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DREENA-B framework: First predictions of R_{AA} and v_2 within dynamical energy loss formalism in evolving QCD medium

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ABSTRACT

Dynamical energy loss formalism allows generating state-of-the-art suppression predictions in finite size QCD medium, employing a sophisticated model of high-$p_\perp$ parton interactions with QGP. We here report a major step of introducing medium evolution in the formalism though $1+1D$ Bjorken ("B") expansion, while preserving all complex features of the original dynamical energy loss framework. We use this framework to provide joint R_{AA} and v_2 predictions, for the first time within the dynamical energy loss formalism in evolving QCD medium. The predictions are generated for a wide range of high $p_\perp$ observables, i.e. for all types of probes (both light and heavy) and for all centrality regions in both $Pb+Pb$ and $Xe+Xe$ collisions at the LHC. Where experimental data are available, DREENA-B framework leads to a good joint agreement with v_2 and R_{AA} data. Such agreement is encouraging, i.e. may lead us closer to resolving v_2 puzzle (difficulty of previous models to jointly explain R_{AA} and v_2 data), though this still remains to be thoroughly tested by including state-of-the-art medium evolution within DREENA framework. While introducing medium evolution significantly changes v_2 predictions, R_{AA} predictions remain robust and moreover in a good agreement with the experimental data; R_{AA} observable is therefore suitable for calibrating parton-medium interaction model, independently from the medium evolution. Finally, for heavy flavor, we observe a strikingly similar signature of the dead-cone effect on both R_{AA} and v_2 - we also provide a simple analytical understanding behind this result. Overall, the results presented here indicate that DREENA framework is a reliable tool for QGP tomography.

1. Introduction

It is by now established that quark-gluon plasma (QGP), being a new state of matter [1,2] consisting of interacting quarks, antiquarks and gluons, is created in ultra-relativistic heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Energy loss of rare high $p_\perp$ particles, which are created in such collisions and which transverse QGP, is considered to be an excellent probe of this form of matter [3–6]. Such energy loss is reflected through different observables, most importantly angular averaged (R_{AA}) [7–14] and angular differential (v_2) [15–22] nuclear modification factor, which can be measured and predicted for both light and heavy flavor probes. Therefore, comparing a comprehensive set of predictions, created under the same model and parameter set, with the corresponding experimental data, allows for systematical investigation of QCD medium properties, i.e. QGP tomography.

We previously showed that the dynamical energy loss formalism [23–25] provides an excellent tool for such tomography. In particular, we demonstrated that the formalism shows a very good agreement [27–30] with a wide range of R_{AA} data, coming from different experiments, collision energies, probes and centralities. Recently, we also used this formalism to generate first v_2 predictions, within DREENA-C framework [26], where DREENA stands for Dynamical Radiative and Elastic ENergy loss Approach, and "C" denotes constant temperature QCD medium. These predictions were compared jointly with R_{AA} and v_2 data, showing a very good agreement with R_{AA} data, while visibly overestimating v_2 data. This overestimation also clearly differentiates the dynamical energy loss from other models, which systematically underestimated the v_2 data, leading to the so called v_2 puzzle [31–33]. On the other hand, it is also clear that v_2 predictions have to be further improved - in particular v_2 was shown to be sensitive to medium evolution, while in DREENA-C medium evolution

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was introduced in the simplest form, through constant medium temperature. This problem then motivated us to introduce medium evolution in DREENA framework.

While several existing energy loss approaches already contain a sophisticated medium evolution, they employ simplified parton energy loss models. On the other hand, our dynamical energy loss formalism corresponds to the opposite "limit", where constant (mean) medium temperature was assumed, combined with a sophisticated model of parton-medium interactions, which includes: *i)* QCD medium composed of dynamical (i.e. moving) scattering centers, which is contrary to the widely used static scattering centers approximation, *ii)* finite size QCD medium, *iii)* finite temperature QCD medium, modeled by generalized HTL approach [34,35], naturally regularizing all infrared and ultraviolet divergencies [23–25,36]. *iv)* collisional [25] and radiative [23] energy losses, calculated within the same theoretical framework, *v)* finite parton mass, making the formalism applicable to both light and heavy flavor, *vi)* finite magnetic [37] mass and running coupling [27].

Note that we have previously showed that all the ingredients stated above are important for accurately describing experimental data [38]. Consequently, introducing medium evolution in the dynamical energy loss, is a major step in the model development, as all components in the model have to be preserved, and no additional simplifications should be used in the numerical procedure. In addition to developing the energy loss expressions with changing temperature, we also wanted to develop a framework that can efficiently generate a set of predictions for all types of probes and all centrality regions. That is, we think that for a model to be realistically compared with experimental data, the comparison should be done for a comprehensive set of light and heavy flavor experimental data, through the same numerical framework and the same parameter set. To implement this principle, we also had to develop a numerical framework that can efficiently (i.e. in a reasonably short time frame) generate such predictions, which will be presented in this paper.

We will start the task of introducing the medium evolution in the dynamical energy loss formalism with DREENA-B framework presented here, where "B" stands for Bjorken. In this framework, QCD medium is modeled by the ideal hydrodynamical $1+1D$ Bjorken expansion [39], which has a simple analytical form of temperature (T) dependence. This simple T dependence will be used as an intermediate between constant (mean) temperature DREENA-C framework and the full evolution QGP tomography tool. While, on one hand, inclusion of Bjorken expansion in DREENA framework is a major task (having in mind complexity of our model, see above), it on the other hand significantly simplifies the numerical procedure compared to full medium evolutions. This will then allow step-by-step development of full QGP tomography framework, and assessing improvements in the predictions when, within the same theoretical framework, one is transitioning towards more complex QGP evolution models within the dynamical energy loss framework.

2. Computational framework

To calculate the quenched spectra of hadrons, we use the generic pQCD convolution, while the assumptions are provided in [27]:

$$\frac{E_f d^3\sigma}{dp_f^3} = \frac{E_i d^3\sigma(Q)}{dp_i^3} \otimes P(E_i \to E_f) \otimes D(Q \to H_Q) \otimes f(H_Q \to e, J/\psi), \tag{1}$$

where "i" and "f", respectively, correspond to "initial" and "final", Q denotes quarks and gluons (partons). $E_i d^3\sigma(Q)/dp_i^3$ denotes the initial parton spectrum, computed at next to leading order [40] for light and heavy partons. $D(Q \to H_Q)$ is the fragmentation function of parton Q to hadron H_Q; for charged hadrons, D and B mesons we use DSS [41], BCFY [42] and KLP [43] fragmentation functions, respectively. $P(E_i \to E_f)$ is the energy loss probability, generalized to include both radiative and collisional energy loss in a realistic finite size dynamical QCD medium in which the temperature is changing, as well as running coupling, path-length and multi-gluon fluctuations. In below expressions, running coupling is introduced according to [27], where the temperature T now changes with proper time τ; the temperature dependence along the jet path is taken according to the ideal hydrodynamical $1+1D$ Bjorken expansion [39]. Partons travel different paths in the QCD medium, which is taken into account through path length fluctuations [44]. Multi-gluon fluctuations take into account that the energy loss is a distribution, and are included according to [27,45] (for radiative energy loss) and [44,46] (for collisional energy loss).

The dynamical energy loss formalism was originally developed for constant temperature QCD medium, as described in detail in [23–25]. We have now derived collisional and radiative energy loss expressions for the medium in which the temperature is changing along the jet path; detailed calculations will be presented elsewhere, while the main results are summarized below.

For the collisional energy loss, we obtain the following analytical expression:

$$\frac{dE_{col}}{d\tau} = \frac{2C_R}{\pi v^2}\alpha_S(ET)\alpha_S(\mu_E^2(T)) \int_0^\infty n_{eq}(|\vec{\mathbf{k}}|, T)d|\vec{\mathbf{k}}| \left(\int_0^{|\vec{\mathbf{k}}|/(1+v)} d|\vec{\mathbf{q}}| \int_{-v|\vec{\mathbf{q}}|}^{v|\vec{\mathbf{q}}|} \omega d\omega + \int_{|\vec{\mathbf{k}}|/(1+v)}^{|\vec{\mathbf{q}}|_{max}} d|\vec{\mathbf{q}}| \int_{|\vec{\mathbf{q}}|-2|\vec{\mathbf{k}}|}^{v|\vec{\mathbf{q}}|} \omega d\omega \right) \left(|\Delta_L(q,T)|^2 \frac{(2|\vec{\mathbf{k}}|+\omega)^2 - |\vec{\mathbf{q}}|^2}{2} + |\Delta_T(q,T)|^2 \frac{(|\vec{\mathbf{q}}|^2-\omega^2)((2|\vec{\mathbf{k}}|+\omega)^2+|\vec{\mathbf{q}}|^2)}{4|\vec{\mathbf{q}}|^4}(v^2|\vec{\mathbf{q}}|^2-\omega^2) \right). \tag{2}$$

Here E is initial jet energy, τ is the proper time, T is the temperature of the medium, α_S is running coupling [27] and $C_R = \frac{4}{3}$. k is the 4-momentum of the incoming medium parton, v is the velocity of the incoming jet and $q = (\omega, \vec{\mathbf{q}})$ is the 4-momentum of the gluon. $n_{eq}(|\vec{\mathbf{k}}|, T) = \frac{N}{e^{|\vec{\mathbf{k}}|/T}-1} + \frac{N_f}{e^{|\vec{\mathbf{k}}|/T}+1}$ is the equilibrium momentum distribution [47] at temperature T including quarks and gluons (N and N_f are the number of colors and flavors, respectively). $\Delta_L(T)$ and $\Delta_T(T)$ are effective longitudinal and transverse gluon propagators [48]:

$$\Delta_L^{-1}(T) = \vec{\mathbf{q}}^2 + \mu_E(T)^2(1 + \frac{\omega}{2|\vec{\mathbf{q}}|}\ln|\frac{\omega-|\vec{\mathbf{q}}|}{\omega+|\vec{\mathbf{q}}|}|), \tag{3}$$

$$\Delta_T^{-1}(T) = \omega^2 - \vec{\mathbf{q}}^2 - \frac{\mu_E(T)^2}{2} - \frac{(\omega^2-\vec{\mathbf{q}}^2)\mu_E(T)^2}{2\vec{\mathbf{q}}^2}(1 + \frac{\omega}{2|\vec{\mathbf{q}}|}\ln|\frac{\omega-|\vec{\mathbf{q}}|}{\omega+|\vec{\mathbf{q}}|}|), \tag{4}$$

while the electric screening (the Debye mass) $\mu_E(T)$ can be obtained by self-consistently solving the expression [49] (n_f is number of the effective degrees of freedom, Λ_{QCD} is perturbative QCD scale):

$$\frac{\mu_E(T)^2}{\Lambda_{QCD}^2}\ln\left(\frac{\mu_E(T)^2}{\Lambda_{QCD}^2}\right)=\frac{1+n_f/6}{11-2/3\,n_f}\left(\frac{4\pi T}{\Lambda_{QCD}}\right)^2. \tag{5}$$

The gluon radiation spectrum takes the following form:

$$\begin{aligned}\frac{dN_{\rm rad}}{dx d\tau}=&\int\frac{d^2k}{\pi}\frac{d^2q}{\pi}\frac{2\,C_R C_2(G)\,T}{x}\frac{\alpha_s(E\,T)\,\alpha_s(\frac{\mathbf{k}^2+\chi(T)}{x})}{\pi}\\&\times\frac{\mu_E(T)^2-\mu_M(T)^2}{(\mathbf{q}^2+\mu_M(T)^2)(\mathbf{q}^2+\mu_E(T)^2)}\\&\times\left(1-\cos\frac{(\mathbf{k}{+}\mathbf{q})^2+\chi(T)}{xE^+}\,\tau\right)\frac{(\mathbf{k}{+}\mathbf{q})}{(\mathbf{k}{+}\mathbf{q})^2+\chi(T)}\\&\times\left(\frac{(\mathbf{k}{+}\mathbf{q})}{(\mathbf{k}{+}\mathbf{q})^2+\chi(T)}-\frac{\mathbf{k}}{\mathbf{k}^2+\chi(T)}\right),\end{aligned} \tag{6}$$

where $C_2(G)=3$ and $\mu_M(T)$ is magnetic screening. $\mathbf{k}$ and $\mathbf{q}$ are transverse momenta of radiated and exchanged (virtual) gluon, respectively. $\chi(T)\equiv M^2x^2+m_E(T)^2/2$, where x is the longitudinal momentum fraction of the jet carried away by the emitted gluon, M is the mass of the quark or gluon jet and $m_g(T)=\mu_E(T)/\sqrt{2}$ is effective gluon mass in finite temperature QCD medium [36]. We also recently abolished the soft-gluon approximation [50], for which we however showed that it does not significantly affect the model results; consequently, this improvement is not included in DREENA-B, but can be straightforwardly implemented in the future DREENA developments, if needed.

Note that, as a result of introducing medium evolution, the dynamical energy loss formalism now explicitly contains changing temperature in the energy loss expression. This is contrary to most of the other models, in which temperature evolution is introduced indirectly, through transport coefficient $\hat{q}$ or gluon rapidity density $\frac{dN_g}{dy}$ (see [51] and references therein). This feature makes the dynamical energy loss a natural framework to incorporate diverse temperature profiles as a starting point for QGP tomography. As the first (major) step, we will below numerically implement this possibility through Bjorken $1+1D$ expansion [39].

Regarding the numerical procedure, computation efficiency of the algorithm implemented in DREENA-C framework [26] was already two orders of magnitude higher with respect to the basic (unoptimized) brute-force approach applied in [27]. However, straightforward adaptation of the DREENA-C code to the case of the Bjorken type evolving medium was not sufficient. This was dominantly due to additional integration over proper time τ, which increased the calculation time for more than two orders of magnitude. The computation of e.g. radiative energy losses alone, for a single probe, took around 10 hours on the available computer resources (a high performance workstation). Taking into account that it requires $\sim 10^3$ such runs to produce the results presented in this paper, it is evident that a substantial computational speedup was necessary.

The main algorithmic tool that we used to optimize the calculation was a combination of sampling and tabulating various intermediate computation values and their subsequent interpolation. We used nonuniform adaptive grids of the sampling points, denser in the parts of the parameter volume where the sampled function changed rapidly. Similarly, the parameters used for the numerical integration (the number of quasi-Monte Carlo sampling points and the required accuracy) were also suitably varied throughout the parameter space. Finally, while the computation in DREENA-C was performed in a software for symbolic computation, the new algorithm was redeveloped in C programming language. The combined effect of all these improvements was a computational speedup of almost three orders of magnitude, which was a necessary prerequisite for both current practical applicability and future developments of DREENA framework.

Regarding the parameters, we implement Bjorken $1+1D$ expansion [39], with commonly used $\tau_0=0.6$ fm [52,53], and initial temperatures for different centralities calculated according to $T_0\sim(dN_{ch}/dy/A_\perp)^{1/3}$ [54], where dN_{ch}/dy is charged multiplicity and $A_\perp$ is overlap area (based on the Glauber model nuclear overlap function) for specific collision system and centrality. We use this equation, starting from $T_0=500$ MeV in 5.02 TeV $Pb+Pb$ most central collisions at the LHC, which is estimated based on average medium temperature of 348 MeV in these collisions, and QCD transition temperature of $T_c\approx 150$ MeV [55]. Note that the average medium temperature of 348 MeV in most central 5.02 TeV $Pb+Pb$ collisions comes from [28] the effective temperature (T_{eff}) of 304 MeV for 0-40% centrality 2.76 TeV Pb+Pb collisions at the LHC [56] experiments (as extracted by ALICE). Once T_0s for most central $Pb+Pb$ collisions are fixed, T_0 for both different centralities and different collision systems ($Xe+Xe$ and $Pb+Pb$) are obtained from the expression above.

Other parameters used in the calculation remain the same as in DREENA-C [26]. In particular, the path-length distributions for both $Xe+Xe$ and $Pb+Pb$ are calculated following the procedure described in [57], with an additional hard sphere restriction $r<R_A$ in the Woods-Saxon nuclear density distribution to regulate the path lengths in the peripheral collisions. Note that the path-length distributions for $Pb+Pb$ are explicitly provided in [26]; we have also checked that, for each centrality, our obtained eccentricities remain within the standard deviation of the corresponding Glauber Monte Carlo results [58]. For $Xe+Xe$, it is straightforward to show that $Xe+Xe$ and $Pb+Pb$ distributions are the same up to rescalling factor ($A^{1/3}$, where A is atomic mass number), as we discussed in [59]. Furthermore, the path-length distributions correspond to geometric quantity, and are therefore the same for all types of partons (light and heavy). For QGP, we take $\Lambda_{QCD}=0.2$ GeV and $n_f=3$. As noted above, temperature dependent Debye mass $\mu_E(T)$ is obtained from [49]. For light quarks and gluons, we, respectively, assume that their effective masses are $M\approx\mu_E(T)/\sqrt{6}$ and $m_g\approx\mu_E(T)/\sqrt{2}$ [36]. The charm and bottom masses are $M=1.2$ GeV and $M=4.75$ GeV, respectively. Magnetic to electric mass ratio is extracted from non-perturbative calculations [60,61], leading to $0.4<\mu_M/\mu_E<0.6$ - this range of screening masses leads to presented uncertainty in the predictions. We note that no fitting parameters are used in the calculations, that is, all the parameters correspond to standard literature values.

