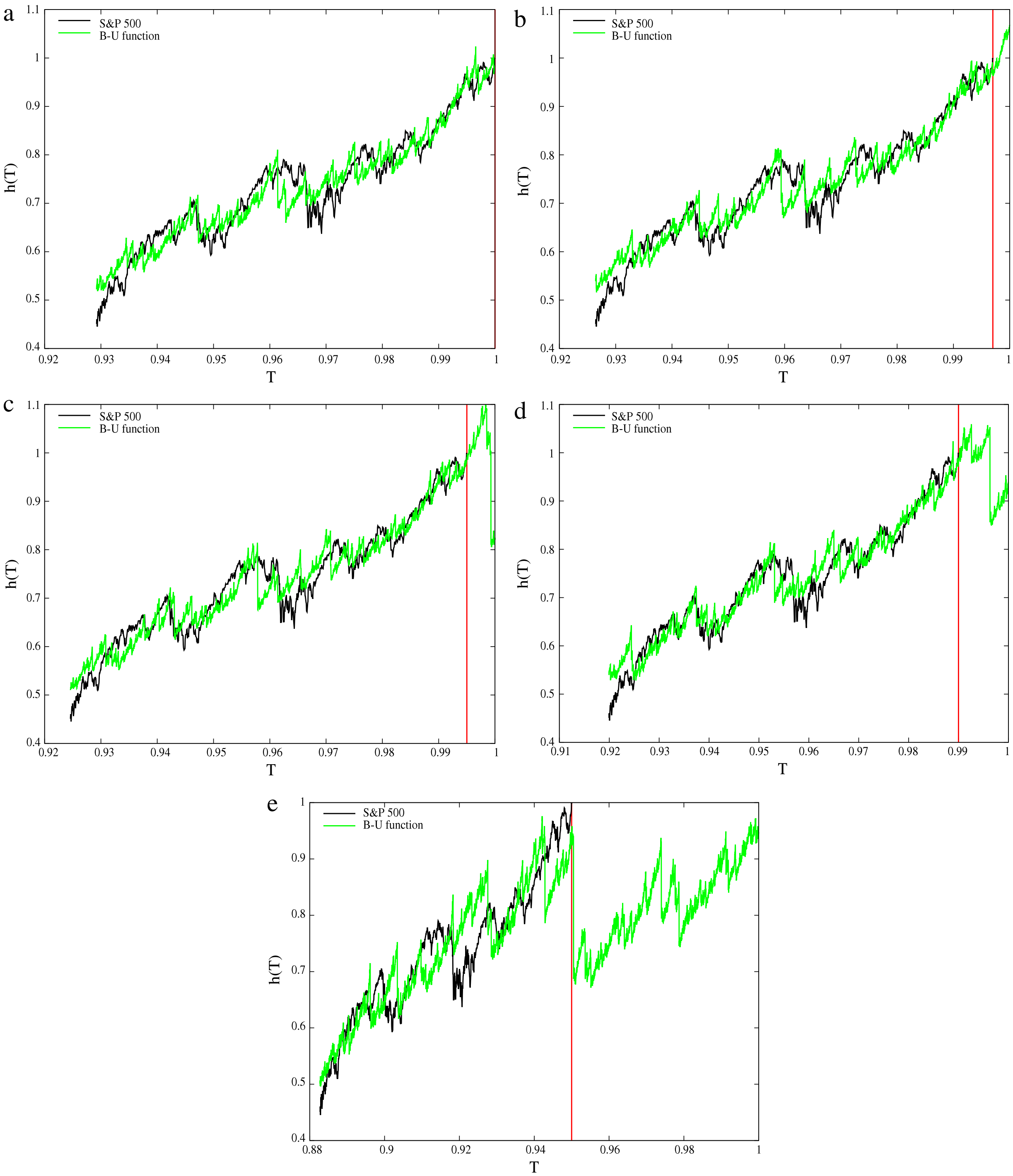


Fig. 7. Fit of the fractal Besicovitch–Ursell function (green, gray line) to the S&P 500 data (black line) for the current financial growth normalized to: (a) $K = 1$; (b) $K = 0.997$; (c) $K = 0.995$; (d) $K = 0.99$; and (e) $K = 0.95$. The fitting function (3) is extended to the time $T = 1$, giving a prediction for the remaining $\Delta T = \frac{1-K}{K} \times 23\,271$ days (part of the curve to the right of the vertical line). Values of the r.m.s. errors of the fits are given in Table 3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

is not possible to obtain good fits and the predictions are also wrong, as illustrated in Fig. 6. Here the final time is taken to be $\Delta T = 422$ before the peak, corresponding to $K = 0.98$, but it was normalized to 0.97. We can notice that there are enhanced deviations between the fitting function and the financial data. Values of the r.m.s. errors given in Table 2 confirm the difference in the quality of the fits shown in Figs. 5 and 6.

Values of the fitting parameters for the fits from Figs. 5 and 6 are given in Table 2. When the factor K accurately determines the time distance to the end of the growth, there is a striking similarity between the parameters of the fits obtained for various values of K.

It would be interesting to apply this procedure to the current financial growth and to see what predictions could be made. Here we present the fits of the S&P 500 data starting at the minimum reached during the preceding crash (16 march 2009) and ending with 20 September 2013. Fitting parameters are limited to the intervals obtained for previous crisis (Table 1). We have examined the fits for various values of normalizing factor K and the values of the obtained fitting parameters are given in Table 3. Normalizing the last date to $K < 1$ assumes that the peak is not reached. What we are searching for is the best fit that can be obtained. Quality of the fits is estimated on the basis of the number of maximum and minimum matches and also on the calculated r.m.s. errors of the fits given in Table 3. Extending the fitting function to unity allows drawing some conclusions about the future behavior of the S&P 500 values. Value of the normalizing factor enables the estimation of time remaining to the end of the growth. Fig. 7(a)–(e) show the fits obtained for the normalizing factor $K = 1, 0.997, 0.995, 0.99$, and 0.95. Vertical lines correspond to the time of the last available data. Green (gray) lines extended to unity show the predicted behavior of the financial index.

Fitting functions describing the financial growths preceding the 1987, 2000 and 2007 crashes display a peak that is at the same place on the time-scale as the maximum of the index value reached during the growth. This peak is followed by a sharp decrease corresponding to the burst. Analysis of the fits of the current financial growth, by taking into account the number of maximum and minimum matches and the r.m.s. errors of the fits, suggests that the growth is in its late stage. Comparing with the results obtained for the previous financial bubbles, it could be estimated that the end of the growth is a matter of months rather than years.

4. Concluding remarks

Presented analysis confirms that the financial markets self-organize themselves to produce deviations from market equilibrium—the bubbles followed by financial crashes in such a way that the development of these bubbles always follows essentially the same route.

We have demonstrated that the fits of the simple fractal function we used can reproduce complicated behavior of the S&P 500 index in a natural way. When the fitting interval comprises the complete growth, there is a remarkable coincidence between the fitting function and the real data. Moving the final data backward in time, causes growing deviations between these two curves. However, if the data ending a certain period before the peak are normalized by the corresponding factor, the coincidence is recovered. Applying this procedure to the data for the current financial growth and searching for the value of the normalizing factor that gives the best fit, it is possible to make some predictions of the behavior of financial indexes and to estimate the time when the index value begins to fall.

Future investigation will be focused on improving the fitting procedure and refining the predictions. We also intend to apply this procedure to anti-bubbles, i.e. falling of the financial index during the financial crisis. Other stock indexes such as DAX, NASDAQ etc., could also be analyzed using fractal functions.

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