3. Results and discussion

In this section, we will present joint R_{AA} and v_2 predictions for light (charged hadrons) and heavy (D and B mesons) flavor in $Pb+Pb$ and $Xe+Xe$ collisions at the LHC, obtained by DREENA-B framework. Based on the path-length distributions from Figure 1 in [26], we will, in Figs. 1 to 2, show R_{AA} and v_2 predictions for light and heavy flavor, in 5.02 TeV $Pb+Pb$ and 5.44 TeV $Xe+Xe$ collisions, at different centralities. We start by presenting charged hadrons predictions, where R_{AA} data are available for both $Pb+Pb$ and $Xe+Xe$, while v_2 data exist for $Pb+Pb$ collisions. Comparison of our joint predictions with experimental data is shown in Fig. 1, where 1*st* and 2*nd* columns correspond, respectively, to R_{AA} and v_2 predictions at $Pb+Pb$, while 3*rd* and 4*th* columns present equivalent predictions/data for $Xe+Xe$ collisions at the LHC. From this figure, we see that DREENA-B is able to well explain joint R_{AA} and v_2 data. For 5.44 TeV $Xe+Xe$ collisions at the LHC, we observe good agreement of our predictions with prelim-

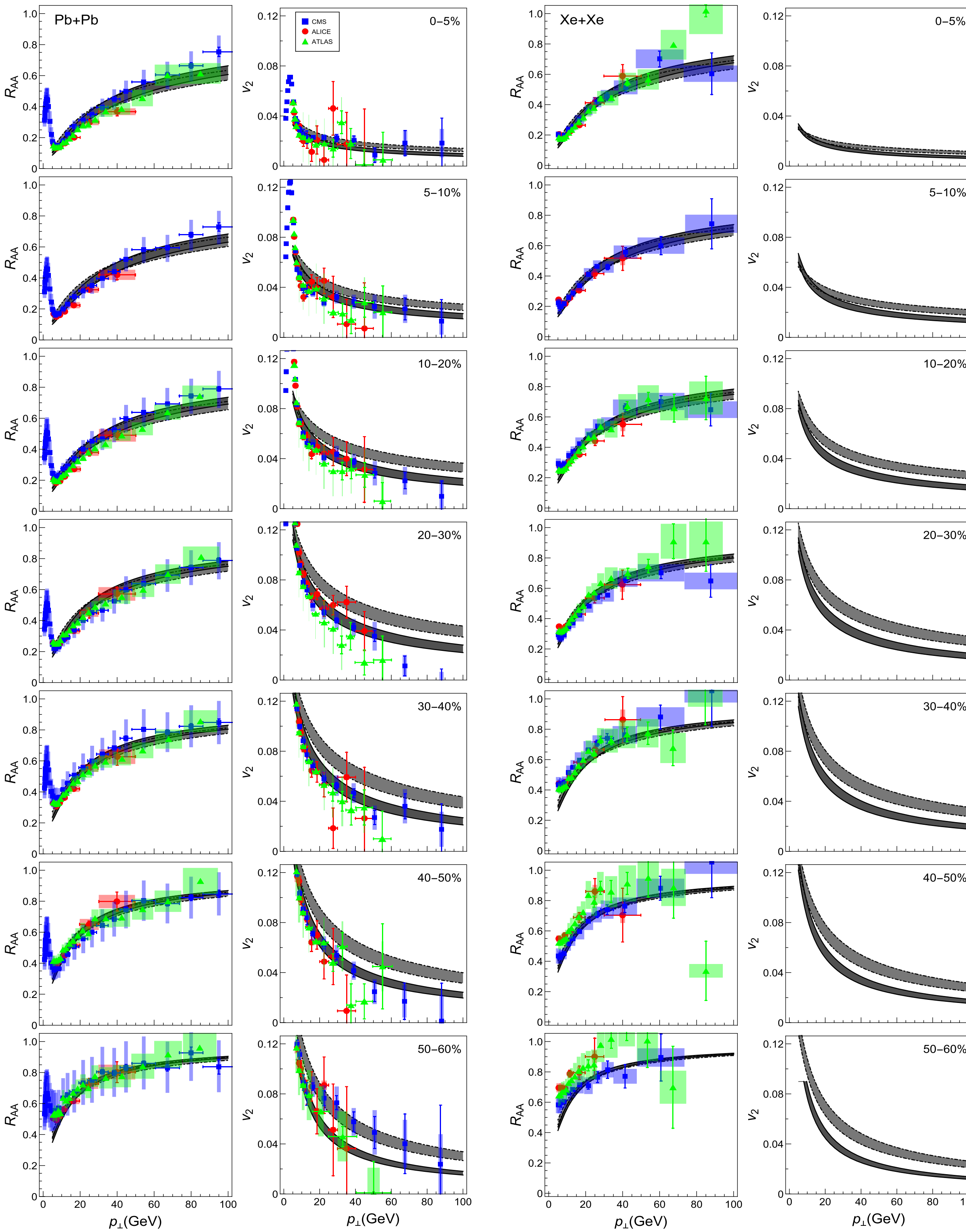

Fig. 1. *First column:* R_{AA} *vs.* $p_\perp$ predictions are compared with 5.02 TeV $Pb+Pb$ ALICE [7], ATLAS [8] and CMS [9] $h^\pm$ experimental data. *Second column:* Equivalent comparison for v_2 *vs.* $p_\perp$ (data [15–17]). *Third column:* R_{AA} *vs.* $p_\perp$ predictions are compared with 5.44 TeV $Xe+Xe$ ALICE [62], ATLAS [63] and CMS [64] preliminary data. *Fourth column:* Equivalent predictions for v_2 *vs.* $p_\perp$. ALICE, ATLAS and CMS data are respectively represented by red circles, green triangles and blue squares, while centrality regions are indicated in the relevant subfigures. Full and dashed curves correspond to, respectively, DREENA-B and DREENA-C frameworks. The gray band boundaries correspond to $\mu_M/\mu_E = 0.4$ and $\mu_M/\mu_E = 0.6$.

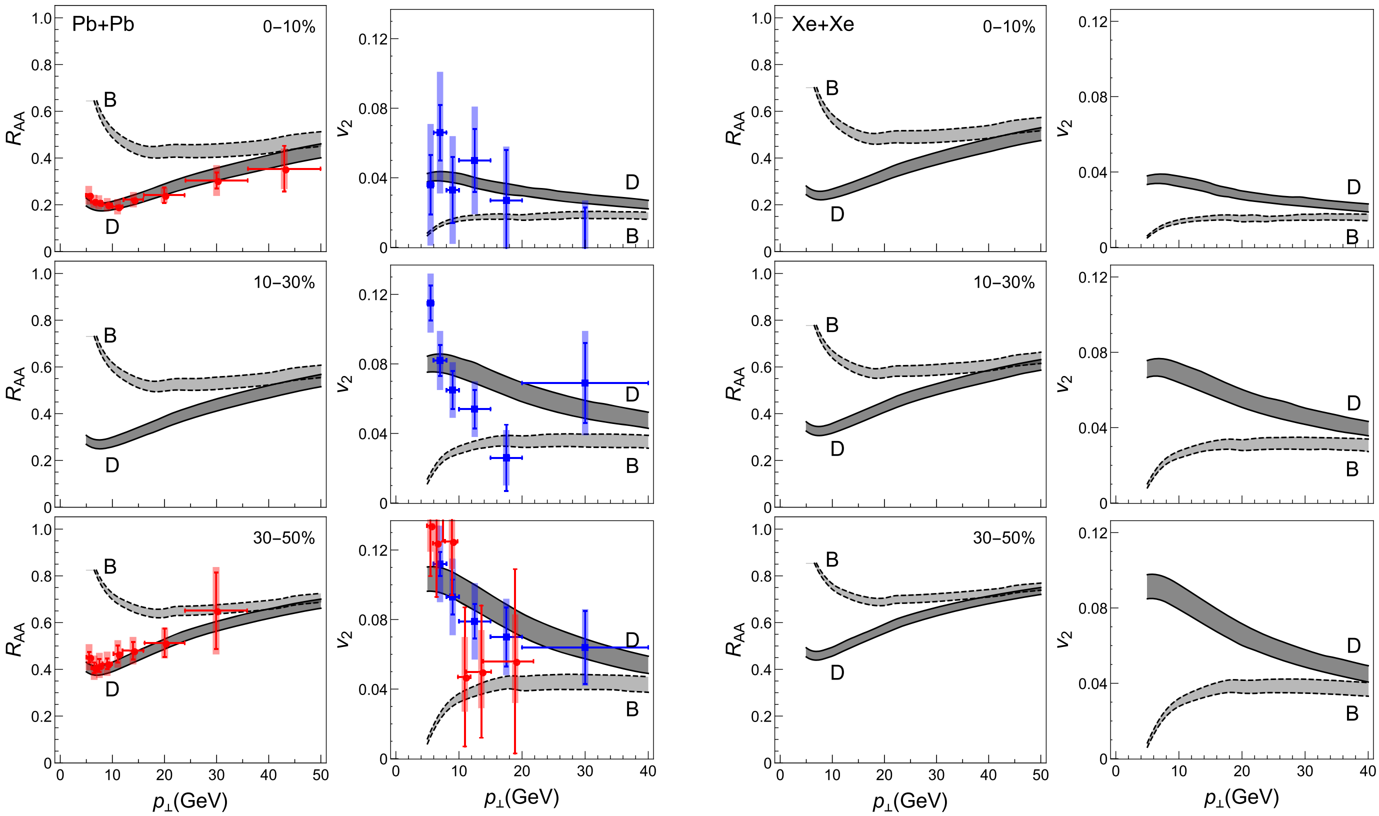

Fig. 2. *First column:* Theoretical predictions for D and B meson R_{AA} *vs.* $p_\perp$ are compared with the available 5.02 TeV $Pb+Pb$ ALICE [10] (red circles) D meson experimental data. *Second column:* v_2 *vs.* $p_\perp$ predictions are compared with 5.02 TeV $Pb+Pb$ ALICE [19] (red circles) and CMS [20] (blue squares) D meson experimental data. *Third and fourth column:* Heavy flavor R_{AA} and v_2 *vs.* $p_\perp$ predictions are, respectively, provided for 5.44 TeV $Xe+Xe$ collisions at the LHC. First to third row, respectively, correspond to $0-10\%$, $10-30\%$ and $30-50\%$ centrality regions. The gray band boundaries correspond to $\mu_M/\mu_E=0.4$ and $\mu_M/\mu_E=0.6$.

inary R_{AA} data from ALICE, ATLAS and CMS data (where we note that these predictions were generated, and posted on arXiv, before the data became available), except for high centrality regions, where our predictions do not agree with ALICE (and also partially with ATLAS) data; however, note that in these regions ALICE, ATLAS and CMS data also do not agree with each other.

Furthermore, comparison of predictions obtained with DREENA-B and DREENA-C frameworks in Fig. 1, allows to directly assess the importance of inclusion of medium evolution on different observables, as the main difference between these two frameworks is that DREENA-B contains Bjorken evolution, while DREENA-C accounts for evolution in the simplest form (through constant mean temperature). We see that inclusion of Bjorken evolution has a negligible effect on R_{AA}, while having a significant effect on v_2. That is, it keeps R_{AA} almost unchanged, while significantly decreasing v_2. Consequently, small effect on R_{AA}, supports the fact that R_{AA} is weakly sensitive to medium evolution, making R_{AA} an excellent probe of jet-medium interactions in QGP; i.e. in QGP tomography, R_{AA} can be used to calibrate parton medium interaction models. On the other hand, medium evolution clearly influences v_2 predictions, in line with previous conclusions [65,66]; this sensitivity makes v_2 an ideal probe to constrain QGP medium parameters also from the point of high $p_\perp$ measurements (in addition to constraining them from low $p_\perp$ predictions and data).

In Fig. 2, we provide joint predictions for D and B meson R_{AA} (left panel) and v_2 (right panel) predictions for both 5.02 TeV $Pb+Pb$ and 5.44 TeV $Xe+Xe$ collisions at the LHC. Predictions are compared with the available experimental data. For D mesons, we again observe good joint agreement with the available R_{AA} and v_2 data. For B mesons (where the experimental data are yet to become available), we predict a notable suppression (see also [27, 67]), which is consistent with non-prompt J/Ψ R_{AA} measurements [68] (indirect probe of b quark suppression). Additionally, we predict non-zero v_2 for higher centrality regions. This does not necessarily mean that heavy B meson flows, since we here show predictions for high $p_\perp$, and flow is inherently connected with *low* $p_\perp$ v_2. On the other hand, high $p_\perp$ v_2 is connected with the difference in the B meson suppression for different (in-plane and out-of-plane) directions, leading to our predictions of non zero v_2 for *high* $p_\perp$ B mesons. Additionally, by comparing D and B meson v_2s in Fig. 2, we observe that their difference is large and that it qualitatively exhibits the same dependence on $p_\perp$ as R_{AA}. This v_2 comparison therefore presents additional important prediction of the heavy flavor dead-cone effect in QGP, where a strikingly similar signature of this effect is observed for R_{AA} and v_2.

The predicted similarity between R_{AA} and v_2 dead-cone effects can be analytically understood by using simple scaling arguments. Fractional energy loss can be estimated as [26]

$$\Delta E/E \sim \eta T^a L^b, \tag{7}$$

where a, b are proportionality factors, T and L are, respectively, the average temperature of the medium and the average path-length traversed by the jet. η is a proportionality factor that depends on initial jet mass M and transverse momentum $p_\perp$.

Under the assumption of small fractional energy loss, we can make the following estimate [26]:

$$R_{AA} \approx 1-\xi(M,p_\perp)T^a L^b,$$
$$v_2 \approx \xi(M,p_\perp)\frac{(T^a L^{b-1}\Delta L - T^{a-1}L^b \Delta T)}{2}, \tag{8}$$

where ΔL and ΔT are, respectively, changes in average path-lengths and average temperatures along out-of-plane and in-plane directions. $\xi = (n-2)\eta/2$, where n is the steepness of the initial momentum distribution function.

The difference between R_{AA} and v_2 for D and B mesons then becomes:

$$R^B_{AA} - R^D_{AA} \approx (\xi(M_c, p_\perp) - \xi(M_b, p_\perp))\, T^a L^b,$$
$$v^D_2 - v^B_2 \approx (\xi(M_c, p_\perp) - \xi(M_b, p_\perp)) \times \frac{(T^a L^{b-1}\Delta L - T^{a-1} L^b \Delta T)}{2}, \quad (9)$$

where M_c and M_b are charm and bottom quark masses respectively. From Eq. (9), we see the same mass dependent prefactor for both R_{AA} and v_2 comparison, intuitively explaining our predicted dead-cone effect similarity for high-$p_\perp$ R_{AA} and v_2.

4. Summary

Overall, we see that comprehensive joint R_{AA} and v_2 predictions, obtained with our DREENA-B framework, lead to a good agreement with all available light and heavy flavor data. This is, to our knowledge, the first study to provide such comprehensive predictions for high $p_\perp$ observables. In the context of v_2 puzzle, this study presents a significant development, as the other models were not able to achieve this agreement without introducing new phenomena [69]. However, for more definite conclusions, the inclusion of more complex QGP evolution within DREENA framework is needed, which is our main ongoing - but highly non-trivial - task, due to the complexity of underlying energy loss formalism.

As an outlook, for $Xe + Xe$, we also showed an extensive set of predictions for both R_{AA} and v_2, for different flavors and centralities, to be compared with the upcoming experimental data. Reasonable agreement with these data would present a strong argument that the dynamical energy loss formalism can provide a reliable tool for precision QGP tomography. Moreover, such comparison between predictions and experimental data can also confirm interesting new patterns in suppression data, such as our prediction of strikingly similar signature of the dead-cone effect between R_{AA} and v_2 data.

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Quantum dynamics of the small-polaron formation in a superconducting analog simulator

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We propose a scheme for investigating the nonequilibrium aspects of small-polaron physics using an array of superconducting qubits and microwave resonators. This system, which can be realized with transmon or gatemon qubits, serves as an analog simulator for a lattice model describing a nonlocal coupling of a quantum particle (excitation) to dispersionless phonons. We study its dynamics following an excitation-phonon (qubit-resonator) interaction quench using a numerically exact approach based on a Chebyshev-moment expansion of the time-evolution operator of the system. We thereby glean heretofore unavailable insights into the process of the small-polaron formation resulting from strongly momentum-dependent excitation-phonon interactions, most prominently about its inherent dynamical timescale. To further characterize this complex process, we evaluate the excitation-phonon entanglement entropy and show that initially prepared bare-excitation Bloch states here dynamically evolve into small-polaron states that are close to being maximally entangled. Finally, by computing the dynamical variances of the phonon position and momentum quadratures, we demonstrate a pronounced non-Gaussian character of the latter states, with a strong antisqueezing in both quadratures.

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I. INTRODUCTION

Recent progress in superconducting (SC) circuits [1,2] has enabled significant strides in the realm of analog quantum simulation [3]. An overwhelming majority of proposals for simulating various physical systems using SC circuits is based on arrays of transmon qubits and microwave resonators in state-of-the-art circuit quantum electrodynamics (circuit-QED) setups [4,5]. Examples include simulators of quantum spin- and spin-boson-type systems, interacting fermions/bosons, topological states of matter, to name but a few [6–8]. In particular, SC simulators of *small-polaron* (SP) models [9–11] have proven superior to their counterparts based on trapped ions [12,13], cold polar molecules [14,15], and Rydberg atoms/ions [16]. Yet, the existing theoretical proposals for simulating SP physics—not only those based on SC circuits—solely address its static aspects.

The SP concept captures the physical situation often found in narrow-band semiconductors and insulators, where the motion of an itinerant excitation—an excess charge carrier (electron, hole) or an exciton—may get hindered by a potential well resulting from the host-crystal lattice displacements [17]. The ensuing SP formation [18], accompanied by the phonon "dressing" of the excitation and an increase in its effective band mass, represents the most striking consequence of strong, short-ranged excitation-phonon (e-ph) coupling [19]. Yet, some important issues—e.g., how long it takes for a SP quasiparticle to form following an e-ph interaction quench (i.e., a sudden switching-on of the e-ph interaction in a previously uncoupled system)—remain ill-understood as of this writing [20,21]. On the theoretical side, this fundamental issue remains unresolved even in the simplest case of purely local e-ph coupling captured by the time-honored Holstein model [22–24]. On the experimental side, studies of the dynamics of polaron formation became possible with advances in ultrafast time-resolved spectroscopies, typically yielding formation times of less than a picosecond [25].

The compelling need to understand the microscopic mechanisms of charge-carrier transport in complex electronic materials, such as crystalline organic semiconductors [26,27], semiconducting counterparts of graphene [28], or cuprates [29,30], prompted investigations of models with strongly momentum-dependent (nonlocal) e-ph interactions [31]. Such interactions, whose corresponding vertex functions have explicit dependence on both the excitation and phonon quasimomenta, are exemplified by the Peierls-type coupling (also known as Su-Schrieffer-Heeger or off-diagonal coupling [32]) that accounts for the dependence of effective excitation hopping amplitudes upon phonon states [33,34]. Aside from their significance for describing transport properties of materials, such couplings have fundamental importance. Namely, they do not obey the Gerlach-Löwen theorem, a formal result that rules out the existence of nonanalytical features in the ground-state-related single-particle properties for certain classes of coupled e-ph models [35].

In this paper, motivated by the aforementioned dearth of studies pertaining to the dynamics of SP formation, we explore this complex phenomenon using an analog simulator that consists of SC qubits and microwave resonators. Adjacent qubits are coupled in this system through a coupler circuit that contains three Josephson junctions (JJs). This system, based on transmon qubits, was proposed in the past by one of us and collaborators for the purpose of simulating static properties of SPs that originate from nonlocal e-ph interactions [10]. Apart from transmons, this system can also be realized with semiconductor-nanowire-based gatemon qubits [36–38].

We analyze the time evolution of SP states ensuing from initially prepared bare-excitation Bloch states with different quasimomenta. We do so by combining exact numerical diagonalization of the effective e-ph Hamiltonian of the system and the Chebyshev-propagator method [39,40] for computing its dynamics. We determine how the SP formation time after an e-ph interaction quench depends on the initial bare-excitation quasimomentum and the e-ph coupling strength. We then further characterize SP formation by evaluating the e-ph entanglement entropy and showing that at the onset of the SP regime it reaches values close to those of maximally entangled states. In addition, we evaluate the dynamical variances of the phonon position and momentum quadratures in SP states and demonstrate their pronounced non-Gaussian character, with a substantial antisqueezing in both quadratures. Our findings can be verified in the proposed simulator—once realized—by measuring the photon number and the attendant squeezing in the resonators.

The remainder of this paper is organized as follows. In Sec. II we present the layout of our analog simulator and its underlying Hamiltonian. Section II B is set aside for the effective e-ph Hamiltonian of the system, followed by a discussion of typical parameter regimes and the salient features of the SP ground state of the system. In Sec. III we provide a brief outline of the strategy that we employ to study the system dynamics and introduce some relevant timescales in the problem at hand. In Sec. IV we present the obtained results for the SP formation time, as well as those found for the e-ph entanglement entropy and the dynamical variances of the phonon position and momentum quadratures. We summarize the paper and conclude with some general remarks in Sec. V. Some involved mathematical derivations, as well as a description of basic aspects of the numerical method used and our own implementation thereof, are relegated to Appendices A–C.

II. SYSTEM AND ITS HAMILTONIAN

A. Layout of the analog simulator

The proposed simulator, depicted in Fig. 1, consists of SC qubits (Q_n) with the energy splitting ε_z, microwave resonators (R_n) with the photon frequency ω_c, and coupler circuits (B_n) with three JJs ($n = 1,\dots,N$). Through the Jordan-Wigner mapping [41] the pseudospin-1/2 degree of freedom of qubits, represented by the operators $\boldsymbol{\sigma}_n$, plays the role of a spinless-fermion excitation; photons in the resonators, created by the operators $a_n^\dagger$, mimic dispersionless phonons. The nth repeating unit of this system is described by the Hamiltonian $H_n = H_n^0 + H_n^J$. Its noninteracting part H_n^0 reads

$$H_n^0 = \frac{\varepsilon_z}{2}\sigma_n^z + \hbar\omega_c a_n^\dagger a_n. \tag{1}$$

The Josephson energy of the coupler circuit B_n [42], a generalization of a SQUID loop, is given by

$$H_n^J = -\sum_{i=1}^{3} E_J^i \cos\varphi_n^i, \tag{2}$$

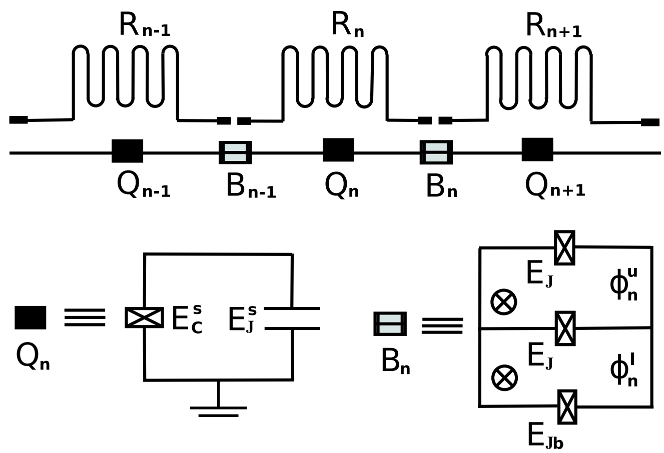

FIG. 1. Schematic diagram of the analog-simulator circuit containing SC qubits Q_n (with charging and Josephson energies E_C^s and E_J^s, respectively), resonators R_n, and coupler circuits B_n with three Josephson junctions ($n = 1,\dots,N$). ϕ_n^l and ϕ_n^u are total fluxes threading the lower and upper loops of B_n, respectively. Qubit Q_n interacts with its neighbors through circuits B_{n-1} and B_n.

where φ_n^i are the respective phase drops on the three JJs and E_J^i their corresponding Josephson energies; we henceforth assume that $E_J^1 = E_J^2 \equiv E_J$ and $E_J^3 = E_{Jb} \neq E_J$.

The qubit and resonator degrees of freedom are coupled through the flux of the resonator modes that pierces the upper loops of coupler circuits. The Josephson-coupling energy of the latter circuits, as demonstrated in what follows, can be expressed as an XY-type (flip-flop) coupling between adjacent qubits with the coupling strength that dynamically depends on the resonator (i.e., photon) degrees of freedom. As a result, this indirect inductive-coupling mechanism effectively gives rise to a qubit-resonator interaction. In addition, coupler circuits are also driven by a microwave radiation (ac flux) and subject to an external dc flux. The required ac fluxes can be generated by microwave-pumped control wires situated in the vicinity of the respective loops, while the dc flux can be supplied through currents in appropriately placed separate control wires.

Let ϕ_n^u and ϕ_n^l be the respective total magnetic fluxes in the upper and lower loops of B_n (cf. Fig. 1), both expressed in units of $\Phi_0/2\pi$, where $\Phi_0 \equiv hc/(2e)$ is the flux quantum. The upper-loop flux ϕ_n^u includes the ac-driving contribution $\pi\cos(\omega_0 t)$ and one that stems from the resonator modes, i.e.,

$$\phi_n^u = \pi\cos(\omega_0 t) + \phi_{n,\mathrm{res}}, \tag{3}$$

where $\phi_{n,\mathrm{res}}$ is given by

$$\phi_{n,\mathrm{res}} = \delta\theta[(a_{n+1} + a_{n+1}^\dagger) - (a_n + a_n^\dagger)]. \tag{4}$$

Here $\delta\theta = [2eA_{\mathrm{eff}}/(\hbar d_0 c)](\hbar\omega_c/C_0)^{1/2}$, where A_{eff} is the effective coupling area, C_0 the capacitance of the resonator, and d_0 the effective spacing in the resonator [43]. The lower-loop flux ϕ_n^l also comprises an ac contribution given by $-(\pi/2)\cos(\omega_0 t)$, with the same frequency as the ac part of ϕ_n^u but a different amplitude. In addition, it includes a dc part ϕ_{dc}—apart from ω_0 the only tunable parameter in the system and its main experimental knob. Thus, ϕ_n^l is given by

$$\phi_n^l = -\frac{\pi}{2}\cos(\omega_0 t) + \phi_{\rm dc}. \tag{5}$$

It should be stressed that the amplitudes of the two ac-driving terms are chosen in such a way as to ensure that the phase drops φ_n^3 on the bottom JJs do not have an explicit time dependence [10].

The time dependence of the ac-driving terms makes it natural to carry out further analysis in the rotating frame of the drive. While this change of frames leads to a shift in the phonon frequency ($\omega_c \rightarrow \delta\omega \equiv \omega_c - \omega_0$), it also renders the Josephson-coupling term time dependent. Yet, it can easily be shown that this time dependence can be neglected due to its rapidly oscillating character. The remaining part of that term reads

$$\bar{H}_n^J = -2\big[t_r - \tfrac{1}{2}E_J J_1(\pi/2)\phi_{n,\rm res}\big]\cos(\varphi_n - \varphi_{n+1}). \tag{6}$$

Here φ_n is the gauge-invariant phase variable of the SC island of the nth qubit [1], $J_n(x)$ are Bessel functions of the first kind, and $t_r = (E_{Jb}/2)(1+\cos\phi_{\rm dc})$, where E_{Jb} is chosen to be given by $2E_J J_0(\pi/2)$.

In the regime of relevance for transmon/gatemon qubits ($E_J^s \gg E_C^s$, where E_C^s and E_J^s are the charging and Josephson energies of a single qubit, respectively) $\cos(\varphi_n - \varphi_{n+1})$ can be expanded up to the second order in $\varphi_n - \varphi_{n+1}$. By switching to the pseudospin operators $\boldsymbol{\sigma}_n$, it can be recast (up to an immaterial additive constant) as $\delta\varphi_0^2[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ - (\sigma_n^z + \sigma_{n+1}^z)/2]$. Here $\delta\varphi_0^2 \equiv (2E_C^s/E_J^s)^{1/2}$, hence $\delta\varphi_0^2 \sim 0.15$ for a typical transmon ($E_J^s/E_C^s \sim 100$) and $\delta\varphi_0^2 \sim 0.28$ for a typical gatemon ($E_J^s/E_C^s \sim 25$).

While the original proposal for simulating SP physics with strongly momentum-dependent e-ph interactions (Ref. [10]) envisioned the use of transmons—the most widely used type of SC qubits, with superior coherence properties—it is worthwhile to stress that the system under consideration can also be realized with gatemon qubits [36–38]. The gatemon is a superconductor–normal-metal–superconductor-type device where an electrostatic gate depletes carriers in a semiconducting weak-link region. This allows one to tune the energy of its JJ, and in turn control the qubit frequency [36]. Because it does not require an external-flux control, this gate-voltage-controlled counterpart of the transmon has a reduced dissipation by a resistive control line and is particularly suitable for use in an external magnetic field.

Both types of SC qubits under consideration have some advantages with regard to their use in the proposed analog simulator. On the one hand, the fact that gatemons do not require external-flux control makes them the prefferred choice for our present purposes, where the use of external magnetic fluxes is essential (recall Sec. II above). On the other hand, some other aspects, e.g., their larger anharmonicity (see Sec. IV B below) and slightly better coherence properties (for a comparison of coherence properties of various SC-qubit types, see Ref. [2]) favor the use of transmons. For the sake of completeness, it is worthwhile to add that analog simulators of nonlocal e-ph couplings based on other types of SC qubits, e.g., flux qubits that have large anharmonicities, can also be envisaged, as previously proposed for the local-coupling Holstein model [11].

B. Effective coupled excitation-phonon Hamiltonian

It is pertinent to switch at this point to the spinless-fermion representation of the qubit (pseudospin-1/2) degrees of freedom. The underlying Jordan-Wigner transformation implies that [41]

$$\begin{aligned}\sigma_n^z &\rightarrow 2c_n^\dagger c_n - 1,\\ \sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ &\rightarrow c_n^\dagger c_{n+1} + \text{H.c.}\end{aligned} \tag{7}$$

As a consequence, the noninteracting (free) part $H_{\rm f}$ of the effective system Hamiltonian, to be denoted as $H_{\rm eff} = H_{\rm f} + H_{\text{e-ph}}$ in the following, includes the excitation-hopping and free-phonon terms

$$H_{\rm f} = -t_0(\phi_{\rm dc})\sum_{n=1}^{N}(c_n^\dagger c_{n+1} + \text{H.c.}) + \hbar\delta\omega\sum_{n=1}^{N} a_n^\dagger a_n, \tag{8}$$

where $t_0(\phi_{\rm dc}) \equiv 2\delta\varphi_0^2\, t_r(\phi_{\rm dc})$ is the $\phi_{\rm dc}$-dependent bare-excitation-hopping integral. (Note that the σ_n^z terms from H_n^0 and $\bar{H}_n^J$ are omitted as they correspond to a constant band-energy offset for spinless fermions.) Similarly, the interacting part of $H_{\rm eff}$ captures two different mechanisms of nonlocal e-ph interaction and is given by

$$\begin{aligned}H_{\text{e-ph}} = g\hbar\delta\omega l_0^{-1}\sum_{n=1}^{N}[&(c_n^\dagger c_{n+1} + \text{H.c.})(u_{n+1} - u_n)\\ &- c_n^\dagger c_n(u_{n+1} - u_{n-1})],\end{aligned} \tag{9}$$

where g is the dimensionless coupling strength and $u_n \equiv l_0(a_n + a_n^\dagger)$ the local Einstein-phonon displacement at site n, with l_0 being the phonon zero-point length. The first term of $H_{\text{e-ph}}$ corresponds to the Peierls e-ph coupling mechanism, which captures the lowest-order (linear) dependence of the excitation hopping amplitude between sites n and $n+1$ on the difference $u_{n+1} - u_n$ of the respective phonon displacements [34,44]. The other one is the breathing-mode term [30], a density-displacement-type mechanism which accounts for the antisymmetric coupling of the excitation density $c_n^\dagger c_n$ at site n with the phonon displacements on the adjacent sites $n \pm 1$. In other words, it captures a nonlocal phonon-induced modulation of the excitation's on-site energy (by contrast to Holstein coupling which describes the local phonon-induced modulation of the same energy).

By transforming the e-ph coupling Hamiltonian to its generic momentum-space form

$$H_{\text{e-ph}} = N^{-1/2}\sum_{k,q}\gamma_{\text{e-ph}}(k,q)c_{k+q}^\dagger c_k(a_{-q}^\dagger + a_q), \tag{10}$$

it is straightforward to verify that its corresponding vertex function is given by

$$\gamma_{\text{e-ph}}(k,q) = 2ig\hbar\delta\omega[\sin k + \sin q - \sin(k+q)], \tag{11}$$

where quasimomenta are expressed in units of the inverse lattice period. Because this vertex function depends on both the excitation (k) and phonon (q) quasimomenta the Hamiltonian

$H_{\rm eff}$ does not belong to the realm of validity of the Gerlach-Löwen theorem [35]. As demonstrated in Ref. [10] its ground state displays a level-crossing-type sharp transition at a critical value of the effective coupling strength $\lambda_{\rm eff} \equiv 2g^2\hbar\delta\omega/t_0$. While for $\lambda_{\rm eff} < \lambda_{\rm eff}^{\rm c}$ the ground state is the (nondegenerate) $K = 0$ eigenvalue of the total quasimomentum operator

$$K_{\rm tot} = \sum_k k c_k^\dagger c_k + \sum_q q a_q^\dagger a_q, \tag{12}$$

for $\lambda_{\rm eff} \geqslant \lambda_{\rm eff}^{\rm c}$ it is twofold degenerate and corresponds to $K = \pm K_{\rm gs}$ (where $K_{\rm gs} \neq 0$ and saturates at $\pi/2$ for sufficiently large $\lambda_{\rm eff}$). In this regime the single-particle dispersion corresponding to the SP Bloch band has mutually symmetric minima at two nonzero quasimomenta, which are here incommensurate with the period of the underlying lattice, a rare occurrence in other physical systems [45].

Aside from a nonanalyticity in the ground-state energy of the system, the aforementioned sharp transition is manifested by analogous features in the ground-state quasiparticle residue (spectral weight) $Z_{\rm gs} \equiv Z_{k=K_{\rm gs}}$, where $Z_k \equiv |\langle\Psi_k|\psi_k\rangle|^2$ is the module squared of the overlap between the bare-excitation Bloch state

$$|\Psi_k\rangle = c_k^\dagger|0\rangle_{\rm e} \otimes |0\rangle_{\rm ph} \tag{13}$$

and the (dressed) Bloch state $|\psi_k\rangle$ of the coupled e-ph system corresponding to the same quasimomentum ($K = k$). [Note that $|0\rangle_{\rm e}$ and $|0\rangle_{\rm ph}$ on the right-hand side (RHS) of the last equation stand for the excitation and phonon vacuum states, respectively.] Another quantity characterizing the SP ground state, which shows a nonanalyticity at $\lambda_{\rm eff} = \lambda_{\rm eff}^{\rm c}$, is the phonon-number expectation value in the ground state $|\psi_{\rm gs}\rangle \equiv |\psi_{K=K_{\rm gs}}\rangle$:

$$\bar{N}_{\rm ph} = \langle\psi_{\rm gs}|\sum_{n=1}^{N} a_n^\dagger a_n|\psi_{\rm gs}\rangle. \tag{14}$$

The system at hand has another peculiar property, namely, it is straightforward to demonstrate that the $k = 0$ bare-excitation Bloch state $|\Psi_{k=0}\rangle$ [cf. Eq. (13)] is an eigenstate of $H_{\rm eff}$ for an arbitrary $\lambda_{\rm eff}$, a direct consequence of the fact that the e-ph vertex function [cf. Eq. (11)] has the property that $\gamma_{\text{e-ph}}(k = 0, q) = 0$ for any q. In particular, for $\lambda_{\rm eff} < \lambda_{\rm eff}^{\rm c}$ this state represents the ground state of $H_{\rm eff}$.

The relation between the dimensionless coupling strength g and the system-specific parameters reads

$$g\hbar\delta\omega = \delta\varphi_0^2 E_J J_1(\pi/2)\delta\theta. \tag{15}$$

It is worthwhile to notice that g does not depend on the tunable system parameters ($\omega_0, \phi_{\rm dc}$) and we specify it by fixing the product of $\delta\varphi_0^2$ and E_J on the RHS of the last equation: $\delta\varphi_0^2 E_J/2\pi\hbar = 100$ GHz. Given that the typical magnitude of $\delta\varphi_0^2$ is twice as large for gatemons compared to transmons, in a transmon-based realization of this system E_J should be taken twice as large to retain the same coupling strength and make further discussion completely general. Unlike g, $\lambda_{\rm eff}$ inherits its dependence on $\phi_{\rm dc}$ from t_0 and is therefore externally tunable:

$$\lambda_{\rm eff}(\phi_{\rm dc}) = g\,\frac{J_1(\pi/2)\delta\theta}{J_0(\pi/2)(1+\cos\phi_{\rm dc})}. \tag{16}$$

For a typical resonator $\delta\theta \sim 3.5 \times 10^{-3}$. Likewise, for $\delta\omega$ we take $\delta\omega/2\pi = 200$–300 MHz. Consequently, for $\delta\omega/2\pi = 200$ MHz (300 MHz) we obtain $\lambda_{\rm eff}^{\rm c} \approx 0.83$ (0.72).

III. DYNAMICS OF SMALL-POLARON FORMATION

A. Interaction quench and initial-state preparation

We study the system dynamics after an e-ph (qubit-resonator) interaction quench at $t = 0$, assuming that the system was initially prepared in the bare-excitation Bloch state $|\Psi_{k=k_0}\rangle$ with quasimomentum k_0. Given that an abrupt change from a bare excitation to a heavily dressed one here takes place for $\lambda_{\rm eff} = \lambda_{\rm eff}^{\rm c}$, a variation of $\phi_{\rm dc}$ from slightly below its critical value to slightly above it is equivalent to an interaction quench in this system.

The initial bare-excitation states can be prepared using a general protocol based on an external driving and the Rabi coupling between the vacuum state and the desired Bloch state [9]. The corresponding preparation time is given by $\tau_{\rm prep} = \pi\hbar/(2\beta_p)$, where β_p is the microwave-pumping amplitude. (Note that an analogous result holds in the case of preparing dressed Bloch states, e.g., a SP ground state, except that in that case the last expression for $\tau_{\rm prep}$ requires another multiplicative factor of $Z_{\rm gs}^{-1}$.) For a typical pumping amplitude $\beta_p/(2\pi\hbar) = 10$ MHz, we obtain $\tau_{\rm prep} = 25$ ns, which is a three orders of magnitude shorter time than currently achievable decoherence times T_2 of the relevant classes of SC qubits [2].

B. Relevant quantities and timescales

In accordance with the discrete translational symmetry of the system under consideration, its effective Hamiltonian commutes with the total quasimomentum operator [cf. Eq. (12)], i.e., $[H_{\rm eff}, K_{\rm tot}] = 0$. Therefore, the system evolves within the eigensubspace of $H_{\rm eff}$ that corresponds to the eigenvalue $K = k_0$ of $K_{\rm tot}$. We compute its state $|\psi(t)\rangle$ at time t for a simulator with $N = 9$ qubits by combining Lanczos-type exact diagonalization [46] of $H_{\rm eff}$ in a symmetry-adapted basis of the truncated Hilbert space of the system (for details, see Appendix A) and the Chebyshev-propagator method [39,40]. The latter relies on expansions of time-evolution operators into finite series of Chebyshev polynomials of the first kind (for general details of this approach and our numerical implementation thereof, see Appendix C).

The knowledge of the state $|\psi(t)\rangle$ of the system at time t allows us to evaluate quantities characterizing the ensuing polaronic character of the dressed excitation. One such quantity is the probability for the system to remain in the initial state $|\Psi_{k=k_0}\rangle$ at time t, given by

$$P_{k_0}(t) = |\langle\psi(t)|\Psi_{k=k_0}\rangle|^2. \tag{17}$$

This quantity, more precisely the matrix element $\langle\psi(t)|\Psi_{k=k_0}\rangle$, is closely related (up to a Fourier transform to the frequency domain) to the momentum-frequency resolved spectral function, a dynamical response function that can be extracted in systems of the present type using a generalization of the Ramsey interference protocol [10]. Another relevant

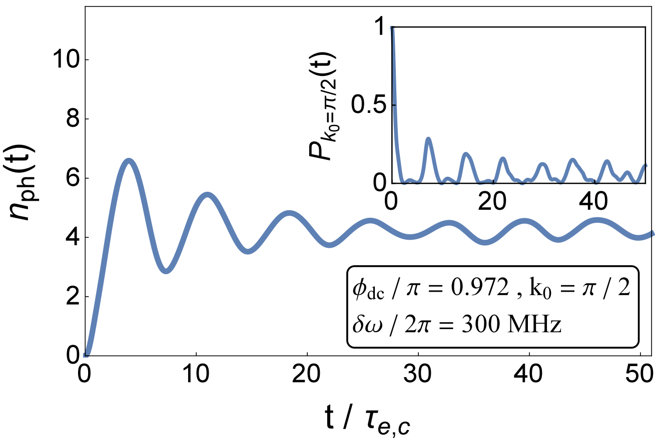

FIG. 2. Expected phonon number after an e-ph interaction quench at $t = 0$ for $k_0 = \pi/2$. Inset: the probability to remain in the initial bare-excitation Bloch state.

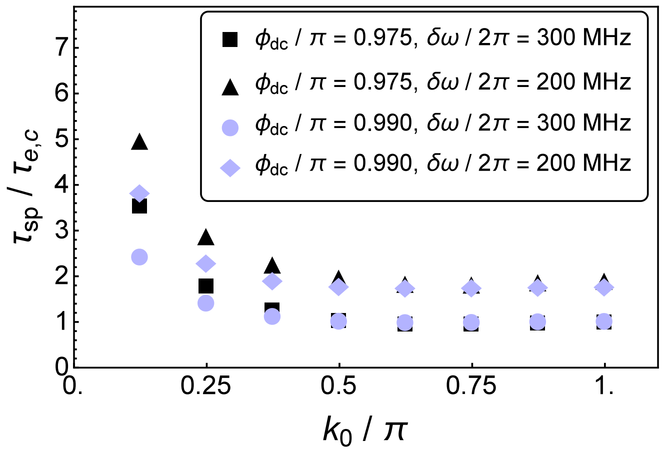

FIG. 3. SP formation time τ_{sp} for varying initial bare-excitation quasimomenta k_0 within the Brillouin zone and different choices of values for ϕ_{dc} and $\delta\omega$.

quantity is the expected total phonon number:

$$n_{ph}(t) = \langle\psi(t)| \sum_{n=1}^{N} a_n^\dagger a_n |\psi(t)\rangle. \tag{18}$$

This observable provides a direct quantitative characterization of the dynamical dressing of an excitation by virtual phonons. In our analog simulator, $n_{ph}(t)$ is amenable to measurement by extracting the photon number on the resonators (for details, see Sec. IV B below).

It is judicious to express the evolution time in units of a timescale closely related to the bare-excitation hopping amplitude $t_0(\phi_{dc})$. Because the latter here depends on the experimental knob ϕ_{dc} by design, we choose this timescale to be set by the critical value $\phi_{dc}^c = 0.972\pi$ of ϕ_{dc} for $\delta\omega/2\pi =$ 300 MHz. Thus, the chosen characteristic timescale is $\tau_{e,c} \equiv \hbar/t_0(\phi_{dc}^c) \approx 0.44$ ns.

One of the most important characteristics of the SP formation process, yet often elusive in solid-state systems exhibiting polaronic behavior [25], is its associated dynamical timescale τ_{sp}. It is pertinent to define it as the time at which the phonon dressing (i.e., the phonon-number expectation value) of an initially bare excitation becomes equal to that of the corresponding SP ground state. In other words, τ_{sp} is defined by the condition

$$n_{ph}(t = \tau_{sp}) = \bar{N}_{ph}, \tag{19}$$

where $\bar{N}_{ph}$ was defined in Eq. (14) above. This ground-state phonon number is in the range 3.9–5.1 (1.8–2) for $\delta\omega/2\pi =$ 200 MHz (300 MHz).

IV. RESULTS AND DISCUSSION

A. Time dependence of n_{ph} and $P_{k_0=\pi/2}$

Typical results of our numerical calculations of $n_{ph}(t)$ and $P_{k_0=\pi/2}(t)$ are presented in Fig. 2. They reflect the fact that the system was initially prepared in the $|\Psi_{k_0=\pi/2}\rangle$ state, which is not an eigenstate of the system Hamiltonian H_{eff} after the quench. In fact, for the concrete choice of parameter values used, this state is a superposition of a multitude of eigenstates of this Hamiltonian, among which the SP ground state with $K_{gs} = \pi/2$ has a weight of only around 0.16. This explains the presence of dynamical recurrences at later times, i.e., a complex oscillatory behavior resulting from the interference of the quantum evolutions of all these eigenstates.

It is instructive to add, for completeness, that our ground-state calculations show that for $\lambda_{eff} = \lambda_{eff}^c$, i.e., at the onset of strong-coupling regime in the system under consideration, there are three discrete states (at each K) below the one-phonon continuum, while for a larger λ_{eff} one can find up to five such states. As a reminder, the one-phonon continuum in a coupled e-ph system with gapped (optical-like) phonon modes, such as, e.g., Einstein-like phonons in the system at hand, originates from the onset of the inelastic-scattering threshold at the energy $E_{gs} + \hbar\omega_{ph}$ (the minimal energy that a dressed excitation ought to have in order to be able to emit a phonon), where E_{gs} is the ground-state energy of the coupled system and $\hbar\omega_{ph}$ the energy of one phonon (in our simulator $\omega_{ph} \to \delta\omega$) [19]. The width of this continuum equals the width of the resulting SP Bloch band. Importantly, the discrete (bound) states below the one-phonon continuum feature as the coherent part, i.e., sharp peaks in the momentum-frequency resolved spectral function [10]. While some details of the dynamics certainly depend on the concrete form of e-ph coupling involved, the increasing number of such discrete (split-off from the continuum) states upon increasing coupling strength results in more complex system dynamics.

B. Small-polaron formation time

The dependence of the SP formation time τ_{sp} on the initial bare-excitation quasimomentum k_0 is illustrated in Fig. 3 (for symmetry-related reasons, it suffices to consider only quasimomenta in one half of the Brillouin zone, i.e., for $0 \leqslant k_0 \leqslant \pi$). τ_{sp} clearly shows an upturn for small k_0, consistent with the fact that it ought to diverge ($\tau_{sp} \to \infty$) as $k_0 \to 0$ because the $k_0 = 0$ bare-excitation Bloch state is an exact eigenstate of H_{eff}. Another important feature that can be inferred from the obtained results is that τ_{sp} depends rather weakly on k_0 for $\pi/2 \leqslant k_0 \leqslant \pi$. This can be contrasted with

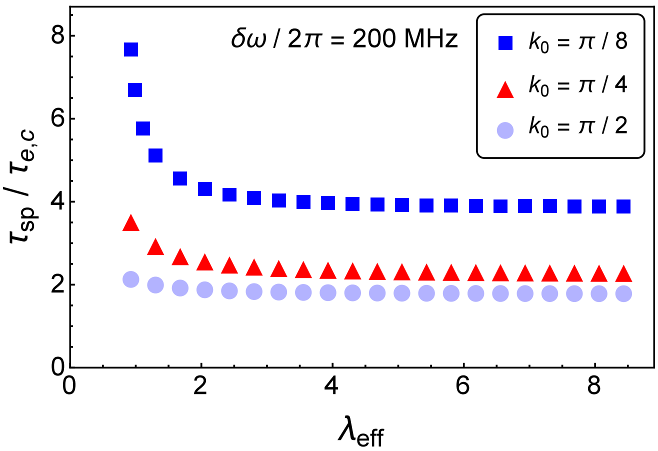

FIG. 4. Dependence of the SP formation time $\tau_{\rm sp}$ on the effective coupling strength $\lambda_{\rm eff}$, shown for $\delta\omega/2\pi = 200$ MHz and different initial quasimomenta k_0.

the Holstein-polaron case [20], where the analogous dynamical timescale strongly depends on the initial bare-excitation quasimomentum.

The obtained dependence of $\tau_{\rm sp}$ on the effective coupling strength $\lambda_{\rm eff}$ is displayed in Fig. 4. While it may seem surprising that $\tau_{\rm sp}$ saturates for $\lambda_{\rm eff}$ above a threshold value, this actually mimics the behavior of the SP indicators (quasiparticle residue, average phonon number) in the ground state. In that case a regime of saturation also sets in for $\lambda_{\rm eff}$ slightly above its critical value at which a nonanalyticity occurs in all ground-state-related quantities [10]. Such a behavior is in stark contrast with that of the momentum-independent Holstein coupling, for which the same quantities change monotonously with the coupling strength.

The variation of the SP formation time—defined by the condition in Eq. (19)—with the effective coupling strength $\lambda_{\rm eff}$ (shown in Fig. 4) results from two competing tendencies. Namely, with increasing $\lambda_{\rm eff}$ phonon dressing of an initially bare excitation becomes faster. However, the average ground-state phonon number $\bar{N}_{\rm ph}$ also becomes larger. In Ref. [20], where the dynamics of the Holstein-polaron formation were investigated, a regime was observed where the formation time grows with $\lambda_{\rm eff}$ (for weak coupling, i.e., small $\lambda_{\rm eff}$) and the one where it decreases (strong coupling, i.e., large $\lambda_{\rm eff}$). Despite the completely different type of e-ph interaction in the system at hand, we also find such regimes. For our typical system parameters, the case with $\delta\omega/2\pi = 200$ MHz (cf. Fig. 4) corresponds to the latter regime, while the case with $\delta\omega/2\pi =$ 300 MHz (results not shown here) is characterized by a slow growth of $\tau_{\rm sp}$ with increasing $\lambda_{\rm eff}$.

The obtained SP formation times $\tau_{\rm sp} \sim (1\text{–}10)\,\tau_{e,c}$ justify *a posteriori* our choice of $\tau_{e,c}$ as the characteristic timescale for the system dynamics. These times are of the order of a few nanoseconds and can be verified in this system through photon-number measurements. This is done by adding an ancilla qubit (far-detuned from the resonator modes), which couples—but exclusively during the measurement itself—to a resonator [9]. The photon number on that resonator can then be extracted by means of a standard quantum non-demolition-measurement readout, which is effectively carried out by measuring the transition frequency of the qubit [47]. The total photon number can then be obtained by adding up those found on individual resonators.

An important issue to address in the context of measuring the photon states in the resonators is the one pertaining to the anharmonicity $\alpha \equiv E_{12} - E_{01}$ of SC qubits, where E_{ij} is the energy difference between states j and i of a single qubit. The anharmonicity determines the minimal pulse duration $t_p \sim \hbar/|\alpha|$ required to avoid leakage into noncomputational single-qubit states. For instance, for a typical transmon with a negative anharmonicity of around 200 MHz, even microwave pulses with durations on the scale of a few nanoseconds are known to be sufficiently frequency selective that one can neglect leakage into higher excited energy levels of the transmon and effectively treat it as a two-level system [5]. For gatemons, whose anharmonicity is slightly smaller than that of transmons [38], similar measurements should also be possible for all but the very shortest SP formation times found.

C. Dynamical variances of the phonon position and momentum quadratures

It is plausible to expect that nonlocal e-ph correlations in this system are reflected through fluctuations within the phonon subsystem, which can be observed via microwave photons in the resonators. To this end, we consider the phonon position and momentum quadratures at an arbitrary, say rth, site (in our system represented by the photon mode on the rth resonator), defined by the operators $x_r \equiv (a_r + a_r^\dagger)/\sqrt{2}$ and $p_r \equiv -i(a_r - a_r^\dagger)/\sqrt{2}$, respectively. We compute their respective dynamical variances $S_x(t)$ and $S_p(t)$, given by

$$
\begin{aligned}
S_x(t) &= \langle\psi(t)|x_r^2|\psi(t)\rangle - \langle\psi(t)|x_r|\psi(t)\rangle^2, \\
S_p(t) &= \langle\psi(t)|p_r^2|\psi(t)\rangle - \langle\psi(t)|p_r|\psi(t)\rangle^2. \qquad (20)
\end{aligned}
$$

(Note that, owing to the discrete translational symmetry of the system, the latter quantities should not depend on r.) The explicit expressions for these variances in our chosen symmetry-adapted basis are provided in Appendix B.

Our numerical evaluation of these dynamical variances shows that S_x dominates over S_p at all times. For instance, in the weakest-coupling case that yields SP ground state in the system at hand, with $\delta\omega/2\pi = 300$ MHz, and $\phi_{\rm dc} = 0.972\pi$ (shown in Fig. 5, which corresponds to $k_0 = \pi/2$), one finds the maximum of $S_x(t)$ to be around 12. The corresponding antisqueezing is as large as 13.8 dB.

What can be inferred from Fig. 5 is that the product $S_x(t)S_p(t)$ of the two dynamical quadrature variances is consistently much larger than 1/4, which illustrates a pronounced non-Gaussian character of fluctuations within the phonon subsystem. This can be ascribed to the nonlocal character of e-ph interaction in this system, with its attendant retardation effects [32].

D. Dynamics of the excitation-phonon entanglement buildup after the quench

It is worthwhile to complement our discussion of the dynamics of SP formation by evaluating the corresponding e-ph entanglement entropy. This quantity proved to be very useful in characterizing ground-state properties of SPs, most

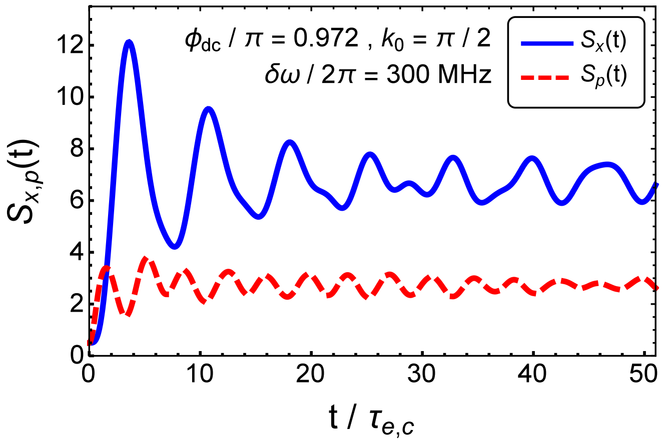

FIG. 5. Typical time dependence of the dynamical variances $S_x(t)$ and $S_p(t)$. The parameter values are indicated in the plot.

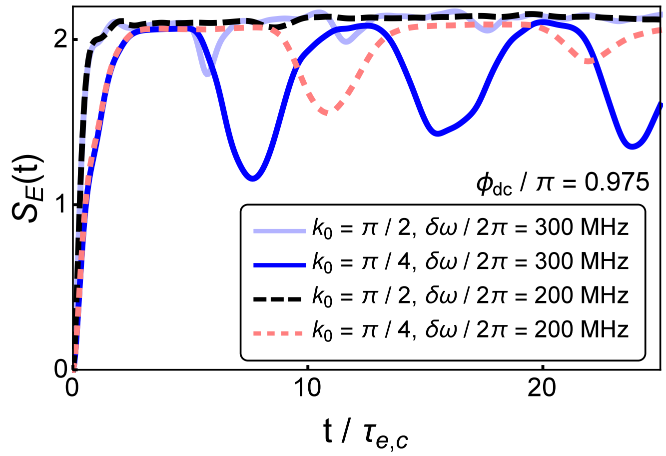

FIG. 6. Time dependence of the e-ph entanglement entropy for $\phi_{\rm dc} = 0.975\pi$ and different choices of values for k_0 and $\delta\omega$.

prominently the onset of sharp SP transitions at a critical e-ph coupling strength, in models with strongly momentum-dependent e-ph interactions [34]. This motivates us to use the same quantity in our present investigation of the SP formation dynamics.

Given that the initial bare-excitation states are of simple-product (separable) character, the e-ph entanglement entropy starts its growth from zero at $t = 0$. The density matrix of our composite (bipartite) e-ph system at time t is given by

$$\rho_{\text{e-ph}}(t) = |\psi(t)\rangle\langle\psi(t)|. \tag{21}$$

The reduced (excitation) density matrix, which has the dimension $N \times N$, is obtained by tracing $\rho_{\text{e-ph}}(t)$ over the phonon basis:

$$\rho_{\rm e}(t) = \mathrm{Tr}_{\rm ph}[\rho_{\text{e-ph}}(t)]. \tag{22}$$

The corresponding e-ph entanglement entropy is defined in terms of this last reduced density matrix as

$$S_{\rm E}(t) = -\mathrm{Tr}_{\rm e}[\rho_{\rm e}(t)\ln\rho_{\rm e}(t)]. \tag{23}$$

The matrix elements of $\rho_{\rm e}(t)$ are computed using Eq. (B25) (for a detailed derivation of those matrix elements, see Appendix B 2). The e-ph entanglement entropy in Eq. (23) can equivalently be expressed in terms of the eigenvalues $\xi_n(t)\,(n = 1,\ldots,N)$ of $\rho_{\rm e}(t)$ (note that $\xi_n > 0$ and $\sum_{n=1}^{N}\xi_n = 1$):

$$S_{\rm E}(t) = -\sum_{n=1}^{N}\xi_n(t)\ln\xi_n(t). \tag{24}$$

Generally speaking, the maximal value that can be reached by this quantity is

$$S_{\text{max-ent}} = \ln N, \tag{25}$$

obtained when $\xi_n = N^{-1}$ for each n (maximally entangled states).

Our explicit evaluation of the e-ph entanglement entropy is illustrated in Figs. 6 and 7, where it is depicted for $\phi_{\rm dc} = 0.975\pi$ and two different initial bare-excitation quasimomenta ($k_0 = \pi/2, \pi/4$) and phonon frequencies ($\delta\omega/2\pi =$ 200, 300 MHz). In particular, Fig. 6 illustrates that the growth of this entanglement entropy from zero at $t = 0$ starts with an abrupt increase in timescales of the order of a few $\tau_{e,c}$. This short-time behavior of the entropy is depicted separately in Fig. 7, from which we can infer that at short times $S_{\rm E}(t)$ depends on k_0, but is essentially independent of $\delta\omega$. The abrupt increase of $S_{\rm E}(t)$ is followed by oscillations at later times. Those oscillations, which are much more pronounced for $k_0 = \pi/4$ than for $k_0 = \pi/2$, are another manifestation of the late-time recurrences, akin to those found in $n_{\rm ph}(t)$ and $P_{k_0=\pi/2}(t)$ (recall Sec. IV A).

Another important feature of the e-ph entanglement entropy, which can be inferred from the obtained results, is that at times $t \approx (2\text{–}3)\tau_{e,c}$, coinciding with the corresponding SP formation times $\tau_{\rm sp}$, this quantity indeed reaches values close to those characterizing maximally entangled states (note that for $N = 9$, we have $S_{\text{max-ent}} = 2.197$). For instance, the respective maximal values of $S_{\rm E}(t)$ obtained for the above case of $k_0 = \pi/2$ are 2.141 for $\delta\omega/2\pi = 200$ MHz and 2.115 for 300 MHz. This is consistent with the results of an earlier study that reached the conclusion that typical SP ground states are essentially maximally entangled [34].

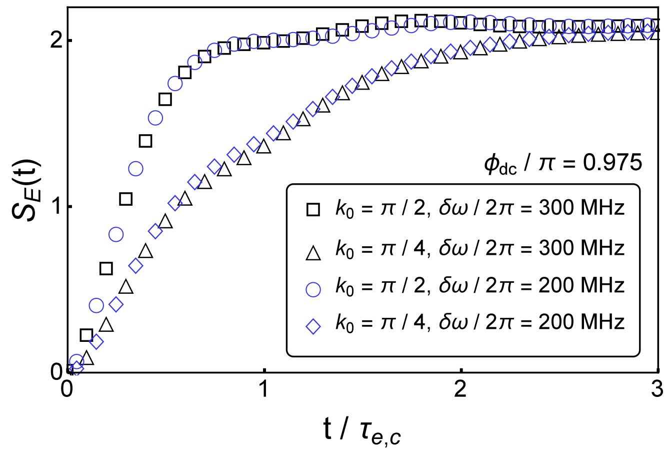

FIG. 7. Short-time behavior of the e-ph entanglement entropy for $\phi_{\rm dc} = 0.975\pi$ and different choices of values for k_0 and $\delta\omega$.

V. SUMMARY AND CONCLUSIONS

To summarize, in this work we explored the dynamics of small-polaron formation in the presence of two different mechanisms of nonlocal excitation-phonon interaction within the framework of an analog simulator. In this simulator, which is based on an array of coupled superconducting qubits (transmons or gatemons) and microwave resonators, the pseudospin degree of freedom of qubits plays the role of spinless-fermion excitation, while photons in the resonators mimic dispersionless phonons. By employing a numerically exact approach—diagonalization of the effective system Hamiltonian combined with the Chebyshev-propagator method for computing its dynamics—we determined the formation time of small polarons that ensue from initially prepared bare-excitation Bloch states following an excitation-phonon (qubit-resonator) interaction quench.

We analyzed how this important dynamical timescale depends on the initial bare-excitation quasimomentum and the effective coupling strength. We then further characterized the system dynamics by evaluating the excitation-phonon entanglement entropy and demonstrated its growth from zero (before the interaction quench) to values close to those inherent to maximally entangled states. Finally, by computing the dynamical variances of the phonon position and momentum quadratures we also demonstrated the non-Gaussian character of small-polaron states resulting from the quench, with a strong antisqueezing in both quadratures.

The present work constitutes a systematic theoretical study of the quantum dynamics of small-polaron formation resulting from strongly momentum-dependent excitation-phonon coupling. Such couplings, in their own right, are of utmost importance for understanding charge-transport mechanisms in several classes of electronic materials. The advanced measurement capabilities of the proposed superconducting analog simulator should allow an accurate verification of our quantitative predictions. To make contact with previous studies of the same phenomenon involving other types of polarons, we compared and contrasted our findings with those pertaining to the small-polaron formation dynamics in the presence of purely local (momentum-independent) Holstein-type excitation-phonon interaction [20,21]. We found, for instance, that in the system at hand, where excitation-phonon coupling itself is strongly momentum-dependent, the small-polaron formation time shows a weaker dependence on the initial bare-excitation quasimomentum than in the Holstein-polaron case.

Several directions of future work can be envisioned. Firstly, the proposed simulator can be utilized for investigations of further nonequilibrium aspects of small-polaron physics, which have so far also been discussed only for Holstein-type excitation-phonon interaction [48–51]. Examples of such aspects include the small-polaron dynamics in the presence of an external electric field [48], as well as the dynamics following a strong oscillatory pulse [49]. Furthermore, while the proposed system serves as a simulator for a one-dimensional excitation-phonon model, the continuously improving scalability of superconducting-qubit systems should allow one to fabricate, in the not-too-distant future, a two-dimensional counterpart of this simulator. Such a system could be used for studying the effects of dimensionality on the formation of small-polaron-type quasiparticles; analogous effects have proven to be quite interesting in the case of Holstein polarons. Finally, a different type of qubit-resonator arrays, featuring effective XXZ-type coupling between qubits [52], would allow an investigation of intersite bipolarons [53], quasiparticles closely related to small polarons. Experimental realization of the proposed system is keenly anticipated.

ACKNOWLEDGMENTS

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APPENDIX A: SYMMETRY-ADAPTED BASIS AND DETAILS OF EXACT DIAGONALIZATION

The Hilbert space of the coupled e-ph system is spanned by states $|n\rangle_e \otimes |\mathbf{m}\rangle_{\mathrm{ph}}$, where $|n\rangle_e \equiv c_n^\dagger|0\rangle_e$ corresponds to the excitation localized at the site n, while

$$|\mathbf{m}\rangle_{\mathrm{ph}} = \prod_{n=1}^{N\otimes} \frac{(b_n^\dagger)^{m_n}}{\sqrt{m_n!}}\,|0\rangle_{\mathrm{ph}}, \tag{A1}$$

where $\mathbf{m} \equiv (m_1,\ldots,m_N)$ are the phonon occupation numbers at different sites. We restrict ourselves to the truncated phonon Hilbert space that includes states with the total number of phonons $m = \sum_{n=1}^N m_n$ not larger than M, where $0 \leqslant m_n \leqslant m$. Accordingly, the total Hilbert space of the system has the dimension $D = D_{\mathrm{e}} D_{\mathrm{ph}}$, where $D_{\mathrm{e}} = N$ and $D_{\mathrm{ph}} = (M+N)!/(M!N!)$.

The dimension of the Hamiltonian matrix to be diagonalized can be further reduced by exploiting the discrete translational symmetry of the system, whose mathematical expression is the commutation $[H_{\mathrm{eff}}, K_{\mathrm{tot}}] = 0$ of operators H_{eff} and K_{tot}. This permits diagonalization of H_{eff} in Hilbert-space sectors corresponding to the eigensubspaces of K_{tot}, where the dimension of each of those K sectors of the total Hilbert space coincides with that of the truncated phonon space, i.e., $D_K = D_{\mathrm{ph}}$. To this end, we utilize the symmetry-adapted basis

$$|K,\mathbf{m}\rangle = N^{-1/2} \sum_{n=1}^{N} e^{iKn}\, \mathcal{T}_n(|1\rangle_{\mathrm{e}} \otimes |\mathbf{m}\rangle_{\mathrm{ph}}), \tag{A2}$$

with $\mathcal{T}_n$ being the (discrete) translation operators whose action ought to comply with the periodic boundary conditions. The last equation can be recast as

$$|K,\mathbf{m}\rangle = N^{-1/2} \sum_{n=1}^{N} e^{iKn}\, |n\rangle_{\mathrm{e}} \otimes \mathcal{T}_n^{\mathrm{ph}} |\mathbf{m}\rangle_{\mathrm{ph}}, \tag{A3}$$

where the operators $\mathcal{T}_n^{\mathrm{ph}}$ represent the action of discrete translations in the phonon Hilbert space. Note that, if $|\mathbf{m}\rangle_{\mathrm{ph}}$ is defined by a set of occupation numbers

$$|\mathbf{m}\rangle_{\mathrm{ph}} = |m_1, m_2, \ldots, m_N\rangle_{\mathrm{ph}}, \tag{A4}$$

then $\mathcal{T}_n^{\rm ph}\,|\mathbf{m}\rangle_{\rm ph}\equiv|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle$ is given by

$$\left|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=|m_{N-n+1},m_{N-n+2},\ldots,m_{N-n}\rangle_{\rm ph}. \tag{A5}$$

In general, in terms of the original phonon occupation numbers, the rth occupation number in $|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle$ is given by $m_{s(r,n)}$, where the site index $s(r,n)$ is defined by

$$s(r,n)\equiv\begin{cases}N-n+r, & \text{for } r\leqslant n\\ r-n, & \text{for } r>n.\end{cases} \tag{A6}$$

Regarding the ground-state calculations, we follow an established phonon Hilbert-space truncation procedure [23] whereby the system size (N) and the maximum number of phonons retained (M) are increased until the convergence for the ground-state energy and the phonon distribution is reached. Our adopted convergence criterion is that the relative error in the ground-state energy and the phonon distribution upon further increase of N and M is not larger than 10^{-4}. The adopted criterion is here satisfied for the system size $N=9$ (with periodic boundary conditions) and requires the total of $M=10$ phonons.

APPENDIX B: MATRIX ELEMENTS AND EXPECTATION VALUES

1. Derivation of the matrix elements of relevant observables

In what follows, we first derive the expressions for the expectation values of a generic observable with respect to the state $|\psi(t)\rangle$ of the system at time t. In view of our use of the symmetry-adapted basis [cf. Eq. (A2)], we do so by deriving the matrix elements of the same observables in that basis. We then specialize to the relevant observables for our present work, the total phonon (photon) number [defined by Eq. (18)], as well as the variances of the phonon position and momentum quadratures corresponding to an arbitrary site [defined by Eq. (20)].

We start from the decomposition of the state $|\psi(t)\rangle$ in the symmetry-adapted basis [defined in Eq. (A2) above]

$$|\psi(t)\rangle=\sum_{\mathbf{m}}C_{\mathbf{m}}^K(t)|K,\mathbf{m}\rangle, \tag{B1}$$

where the expansion coefficients $C_{\mathbf{m}}^K(t)$ can be obtained through our computation of the state evolution. For an arbitrary observable A we then have

$$\langle\psi(t)|A|\psi(t)\rangle=\sum_{\mathbf{m},\mathbf{m}'}C_{\mathbf{m}'}^{K*}(t)C_{\mathbf{m}}^K(t)\langle K,\mathbf{m}'|A|K,\mathbf{m}\rangle, \tag{B2}$$

which, with already known expansion coefficients, leaves us with the task of calculating the matrix elements $\langle K,\mathbf{m}'|\,A\,|K,\mathbf{m}\rangle$ for the relevant observables.

Assuming, as is the case for our relevant observables, that A depends only on phonon operators, it is straightforward to show, using Eq. (A3), that

$$\langle K,\mathbf{m}'|\,A\,|K,\mathbf{m}\rangle=\frac{1}{N}\sum_{n=1}^{N}\left\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'\right|A\left|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle, \tag{B3}$$

where in deriving this last result we made use of the fact that ${}_{\rm e}\langle n'|n\rangle_{\rm e}=\delta_{nn'}$. Before embarking on further derivations it is useful to note that $\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle$ is independent of n and equals 1 if the two sets of phonon occupation numbers, $\mathbf{m}$ and $\mathbf{m}'$, are completely the same, otherwise it evaluates to zero. In other words,

$$\left\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'\middle|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=\delta_{\mathbf{m},\mathbf{m}'}. \tag{B4}$$

In the simplest case, for $A=a_r^\dagger a_r$, we first note that

$$a_r^\dagger a_r\left|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=m_{s(r,n)}\left|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle, \tag{B5}$$

where $s(i,n)$ is defined by Eq. (A6). By inserting the last result into Eq. (B3) and making use of Eq. (B4) we then easily obtain that

$$\langle K,\mathbf{m}'|a_r^\dagger a_r\,|K,\mathbf{m}\rangle=\frac{\delta_{\mathbf{m},\mathbf{m}'}}{N}\sum_{n=1}^{N}m_n, \tag{B6}$$

where, owing to the discrete translational symmetry of the system, the RHS of the last equation does not explicitly depend on r. [In writing the last equation, we made use of the fact that $\sum_{n=1}^N m_{s(r,n)}\equiv\sum_{n=1}^N m_n$.] By extension, for $A=\sum_{r=1}^N a_r^\dagger a_r$ (total photon number), we get

$$\langle K,\mathbf{m}'|\sum_{r=1}^{N}a_r^\dagger a_r|K,\mathbf{m}\rangle=\delta_{\mathbf{m},\mathbf{m}'}\sum_{n=1}^{N}m_n. \tag{B7}$$

Upon inserting the last result into the general equation (B2), we obtain the desired expectation value

$$\langle\psi(t)|\sum_{r=1}^{N}a_r^\dagger a_r|\psi(t)\rangle=\sum_{n,\mathbf{m}}m_n\left|C_{\mathbf{m}}^K(t)\right|^2. \tag{B8}$$

For $A=a_r$, we first notice that

$$a_r\left|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=\sqrt{m_{s(r,n)}}\left|\mathcal{T}_n^{\rm ph}\mathbf{m}_{(r,-1)}\right\rangle, \tag{B9}$$

where $|\mathcal{T}_n^{\rm ph}\mathbf{m}_{(r,-1)}\rangle$ is the vector obtained by changing the rth occupation number in $|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle$ from $m_{s(r,n)}$ to $m_{s(r,n)}-1$. This implies that

$$\left\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'\middle|a_r\middle|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=\sqrt{m_{s(r,n)}} \tag{B10}$$

provided that the two sets ($\mathbf{m}$ and $\mathbf{m}'$) have the same occupation numbers except at site $s(r,n)$ where $m'_{s(r,n)}$ should be equal to $m_{s(r,n)}-1$; otherwise, $\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'|a_r|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle=0$. The desired matrix element $\langle K,\mathbf{m}'|a_r|K,\mathbf{m}\rangle$ is obtained by combining Eq. (B10) and the general result in Eq. (B3).

In an analogous fashion, for $A=a_r^\dagger$ we obtain

$$\left\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'\middle|a_r^\dagger\middle|\mathcal{T}_n^{\rm ph}\mathbf{m}\right\rangle=\sqrt{m_{s(r,n)}+1} \tag{B11}$$

if the two sets ($\mathbf{m}$ and $\mathbf{m}'$) have the same occupation numbers except at site $s(r,n)$ where $m'_{s(r,n)}$ should be equal to $m_{s(r,n)}+1$; otherwise, $\langle\mathcal{T}_n^{\rm ph}\mathbf{m}'|\,a_r^\dagger\,|\mathcal{T}_n^{\rm ph}\mathbf{m}\rangle=0$. The matrix element $\langle K,\mathbf{m}'|\,a_r^\dagger\,|K,\mathbf{m}\rangle$ sought for is easily obtained by inserting the expression in Eq. (B11) into the general equation (B3).

By combining the derived expressions for $\langle K,\mathbf{m}'|a_r|K,\mathbf{m}\rangle$ and $\langle K,\mathbf{m}'|a_r^\dagger|K,\mathbf{m}\rangle$, we can easily obtain the desired results for $\langle K,\mathbf{m}'|x_r|K,\mathbf{m}\rangle$ and $\langle K,\mathbf{m}'|p_r|K,\mathbf{m}\rangle$.

When $A=x_r^2=(a_r^\dagger+a_r)^2/2$ or $A=p_r^2=-(a_r^\dagger-a_r)^2/2$, we first note that

$$x_r^2\equiv\tfrac{1}{2}\left[2a_r^\dagger a_r+1+(a_r^\dagger)^2+a_r^2\right], \tag{B12}$$

$$p_r^2 \equiv \tfrac{1}{2}\big[2a_r^\dagger a_r + 1 - (a_r^\dagger)^2 - a_r^2\big]. \tag{B13}$$

Repeating the above procedure, to compute the desired matrix elements $\langle K, \mathbf{m}'|(a_r^\dagger)^2|K, \mathbf{m}\rangle$ and $\langle K, \mathbf{m}'|a_r^2|K, \mathbf{m}\rangle$, we have to first determine $\langle \mathcal{T}_n^{\text{ph}}\mathbf{m}'|(a_r^\dagger)^2|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle$ and $\langle \mathcal{T}_n^{\text{ph}}\mathbf{m}'|a_r^2|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle$. It is straightforward to show that, for instance,

$$\big\langle \mathcal{T}_n^{\text{ph}}\mathbf{m}'\big|(a_r^\dagger)^2\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle = \sqrt{[m_{s(r,n)}+1][m_{s(r,n)}+2]} \tag{B14}$$

provided that the two sets ($\mathbf{m}$ and $\mathbf{m}'$) have the same occupation numbers except at site $s(r,n)$ where $m'_{s(r,n)}$ should be equal to $m_{s(r,n)}+2$; otherwise, $\langle K, \mathbf{m}'|(a_r^\dagger)^2|K, \mathbf{m}\rangle = 0$. Similarly, we have that

$$\big\langle \mathcal{T}_n^{\text{ph}}\mathbf{m}'\big|a_r^2\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle = \sqrt{m_{s(r,n)}[m_{s(r,n)}-1]} \tag{B15}$$

if the two sets ($\mathbf{m}$ and $\mathbf{m}'$) have the same occupation numbers except at site $s(r,n)$ where $m'_{s(r,n)}$ should be equal to $m_{s(r,n)}-2$; otherwise, $\langle \mathcal{T}_n^{\text{ph}}\mathbf{m}'|a_r^2|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle = 0$.

With the aid of the expressions for the matrix elements obtained thus far, and using the general expression in Eq. (B2) with the coefficients $C_{\mathbf{m}}^K(t)$ obtained from the computation of the system dynamics, one can straightforwardly obtain the variances $S_x(t)$ and $S_p(t)$ of the position and momentum quadratures [cf. Eq. (20)].

2. Derivation of the matrix elements of the reduced density matrix

In what follows, we derive expressions for the matrix elements of the reduced density matrix assuming that the system under consideration evolves starting from a bare-excitation Bloch state with quasimomentum k_0 at $t = 0$.

We make use of our standard symmetry-adapted basis [cf. Eq. (A3)] for $K = k_0$:

$$|K = k_0, \mathbf{m}\rangle = N^{-1/2}\sum_{n=1}^{N} e^{ik_0 n}|n\rangle_{\text{e}} \otimes \mathcal{T}_n^{\text{ph}}|\mathbf{m}\rangle_{\text{ph}} \tag{B16}$$

and start by expanding the state $|\psi(t)\rangle$ of the system at time t with respect to this basis:

$$|\psi(t)\rangle = \sum_{\mathbf{m}} C_{\mathbf{m}}^{k_0}(t)|k_0, \mathbf{m}\rangle. \tag{B17}$$

The density matrix of our composite (bipartite) e-ph system at time t is given by Eq. (21) and, using the expansion in Eq. (B17), can be expressed as

$$\rho_{\text{e-ph}}(t) = \sum_{\mathbf{m},\mathbf{m}'} C_{\mathbf{m}'}^{k_0*}(t)C_{\mathbf{m}}^{k_0}(t)|k_0, \mathbf{m}\rangle\langle k_0, \mathbf{m}'|. \tag{B18}$$

By now making use of Eq. (A3), i.e., its special case for $K = k_0$, we further obtain

$$\rho_{\text{e-ph}}(t) = N^{-1}\sum_{\mathbf{m},\mathbf{m}'}\sum_{n,n'=1}^{N} e^{ik_0(n-n')}C_{\mathbf{m}'}^{k_0*}(t)C_{\mathbf{m}}^{k_0}(t) \times |n\rangle\langle n'| \otimes \big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle\big\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\big|. \tag{B19}$$

The reduced excitation density matrix is obtained by tracing the last density matrix over the phonon basis [cf. Eq. (22)]. Let $\mathbf{m}''$ be the dummy index for the phonon basis states, i.e., the set of all phonon occupation-number configurations. Then we have

$$\rho_{\text{e}}(t) = \sum_{\mathbf{m}''}\langle\mathbf{m}''|\rho_{\text{e-ph}}(t)|\mathbf{m}''\rangle, \tag{B20}$$

which, by inserting $\rho_{\text{e-ph}}(t)$ from Eq. (B19), becomes

$$\rho_{\text{e}}(t) = N^{-1}\sum_{\mathbf{m},\mathbf{m}',\mathbf{m}''}\sum_{n,n'=1}^{N} e^{ik_0(n-n')}C_{\mathbf{m}'}^{k_0*}(t)C_{\mathbf{m}}^{k_0}(t) \times \big\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\big|\mathbf{m}''\big\rangle\big\langle\mathbf{m}''\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle|n\rangle\langle n'|. \tag{B21}$$

We now note that

$$\sum_{\mathbf{m}''}\big\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\big|\mathbf{m}''\big\rangle\big\langle\mathbf{m}''\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle = \big\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle, \tag{B22}$$

where we made use of the completeness relation in the phonon Hilbert space

$$\sum_{\mathbf{m}''}|\mathbf{m}''\rangle\langle\mathbf{m}''| = \mathbb{1}. \tag{B23}$$

Using the result in Eq. (B22), the expression for $\rho_{\text{e}}(t)$ in Eq. (B21) now simplifies to

$$\rho_{\text{e}}(t) = N^{-1}\sum_{\mathbf{m},\mathbf{m}'}\sum_{n,n'=1}^{N} e^{ik_0(n-n')}C_{\mathbf{m}'}^{k_0*}(t)C_{\mathbf{m}}^{k_0}(t) \times \big\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\big|\mathcal{T}_n^{\text{ph}}\mathbf{m}\big\rangle|n\rangle\langle n'|. \tag{B24}$$

From the last equation we readily read off the final expression for the matrix elements of the reduced excitation density matrix:

$$(\rho_{\text{e}})_{nn'}(t) = N^{-1}e^{ik_0(n-n')}\sum_{\mathbf{m},\mathbf{m}'} C_{\mathbf{m}'}^{k_0*}(t)C_{\mathbf{m}}^{k_0}(t)\big\langle\mathbf{m}'\big|\mathcal{T}_{n-n'}^{\text{ph}}\mathbf{m}\big\rangle, \tag{B25}$$

where we made use of the fact that $\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle \equiv \langle\mathbf{m}'|\mathcal{T}_{n-n'}^{\text{ph}}\mathbf{m}\rangle$. It is also useful to note that the final result in Eq. (B25) can more succinctly be recast as

$$(\rho_{\text{e}})_{nn'}(t) = N^{-1}e^{ik_0(n-n')}\langle\psi(t)|\mathcal{T}_{n-n'}^{\text{ph}}|\psi(t)\rangle. \tag{B26}$$

In order to evaluate the matrix element $\langle\mathbf{m}'|\mathcal{T}_{n-n'}^{\text{ph}}\mathbf{m}\rangle$, it is useful to recall Eqs. (A5) and (A6). Note that $\langle\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle = 1$ if all the corresponding phonon occupation numbers in $|\mathcal{T}_n^{\text{ph}}\mathbf{m}\rangle$ and $|\mathcal{T}_{n'}^{\text{ph}}\mathbf{m}'\rangle$ are the same, otherwise this matrix element evaluates to zero.

APPENDIX C: CHEBYSHEV-PROPAGATOR METHOD FOR DYNAMICS

In the following, we briefly recapitulate the essential aspects of the computational technique utilized in the present work, the Chebyshev-propagator method (CPM) [39], followed by some basic details of our concrete implementation thereof. A more detailed introduction into the CPM is provided in Ref. [40].

1. Basics of the CPM

For a system described by the Hamiltonian H, the time-evolution operator $U(t+\delta t, t) = U(\delta t) = e^{-iH\delta t}$ can be expanded in a finite series of N_{C} Chebyshev polynomials of the first kind $T_p(x) = \cos(p \arccos x)$ [39,40]:

$$U(\delta t) = e^{-ib\delta t}\left[c_0(a\delta t) + 2\sum_{p=1}^{N_{\mathrm{C}}} c_p(a\delta t) T_p(\widetilde{H})\right]. \tag{C1}$$

Here $\widetilde{H} = (H-b)/a$ is a rescaled Hamiltonian of the system, where $E_{\min}$ ($E_{\max}$) is the minimal (maximal) eigenvalue of H, $b = (E_{\max} + E_{\min})/2$, and $a = (E_{\max} - E_{\min} + \epsilon)/2$, with $\epsilon = \alpha_c(E_{\max} - E_{\min})$ being introduced to ensure that the rescaled spectrum lies well inside $[-1, 1]$. The expansion coefficients are given by

$$c_p(a\delta t) = \int_{-1}^{1} \frac{T_p(x)e^{-ixt}}{\pi\sqrt{1-x^2}} dx = (-i)^p J_p(a\delta t), \tag{C2}$$

where $J_p(a\delta t)$ is the pth-order Bessel function of the first kind. In cases where the system Hamiltonian does not depend explicitly on time, these expansion coefficients also depend only on the time step δt (but not explicitly on time t), thus it is sufficient to compute them only once.

The recurrence relations for the Chebyshev polynomials [39] can be used to simplify the computation of the state evolution $|\psi(t+\delta t)\rangle = U(\delta t)|\psi(t)\rangle$ from one point on a time grid to the next one. The problem is effectively reduced to the iterative evaluation of vectors $|v_p\rangle \equiv T_p(\widetilde{H})|\psi(t)\rangle$ using the recursive relation

$$|v_{p+1}\rangle = \widetilde{H}|v_p\rangle - |v_{p-1}\rangle, \tag{C3}$$

where $|v_0\rangle = \psi(t)$ and $|v_1\rangle = \widetilde{H}|v_0\rangle$. Evolving the state vector $|\psi(t)\rangle$ from one time step to the next one requires N_{C} matrix-vector multiplications of a given complex vector with a sparse Hamiltonian matrix, a step that for the system evolution from $t=0$ to $t=t_f$ has to be performed $t_f/\delta t$ times.

Given that the CPM requires only the knowledge of two extremal eigenvalues of the system Hamiltonian, it is convenient to combine it with Lanczos-type diagonalization for sparse matrices [46]. The CPM has by now proven to be superior to other direct or iterative integration schemes, in terms of both computational cost and accuracy [54].

2. Implementation details and numerical consistency checks

The results obtained for the system dynamics in this work were based on calculations performed for a system with $N = 9$ qubits, with up to $M = 20$ phonons in the truncated phonon Hilbert space. Thus, the resulting maximal dimension of the truncated phonon Hilbert space (as discussed in Appendix A, this is also the dimension of any K sector of the full Hilbert space) was $D \approx 10^7$ and to make the storage of the nonzero matrix elements possible we used the sparse-matrix form. Reaching the numerical convergence in our dynamics calculations typically required us to use between $N_{\mathrm{C}} = 9$ and $N_{\mathrm{C}} = 14$ Chebyshev polynomials in the expansion given by Eq. (C1). In addition, the smallest time step required for numerical convergence was $\delta t = 0.05\tau_{\mathrm{e,c}}$, i.e., up to 20 time steps were used within the period that corresponds to the physically meaningful (excitation-hopping) timescale $\tau_{\mathrm{e,c}} \approx 0.44$ ns. Our runs included those with the total evolution times t_f as large as $100\tau_{\mathrm{e,c}}$, i.e., with up to 2000 such time steps. In our calculations, ϵ was kept at the fixed value of 10^{-3} (cf. Appendix C 1).

We carried out our CPM-based dynamics calculations on an 8-core, 3.5 GHz Intel Xeon CPU E5-1620 machine, with a total of 32 GB of main memory. The runs that were required to obtain all the results presented in this paper consumed less than 250 CPU hours.

The results were checked for consistency whenever it was possible. In particular, testing for unitarity turned out to be a good measure of convergence of the CPM. Namely, at each iteration step the norm of the evolving state vector was calculated and any deviation from unity larger than 10^{-4} was considered a surefire sign that a higher precision (i.e., either a shorter time step or a larger N_{C}) is needed. Despite the fact that we maintained this unitarity margin of error to be much lower than 10^{-4} throughout our calculations, this was not always sufficient and additional convergence tests, performed by increasing computational precision and confirming the stability of the results, were carried out.

As another internal consistency check, we used the mathematical relation between the expectation values of x_r^2, p_r^2, and the phonon-number operator $a_r^\dagger a_r$ that stems from the identity $x_r^2 + p_r^2 = 2a_r^\dagger a_r + 1$. This relation was satisfied by our data at 10^{-7} precision, which is a highly nontrivial test as the expectation values of x_r^2 and p_r^2 on one hand, and that of $a_r^\dagger a_r$ on the other, were evaluated by completely different and mutually independent means.

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DREENA-C framework: joint R_{AA} and v_2 predictions and implications to QGP tomography

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Abstract

In this paper, we presented our recently developed Dynamical Radiative and Elastic ENergy loss Approach (DREENA-C) framework, which is a fully optimized computational suppression procedure based on our state-of-the-art dynamical energy loss formalism in constant temperature finite size QCD medium. With this framework, we have generated, for the first time, joint R_{AA} and v_2 predictions within our dynamical energy loss formalism. The predictions are generated for both light and heavy flavor probes, and different centrality regions in Pb + Pb collisions at the LHC, and compared with the available experimental data. While R_{AA} predictions agree with experimental data, v_2 predictions qualitatively agree with, but are quantitatively visibly above, the experimental data (in disagreement with other models, which underestimate v_2). Consistently with numerical predictions, through simple analytic analysis, we show that R_{AA} is insensitive to medium evolution (though highly sensitive to energy loss mechanisms), while v_2 is highly sensitive to the evolution. As a major consequence for precision quark-gluon plasma (QGP) tomography, this then leaves a possibility to calibrate energy loss models on R_{AA} data, while using v_2 to constrain QGP parameters that are in agreement with both high and low $p_\perp$ data.

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Keywords: quark-gluon plasma, parton energy loss, heavy quarks, perturbative QCD, QGP tomography, high pt suppression predictions

(Some figures may appear in colour only in the online journal)

1. Introduction

Quark-gluon plasma (QGP) is a new state of matter [1, 2] consisting of interacting quarks, antiquarks and gluons. Such a new state of matter is created in ultra-relativistic heavy ion collisions at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC). Rare high momentum probes, which are created in such collisions and which transverse QGP, are excellent probes of this extreme form of matter [3–5]. Different observables (such as angular averaged nuclear modification factor R_{AA} and angular anisotropy v_2), together with probes with different masses, probe this medium in a different manner. Therefore, comparing comprehensive set of joint predictions for different probes and observables, with available experimental data from different experiments, collision systems and collision energies, allows investigating properties of QCD medium created in these collisions [6–12].

However, to implement this idea, it is necessary to have a model that realistically describes high-$p_\perp$ parton interactions with the medium. With this goal in mind, we developed state-of-the-art dynamical energy loss formalism [13, 14], which includes different important effects (some of which are unique to this model). Namely, (i) the formalism takes into account finite size, finite temperature QCD medium consisting of dynamical (that is moving) partons, contrary to the widely used static scattering approximation and/or medium models with vacuum-like propagators. (ii) The calculations are based on the finite temperature field theory [15, 16], and generalized HTL approach, in which the infrared divergencies are naturally regulated, so the model does not have artificial cutoffs. (iii) Both radiative [13] and collisional [17] energy losses are calculated under the same theoretical framework, applicable to both light and heavy flavor. (iv) The formalism is generalized to the case of finite magnetic [18] mass and running coupling [19], and most recently, we also applied first steps towards removing widely used soft-gluon approximation from radiative energy loss calculations, enhancing the applicability region of this formalism [20]. This formalism was further integrated into numerical procedure [19], which includes initial $p_\perp$ distribution of leading partons [21, 22], energy loss with path-length [23, 24] and multi-gluon [25] fluctuations, and fragmentation functions [26–28], to generate the final medium modified distribution of high $p_\perp$ hadrons. While all the above effects have to be included based on theoretical grounds, it is plausible to ask whether all of these ingredients are necessary for accurately interpreting the experimental data, particularly since other available approaches [29–33] commonly neglect some—or many—of these effects. To address this important issue, in [34], we showed that, while abolishing widely used static approximation is the most important step for accurate suppression predictions, *including all other effects* is necessary for a fine agreement with high-$p_\perp$ R_{AA} (and v_2, not published) data.

To be able to generate predictions that can reasonably explain the experimental data, all ingredients stated above have to be preserved (with no additional simplifications used in the numerical procedure), as all of these ingredients were shown to be important for reliable theoretical predictions of jet suppression [34]. From computational perspective, it is also necessary to develop a framework that can efficiently generate wide set of theoretical predictions, to be compared with a broad range of available (or upcoming) experimental data. We here present DREENA-C (Dynamical Radiative and Elastic ENergy loss Approach) framework, which is the first step towards this goal. Due to the complexity of the underlying parton-medium interaction model, this first step takes into account the medium evolution in its simplest form, through mean (constant) medium temperature (thus 'C' in DREENA-C framework). In addition to presenting the necessary baseline to be compared with future redevelopments of the dynamical energy loss to more fully account for the medium evolution, DREENA-C is also an optimal numerical framework for studying the medium evolution effects on certain observables. That is, as this framework takes into account state-of-the-art parton-medium interaction model, but only rudimental medium evolution, comparison of its predictions with experimental data allows assessing sensitivity of certain variables to QGP evolution.

DREENA-C framework corresponds to, in its essence, the numerical procedure presented in [19], with a major new development that the code is now optimized to use minimal computer resources and produce predictions within more than two orders of magnitude shorter time compared to [19]. Such step is necessary, as all further improvements of the framework, necessarily need significantly more computer time and resources. So, without this development, further improvements, e.g. towards nontrivially evolving QGP medium, would not be realistically possible. That is, DREENA-C framework, addresses the goal of efficiently generating predictions for diverse observables.

Exploiting the ability to generate predictions for a wide range of observables, we will here use DREENA-C framework to, for the first time, present joint R_{AA} and v_2 theoretical predictions within our dynamical energy loss formalism; these predictions will be generated for different experiments (ALICE, CMS and ATLAS), probes (light and heavy) and experimental conditions (wide range of collision centralities). Note that some of our results correspond to true predictions (some centrality intervals for B and D mesons), while for other cases, e.g. for charged hadrons, they correspond to postdictions, as the experimental data are already available. Motivation for generating these predictions is the following: (i) the theoretical models up to now were not able to jointly explain these data, which is known as v_2 puzzle [35, 36]. That is, the models lead to underprediction of v_2, unless new phenomena (e.g. magnetic monopoles) are introduced [37]. (ii) Having this puzzle in mind, and the fact that other available models employ the complementary approach, i.e. combine simplified energy loss models with more sophisticated medium evolutions, this work will enable assessing to what extent state-of-the-art energy loss model, but with simplest QGP evolution, is able to jointly explain R_{AA} and v_2 data. To obtain additional understanding of this important issue, we will bellow complement DREENA-C predictions with analytical estimates. (iii) DREENA-C predictions will establish an important baseline for testing how future introduction of the medium evolution will improve the formalism. Moreover, such step-by-step introduction of different medium evolution effects in the model will also allow to investigate their importance in explaining the experimental data, which is highly relevant for QGP tomography.

2. Methods

The DREENA-C framework is a fully optimized numerical procedure, which contains all ingredients presented in detail in [19]. We below briefly outline the main steps in this procedure.

The quenched spectra of light and heavy flavor observables are calculated according to the generic pQCD convolution:

$$\frac{E_f \mathrm{d}^3\sigma}{\mathrm{d}p_f^3} = \frac{E_i \mathrm{d}^3\sigma(Q)}{\mathrm{d}p_i^3} \otimes P(E_i \to E_f) \otimes D(Q \to H_Q). \tag{1}$$

Subscripts i and f correspond, respectively, to 'initial' and 'final', and Q denotes initial light or heavy flavor jet. $E_i \mathrm{d}^3\sigma(Q)/\mathrm{d}p_i^3$ denotes the initial momentum spectrum, which are computed according to [21, 22], $P(E_i \to E_f)$ is the energy loss probability, computed within the dynamical energy loss formalism [13, 14], with multi-gluon [25], path-length fluctuations [24] and running coupling [19]. $D(Q \to H_Q)$ is the fragmentation function of light and heavy flavor parton Q to hadron H_Q, where for light flavor, D and B mesons we use, DSS [26], BCFY [27] and KLP [28] fragmentation functions, respectively.

Regarding the numerical procedure, a major new development is that the code is now optimized, so that it is two orders of magnitude faster compared to the brute-force approach applied in [19]. Technically, the main optimization method we used was a combination of tabulation and interpolation of values of intermediary functions that appear at various steps of the energy loss calculation. This approach significantly reduces the number of necessary integrations. However, it must be preceded by careful analysis of the behavior of interpolated functions and the function sampling must be tailored to this behavior, so that effectively no loss of precision is introduced. Furthermore, in comparison to the computation of [19], different and better suited methods of numerical integration were used (mostly quasi Monte Carlo integration), producing a large speedup, higher integration precision and stability of the underlying results. Finally, the code was parallelized to take advantage of contemporary multi-core workstations. Furthermore, the optimization also allowed for further improvements of the physical model: (i) due to numerical constraints, in the previous multi-gluon fluctuation procedure, the number of radiated gluons was limited to 3. The procedure is now redeveloped to include the arbitrary number of radiated gluons; the detailed numerical analysis (both from the point of numerical precision and time efficiency) showed that the optimal limit of gluons to be included in the procedure is 4–5. (ii) Both radiative and collisional energy losses are now combined gradually along the traversed path of the parton, unlike in [19], where radiative and collisional losses were accounted separately.

As noted above, we model the medium by assuming a constant average temperature of QGP. We concentrate on the central rapidity region in 5.02 TeV Pb + Pb collisions at the LHC, though we note that these predictions will be applicable for 2.76 TeV Pb + Pb collisions as well, since the predictions for these two collision energies almost overlap [38]. To determine the temperature for each centrality region in 5.02 TeV Pb + Pb collisions, we use [39, 40] $T^3 \sim \frac{\frac{dN_g}{dy}}{A_\perp \bar{L}} \to T = c\left(\frac{\frac{dN_{ch}}{dy}}{A_\perp \bar{L}}\right)^{1/3}$, where $\frac{dN_g}{dy}$ is gluon rapidity density, $A_\perp$ is the overlap area and $\bar{L}$ is the average size of the medium for each centrality region. At mid rapidity, $\frac{dN_g}{dy}$ is directly proportional to experimentally measured charged particle multiplicity $\frac{dN_{\rm ch}}{d\eta}$, which is measured for 5.02 TeV Pb + Pb collisions at the LHC across different centralities [41]. Furthermore, c is a constant, which can be fixed through ALICE measurement of effective temperature for 0%–20% centrality at 2.76 TeV Pb + Pb collisions LHC [42]. For each centrality region, path-length distributions (as well as overlap area $A_\perp$ and average size of the medium $\bar{L}$) are calculated following the procedure described in [23], with an additional hard sphere restriction $r < R_A$ in the Woods–Saxon nuclear density distribution to regulate the path lengths in the peripheral collisions.

In numerical calculations, we use no fitting parameters in generating predictions for comparison with the data, i.e. all the parameters correspond to standard literature values. We consider a QGP with $\Lambda_{\rm QCD} = 0.2$ GeV and $n_f = 3$. The temperature dependent Debye mass μ_E (T) is obtained from [43], while for the light quarks, we assume that their mass is dominated by the thermal mass $M \approx \mu_E/\sqrt{6}$, and the gluon mass is $m_g \approx \mu_E/\sqrt{2}$ [44]. The charm (bottom) mass is $M = 1.2$ GeV ($M = 4.75$ GeV). Finite magnetic mass effect is also included in our framework [18], as various non-perturbative calculations [45, 46] have shown that magnetic mass μ_M is different from zero in QCD matter created at the LHC and RHIC.

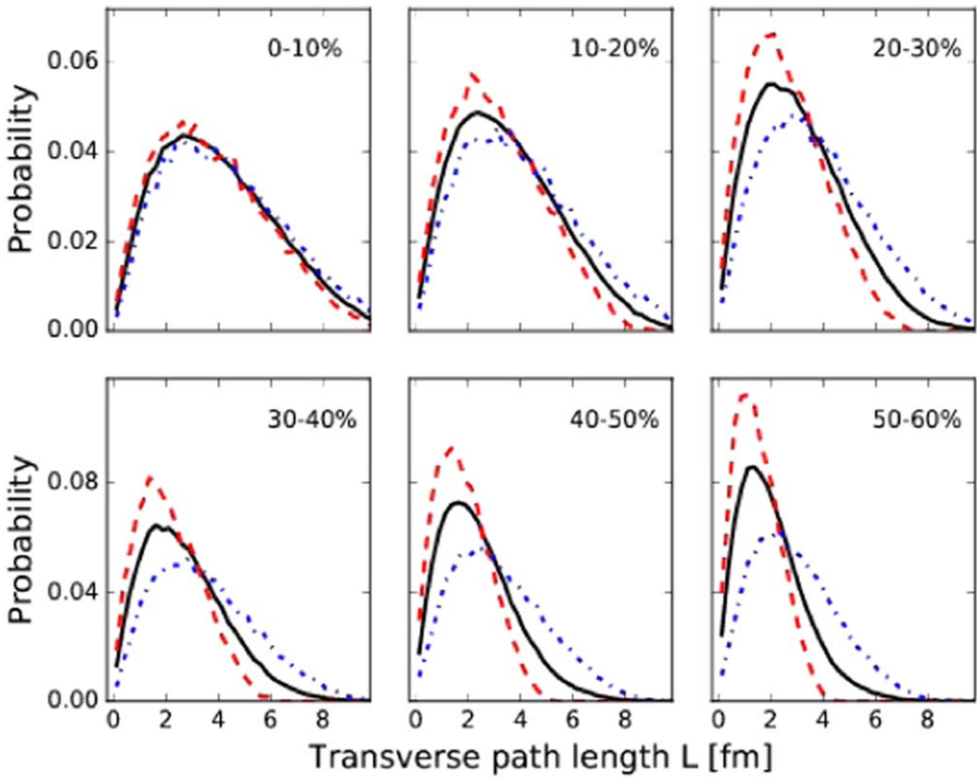

Figure 1. Path-length distributions. Probability distributions for hard parton path lengths in Pb + Pb collisions at $\sqrt{s_{NN}} = 5.02$ TeV for (0–10)%–(50–60)% centrality classes. Solid black curves: the total distributions with all hard partons included are represented; Dashed red curves: the distributions include only in-plane particles ($|\phi| < 15°$ or $||\phi|-180°| < 15°$); dashed–dotted blue curves: the distributions include only out-of-plane partons ($||\phi|-90°| < 15°$).

Magnetic to electric mass ratio is extracted from these calculations to be $0.4 < \mu_M/\mu_E < 0.6$, so presented uncertainty in the predictions comes from this range of screening masses ratio. Note that other uncertainties (e.g. in quark masses or effective temperature), are not included in this study. However, we have checked that uncertainties in the quark masses lead to small (up to 4% for $p_\perp > 8$ GeV, and decreasing with increasing $p_\perp$) difference in the resulting predictions. Regarding effective temperature, as this temperature comes with large error bars, in [47] we presented a detailed study of how this uncertainty affects the R_{AA} calculations. We found that R_{AA} dependence on T is almost linear (and the same for all parton energies and all types of flavor) and does not significantly affect the suppression, concluding that uncertainty in the effective temperature would basically lead to a systematic (constant value) shift in the predictions, i.e. the results presented in this paper would not be affected by this uncertainty.

3. Results and discussion

In this section, we will present joint R_{AA} and v_2 predictions for high $p_\perp$ charged hadrons, D and B mesons in Pb + Pb collisions at the LHC. In figure 1 we first show probability distributions for hard parton path lengths in Pb + Pb collisions for different centralities, obtained by the procedure specified in the previous section. For most central collisions, we observe that in-plane and out-of-plane distributions almost overlap with the total (average) path-length distributions, as expected. As the centrality increases, in-plane and out-of-plane distributions start to significantly separate (in different directions) from average path-length distributions. Having in mind that [48]

$$v_2 \approx \frac{1}{2}\frac{R_{AA}^{\mathrm{in}} - R_{AA}^{\mathrm{out}}}{R_{AA}^{\mathrm{in}} + R_{AA}^{\mathrm{out}}}, \tag{2}$$

this leads to the expectation of v_2 being small in most central collisions and increasing with increasing centrality. Regarding the equation (2) above, note that this estimate presents a conventional way [48–51] to calculate high $p_\perp$ v_2, and it leads to exact result if the higher harmonics v_4, v_6, etc. are zero at high $p_\perp$, and the opening angle (where R_{AA}^{in} and R_{AA}^{out} are evaluated) goes to zero.

Based on path-length distributions from figure 1, we can now calculate average R_{AA}, as well as in-plane and out-of-plane R_{AA}s (R_{AA}^{in} and R_{AA}^{out}), and consequently v_2 for both light and heavy flavor probes and different centralities. We start by generating predictions for charged hadrons, where data for both R_{AA} and v_2 are available. Comparison of our joint predictions with experimental data is shown in figure 2, where left and right panels correspond, respectively, to R_{AA} and v_2. We see good agreement with R_{AA} data, which is also robust, i.e. achieved across wide range of centralities and experiments. Regarding v_2, we surprisingly see that our v_2 predictions are visibly above the data. This is in contrast with other energy loss models which consistently lead to underprediction of v_2, where to resolve this, new phenomena (e.g. magnetic monopoles) were introduced [37]. Despite this quantitative disagreement, we see a reasonable qualitative agreement between the model and the data, i.e. the predictions are just shifted above the data; this will be further discussed below.

In figure 3, we provide predictions for D meson average R_{AA} (left panel) and v_2 (right panel) data, for four different centrality regions. The predictions are compared with the available 5.02 TeV Pb + Pb experimental data. For average R_{AA}, we observe good agreement with the data. Regarding v_2, we observe similar behavior as for charged hadron: i.e. while we obtain a reasonable qualitative agreement with the measurements, quantitatively there is again an unexpected (having in mind predictions of other models) overestimation of the data. Figure 4 shows equivalent predictions as figure 3, only for B mesons. For R_{AA}, we compare our predictions with the available $B^\pm$ [58], B_s^0 [59], non-prompt J/Ψ [60, 61] and non-prompt D_0 [62] data. Note that we can compare B meson predictions with these indirect b quark suppression data, as due to interplay of collisional and radiative energy loss, B meson suppression is almost independent on $p_\perp$ for $p_\perp > 10\,\mathrm{GeV}$ [47], so the fragmentation/decay functions will not play a large role for different types of b quark observables. Also, note that our predictions are provided for mid-rapidity region; for non-prompt D_0 (which are given for $|y < 1|$), we see good agreement between our predictions and the data. For $B^\pm$ and non-prompt J/Ψ, our predictions show qualitatively good agreement, but overprediction of R_{AA} data. This is expected, having in mind that those data are given for $|y \lesssim 2|$, where both experiments show 30%–50% increase in R_{AA} with decreasing rapidity. Our predictions do not agree with B_s^0, but these data come with very large error bars. For v_2, we predict values significantly different from zero for all centrality regions, and see that our predictions agree with the available non-prompt J/Ψ data [61, 63], though we note that these predictions are given with very large error bars. This does not necessarily mean that heavy B meson flows, as flow is inherently connected with *low* $p_\perp$ v_2, and here we show predictions for high $p_\perp$. On

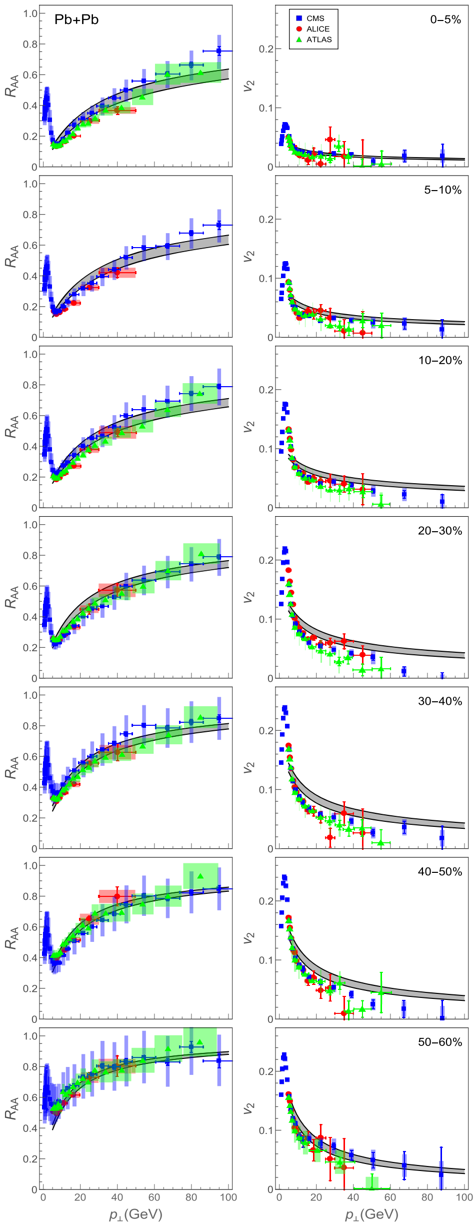

Figure 2. Joint R_{AA} and v_2 predictions for charged hadrons. *Left panels:* theoretical predictions for R_{AA} *versus* $p_\perp$ are compared with ALICE [52] (red circles), CMS [53] (blue squares) and ATLAS [54] (green triangles) charged hadron experimental data for 5.02 TeV Pb + Pb collisions at the LHC. *Right panels:* theoretical predictions for v_2 *versus* $p_\perp$ are compared with ALICE [55] (red circles), CMS [56] (blue squares) and ATLAS [57] (green triangles) charged hadron experimental data for 5.02 TeV Pb + Pb collisions at the LHC. The gray band boundaries correspond to $\mu_M/\mu_E = 0.4$ and $\mu_M/\mu_E = 0.6$. Rows 1–7 correspond to, respectively, 0%–5%, 5%–10%, 10%–20%,…, 50%–60% centrality regions.

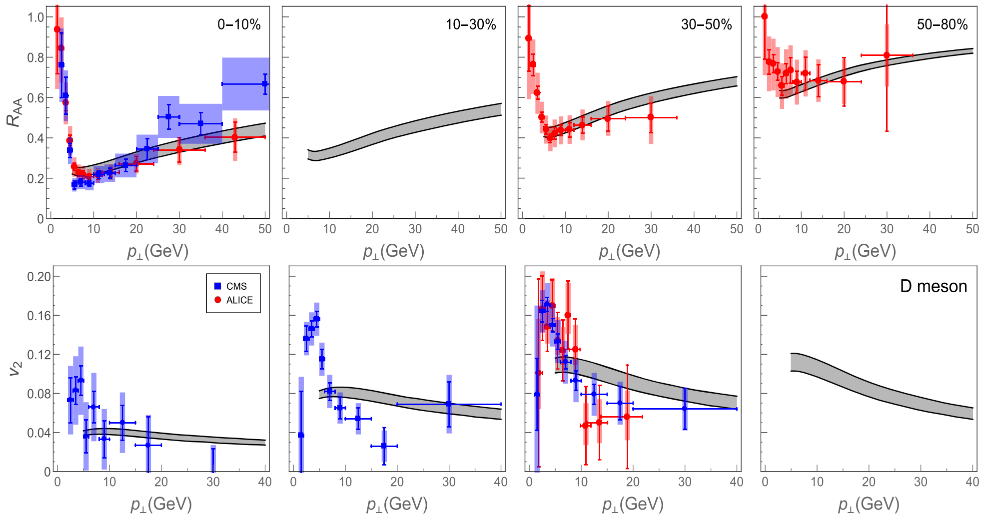

Figure 3. Joint R_{AA} and v_2 predictions for D mesons. *Upper panels:* theoretical predictions for R_{AA} *versus* $p_\perp$ are compared with ALICE [64] (red circles) and CMS [65] (blue squares) D meson experimental data for 5.02 TeV Pb + Pb collisions at the LHC. *Lower panels:* theoretical predictions for v_2 *versus* $p_\perp$ are compared with ALICE [51] (red circles) and CMS [66] (blue squares) D meson experimental data for 5.02TeV Pb + Pb collisions at the LHC. The gray band boundaries correspond to $\mu_M/\mu_E = 0.4$ and $\mu_M/\mu_E = 0.6$. First to fourth column correspond to, respectively, 0%–10%, 10%–30%, 30%–50% and 50%–80% centrality regions.

the other hand, high $p_\perp$ v_2 is connected with the difference in the energy loss (i.e. suppression) for particles going in different (e.g. in-plane and out-of-plane) directions; this difference then leads to our predictions of non-zero v_2 for *high* $p_\perp$ B mesons.

Overall, we see that our predicted R_{AA}s agree well with all measured (light and heavy flavor) data, while our v_2 predictions are consistently above the experimental data. Since our model has sophisticated description of parton-medium interactions, but highly simplified medium evolution model (through average medium temperature), these robust numerical results imply the following: (i) R_{AA} is largely insensitive to the medium evolution, in contrast to its (previously shown [34]) large sensitivity to parton-medium interactions. (ii) v_2 is sensitive to the details of medium evolution. These two conclusions have important implications for QGP tomography, in particular (i) R_{AA} can be used to calibrate parton-medium interaction models, while (ii) v_2 can be used to constrain QGP medium evolution parameters also from the point of high $p_\perp$ data (in addition to constraining them from low $p_\perp$ predictions/measurements). One should note that insensitivity of R_{AA} and sensitivity of v_2 predictions to QGP evolution were also observed by using very different models and numerical frameworks [67, 68]. This then clearly suggests that such (in)sensitivity may be a general phenomenon, but to claim this, one should also gain an analytical understanding, which we provide below. Furthermore, the numerical results presented above also lead to the following questions, which are important from the point of future precision QGP tomography: (i) what is the reason behind the observed overestimation of v_2 within DREENA-C framework, and can expanding medium lead to a better agreement with the experimental data? (ii) Do we expect that B meson v_2 predictions will still be non-zero, once the expanding medium is introduced?

To intuitively approach the issues raised above, we start by noting that, within our dynamical energy loss formalism, $\Delta E/E \sim T^a$ and $\Delta E/E \sim L^b$, where $a, b \to 1$ ($\Delta E/E$ is

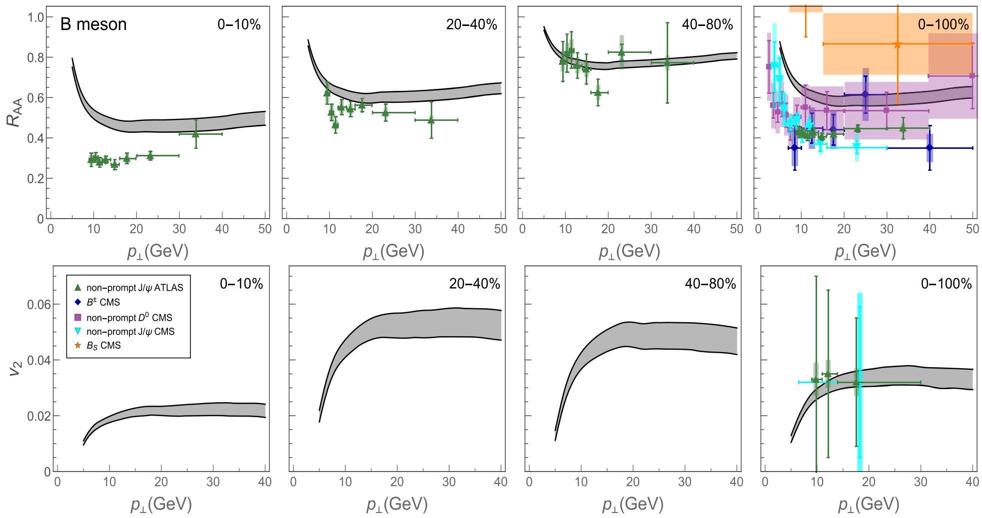

Figure 4. Joint R_{AA} and v_2 predictions for B mesons. *Upper panels*: theoretical predictions for B meson R_{AA} *versus* $p_\perp$ are compared with ATLAS [60] (green triangles), CMS [61] (cyan triangles) non-prompt J/Ψ, and CMS non-prompt D^0 [62] (purple squares), $B^\pm$ [58] (blue diamonds) and B_s^0 [59] (orange stars) experimental data for 5.02 TeV Pb + Pb collisions at the LHC. *Lower panels:* theoretical predictions for B meson v_2 *versus* $p_\perp$ are compared with ATLAS [63] (green triangles) and CMS [61] (cyan triangles) non-prompt J/Ψ for 5.02TeV Pb + Pb collisions at the LHC. The gray band boundaries correspond to $\mu_M/\mu_E = 0.4$ and $\mu_M/\mu_E = 0.6$. First to fourth column correspond to, respectively, 0%–10%, 20%–40%, 40%–80% and 0%–100% centrality regions.

fractional energy loss, T is the average temperature of the medium, while L is the average path-length traversed by the jet). To be more precise, note that both dependencies are close to linear, though a and b are still significantly different from 1 [38]. However, for the purpose of this estimate, let us assume that $a = b = 1$, leading to

$$\Delta E/E \approx \chi TL, \tag{3}$$

where χ is a proportionality factor.

Another commonly used estimate [25] is that

$$R_{AA} \approx \left(1 - \frac{1}{2}\frac{\Delta E}{E}\right)^{n-2}, \tag{4}$$

where n is the steepness of the initial momentum distribution function (i.e. approximate exponent of a power-law of initial momentum distribution $p_\perp^{-n}$), and $\Delta E/E$ is notably smaller than 1.

In the case when fractional energy loss $\Delta E/E \ll 1$, equation (4) becomes

$$R_{AA} \approx \left(1 - \frac{n-2}{2}\frac{\Delta E}{E}\right) \approx (1 - \xi TL), \tag{5}$$

where $\xi = (n-2)\,\chi/2$.

In the DREENA-C approach, T is constant, and the same in in-plane and out-of-plane directions, while $L_{\text{in}} = L - \Delta L$ and $L_{\text{out}} = L + \Delta L$, leading to

$$R_{AA} \approx \frac{1}{2}(R_{AA}^{\text{in}} + R_{AA}^{\text{out}}) \approx \frac{1}{2}(1 - \xi TL_{\text{in}} + 1 - \xi TL_{\text{out}})$$
$$= 1 - \xi T\frac{L_{\text{in}} + L_{\text{out}}}{2} = 1 - \xi TL, \tag{6}$$

and

$$v_2 \approx \frac{1}{2}\frac{R_{AA}^{\text{in}} - R_{AA}^{\text{out}}}{R_{AA}^{\text{in}} + R_{AA}^{\text{out}}} \approx \frac{1}{2}\frac{(1 - \xi TL_{\text{in}}) - (1 - \xi TL_{\text{out}})}{2(1 - \xi TL)}$$
$$= \frac{1}{2}\frac{\xi T\Delta L}{1 - \xi TL} \approx \frac{\xi T\Delta L}{2}. \tag{7}$$

If the medium evolves, and by assuming $1 + 1$D Bjorken time evolution [69] (as qualitatively sufficient for the early time dynamics [70]), the average temperature along in-plane will be larger than along out-of-plane direction [71], leading to $T_{\text{in}} = T + \Delta T$ and $T_{\text{out}} = T - \Delta T$ (where $\Delta L/L \cdot \Delta T/T \ll 1$). By repeating the above procedure in this case, it is straightforward to obtain

$$R_{AA} \approx 1 - \xi TL \tag{8}$$

and

$$v_2 \approx \frac{1}{2}\frac{(1 - \xi T_{\text{in}}L_{\text{in}}) - (1 - \xi T_{\text{out}}L_{\text{out}})}{2(1 - \xi TL)} = \frac{1}{2}\frac{\xi T\Delta L - \xi\Delta TL}{1 - \xi TL}$$
$$\approx \frac{\xi T\Delta L - \xi\Delta TL}{2}. \tag{9}$$

We see that, while v_2 explicitly depends on ΔT and ΔL, R_{AA} does not. Therefore, it follows that, consistently with previous numerical results, R_{AA} can be only weakly sensitive to QGP evolution, while v_2 is quite sensitive to this evolution; note that this is to our knowledge, the first time that analytical argument to sensitivity of R_{AA} and v_2 to medium evolution is provided. Moreover, from equations (7) and (9), we see that introduction of temperature evolution is expected to lower v_2 compared to constant T case. Consequently, an accurate/complete energy loss models, when applied in the context of constant temperature medium should lead to higher v_2 than expected, while introduction of T evolution in such models would lower the v_2 compared to non-evolving case. Based on this, and the fact that previous theoretical approaches were not able to reach high enough v_2 without introducing new phenomena [37], we argue that accurate description of high-$p_\perp$ parton-medium interactions is crucial for accurate description of high-$p_\perp$ experimental data. With regards to this, the above results strongly suggest that the dynamical energy loss formalism has the right features needed to accurately describe jet-medium interactions in QGP, which is crucial for high precision QGP tomography.

Regarding the second question mentioned above, for B meson to have $v_2 \approx 0$, it is straightforward to see that one needs $\Delta T/T \approx \Delta L/L$. Having in mind that $\Delta L/L$ is quite large for larger centralities (see figure 1), $\Delta T/T$ would also have to be about the same magnitude. We do not expect this to happen, based on our preliminary estimates of the temperature changes in in-plane and out-of-plane in $1 + 1$D Bjorken expansion scheme [69]. That is, our expectations is that B meson v_2 will be smaller than presented here, but still significantly larger than zero, at least for large centrality regions. However, this still remains to be tested in the future with the introduction of full evolution model within our framework.

4. Conclusion

In this paper, we introduced the DREENA-C framework, which is a computational suppression procedure based on our dynamical energy loss formalism in finite size QCD medium with constant (mean) medium temperature. This approach, which combines a state-of-the-art energy loss model, but with including QGP evolution in its simplest form, is complementary to other available models that combine simplified energy loss models with more sophisticated medium evolutions. As such, DREENA-C can provide an important insight to what extent the accurate description of high-$p_\perp$ parton-medium interactions *versus* accurate description of medium evolution is necessary for accurately explaining high $p_\perp$ R_{AA} and v_2 measurements.

We here used the DREENA-C framework to, for the first time, generate joint R_{AA} and v_2 predictions for both light and heavy flavor probes and different centrality regions in Pb + Pb collisions at the LHC, and compare them with the available experimental data. We consistently, through both numerical and analytical calculations, obtained that R_{AA} is sensitive to the average properties of the medium, while v_2 is highly sensitive to the details of the medium evolution. Analytical calculations brought another advantage of DREENA-C, as they would likely not be possible in frameworks with more complex medium evolution models, but bring simple and intuitive predictions/explanations for our results, which is necessary for better qualitative and quantitative understanding of the obtained results.

Since different medium evolution profiles have both different average properties and different details of the evolution, in precision QGP tomography, both R_{AA} and v_2 have to be jointly used to extract the QGP properties. The DREENA-C framework presents an optimal starting point for QGP tomography, as R_{AA} predictions (obtained through DREENA-C) can be first used to calibrate the energy loss model itself; that is, DREENA-C is fast (which is important for efficient energy loss calibration), and it does not contain the details of the medium evolution, which could provide an unwanted background for such a purpose. Once this crucial step of accurate description and calibration of parton-medium interactions is achieved, different more-detailed profiles of medium evolution (generated through different bulk medium models and parameters, with and without event by event fluctuations) can be tested (through our future advancement of DREENA framework) to assess which of these profiles provide a simultaneous agreement with both high $p_\perp$ R_{AA} and v_2 data, across wide range of diverse experimental data and without further adjustment of energy loss models. In this way, QGP parameters can be constrained from both low and high $p_\perp$ measurements.

Furthermore, other approaches face difficulties in jointly explaining R_{AA} and v_2 data, where smaller v_2, than experimentally observed, is obtained. In distinction to other approaches, we here obtained an overprediction of v_2, where the analytical estimates moreover show that inclusion of more realistic medium evolution models would lead to better agreement with the data. This, together with the fact that v_2 prediction provided here already qualitatively (though not quantitatively) agree with the data, indicate an important (and highly non-trivial) conclusion that accurate description of high-$p_\perp$ parton interactions with QGP is likely the most important ingredient for generating high-$p_\perp$ predictions. These results therefore strongly suggest that our dynamical energy loss formalism provides a suitable basis for the QGP tomography (outlined above), which is our main future goal.

Acknowledgments

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$O(6)$ algebraic theory of three nonrelativistic quarks bound by spin-independent interactions

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We apply the newly developed theory of permutation-symmetric $O(6)$ hyperspherical harmonics to the quantum-mechanical problem of three nonrelativistic quarks confined by a spin-independent three-quark potential. We use our previously derived results to reduce the three-body Schrödinger equation to a set of coupled ordinary differential equations in the hyper-radius R with coupling coefficients expressed entirely in terms of (i) a few interaction-dependent $O(6)$ expansion coefficients and (ii) $O(6)$ hyperspherical harmonics matrix elements that have been evaluated in our previous paper. This system of equations allows a solution to the eigenvalue problem with homogeneous three-quark potentials, the class of which includes a number of standard *Ansätze* for the confining potentials, such as the Y- and Δ-string ones. We present analytic formulas for the $\mathrm{K} = 2, 3, 4, 5$ shell states' eigenenergies in homogeneous three-body potentials, which we then apply to the Y and Δ strings as well as the logarithmic confining potentials. We also present numerical results for power-law pairwise potentials with the exponent ranging between -1 and $+2$. In the process, we resolve the 25-year-old Taxil and Richard vs Bowler *et al.* controversy regarding the ordering of states in the $\mathrm{K} = 3$ shell, in favor of the former. Finally, we show the first clear difference between the spectra of Δ- and Y-string potentials, which appears in $\mathrm{K} \geq 3$ shells. Our results are generally valid, not just for confining potentials but also for many momentum-independent permutation-symmetric homogenous potentials that need not be pairwise sums of two-body terms. The potentials that can be treated in this way must be square integrable under the $O(6)$ hyperangular integral, the class of which, however, does not include the Dirac δ function.

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I. INTRODUCTION

The nonrelativistic three-quark system has been the basis of our understanding of baryon spectroscopy for more than 50 years; of course, this model also has many limitations, its nonrelativistic character being just one of several. After the November 1974 discovery of charmed hadrons, the nonrelativistic nature stopped being a detriment, at least in the case of heavy quarks. There are, of course, still only comparatively few heavy-quark baryons in Particle Data Group tables, and fewest of all are the triple-heavy ones. That circumstance will not prevent us from trying to understand them, however. Indeed, even if there were no heavy-quark baryons at all, it would still be an important systematic question to answer, if for no other reason than to have a definite benchmark against which to compare relativistic calculations.

Chronologically, at first, all calculations were done with a harmonic oscillator potential, due to its integrability, but with passing time, other "more realistic" potentials, such as the pairwise sum of the Coulomb and linearly rising two-body potentials plus various forms of "strong hyperfine" interactions, have been used in numerical calculations. Such calculations generally involve uncontrolled, sometimes drastic, approximations, such as the introduction of cutoff(s), due to the contact nature of the strong hyperfine interactions, thus leaving open many questions about the level ordering, convergence, and even existence of energy spectra in such calculations [1].

In this, the third in a series of papers, we show that the nonrelativistic three-quark problem does have a well-defined spectrum for a class of (homogeneous) potentials that includes the "standard" confinement potentials. This development is based on two previous (sets of) papers: (1) Refs. [8,9], wherein the three-body permutation symmetry-adapted $O(6)$ hyperspherical harmonics were constructed, and (2) Ref. [10], wherein we applied the said permutation symmetry-adapted $O(6)$ hyperspherical harmonics to the problem of three nonrelativistic identical particles in a homogeneous potential. Here, we present a mathematically well-defined method for solving the three-heavy quarks problem, together with several examples: the $\mathrm{K} = 0, \ldots, 5$ shells. These examples turn out to be (very) instructive, as they clearly mark out the region of applicability of our method.

In spite of the huge amount of literature on the quantum-mechanical three-body bound-state problem, in which the hyperspherical harmonics play a prominent role,

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Refs. [11–14], there are still many open problems related to the general structure of the three-body bound-state spectrum (e.g., the ordering of states, even in the simplest case of three identical particles). [15] The core of the existing difficulties can be traced back to the absence of a systematic construction of permutation-symmetric three-body wave functions. Until recently, see Refs. [8,9], permutation-adapted three-body hyperspherical harmonics in three dimensions were known explicitly only in a few special cases, such as those with total orbital angular momentum $L = 0$, 1, Refs. [13,16].

In this paper, we confine ourselves to the study of factorizable (into hyper-radial and hyperangular parts) three-body potentials that are square integrable [17] (in hyperangles) for technical reasons; for this class of potentials, our method allows closed-form ("analytical") results, at sufficiently small values of the grand angular momentum K (i.e., up to and including the $K \leq 8$ shell). Factorizable potentials include homogenous potentials, which in turn include pairwise sums of two-body power-law potentials, such as the linear (confining) "Δ-string," "Y-string" [19,20], and Coulomb potentials. Lattice QCD studies [21–23] suggest that three static quarks potential is a (linear) combination of the aforementioned three.

Singular potentials, such as the (strong, or electromagnetic) hyperfine interactions, that include the Dirac δ function, even though homogeneous, do not fall into the class of potentials susceptible to this method, as they are not square integrable; therefore, they require special attention and will be treated elsewhere. The spin-orbit potentials generally involve both the spin and the spatial variables for their permutation invariance, which requires special techniques. Simple inhomogenous potentials can only be treated numerically, however, using our method.

Strictly speaking, our (present) results are applicable only to three-equal-heavy quark systems, not one of which has been created in experiment, thus far (which does not mean that some are not forthcoming). This condition limits the method's applicability to c^3 and b^3 baryons only. Of course, in these two cases, there is no flavor multiplicity, and we may drop the $SU_{FS}(6)$ and $SU_F(3)$ labels. Nevertheless, we have kept the full $SU_{FS}(6)$ and $SU_F(3)$ labels, in the hope that in the future the present methods can and will be extended to (a) two identical and one distinct heavy-quark systems, such as the c^2b and b^2c, and (b) (semi)relativistic three-light quark systems.

This paper is divided into six sections and two Appendices. After the present Introduction, in Sec. II, we show how the Schrödinger equation for three particles in a homogenous/factorizable potential can be reduced to a single differential equation and an algebraic/numerical problem for their coupling strengths. In Sec. III, we defined the Y-string and Δ-string, the QCD Coulomb, and the logarithmic potentials and calculated the four lowest $O(6)$ hyperspherical harmonics expansion coefficients that are relevant to $K \leq 5$ shell states. In Sec. IV, we calculate the $K = 2$, 3, 4, 5 shells' level splittings in terms of four parameters that characterize the three-body potential. In Sec. V, we discuss our results, and in Sec. VI, we summarize and draw conclusions. The details of calculations are shown in Appendix B.

II. THREE-BODY PROBLEM IN HYPERSPHERICAL COORDINATES

In this section, we shall closely follow the treatment of the nonrelativistic three-body problem presented in Ref. [10].

The three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ can be transcribed from the Euclidean relative position (Jacobi) vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2})$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3})$, into hyperspherical coordinates as $\Psi(R, \Omega_5)$, where $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyper-radius and five angles Ω_5 that parametrize a hypersphere in the six-dimensional Euclidean space. Three ($\Phi_i; i = 1, 2, 3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyperangles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g., $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. As we saw above, one linear combination of the two variables $\boldsymbol{\rho}^2$ and $\boldsymbol{\lambda}^2$ is already taken by the hyper-radius R, so the shape space is two dimensional, and topologically equivalent to the surface of a three-dimensional sphere.

There are two traditional ways to parametrize this sphere: (1) the standard Delves choice [11] of hyperangles (χ, θ), which somewhat obscures the full S_3 permutation symmetry of the problem, and (2) the Iwai, Ref. [14], hyperangles (α, ϕ): $(\sin\alpha)^2 = 1 - (\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2})^2$, $\tan\phi = (\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2})$, reveal the full S_3 permutation symmetry of the problem: the angle α does not change under permutations, so all permutation properties are encoded in the ϕ dependence of the wave functions. We shall use the latter choice, as it leads to permutation-adapted hyperspherical harmonics, as explained in Refs. [8,9], in which specific hyperspherical harmonics used here are displayed.

We expand the wave function $\Psi(R, \Omega_5)$ in terms of hyperspherical harmonics $\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)$, $\Psi(R, \Omega_5) = \sum_{\mathrm{K},[\mathrm{m}]}\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)$, where K together with $[\mathrm{m}] = [Q, \nu, L, L_z = m]$ constitutes the complete set of hyperspherical quantum numbers: K is the hyperspherical angular momentum, L is the (total orbital) angular momentum, $L_z = m$ its projection on the z axis, Q is the Abelian quantum number conjugated with the Iwai angle ϕ, and ν is the multiplicity label that distinguishes between hyperspherical harmonics with the remaining four quantum numbers that are identical; see Ref. [8,9].

The hyperspherical harmonics turn the Schrödinger equation into a set of (infinitely) many coupled equations,

TABLE I. Expansion coefficients v_{KQ} of the Y- and Δ-string as well as of the Coulomb and logarithmic potentials in terms of $O(6)$ hyperspherical harmonics $\mathcal{Y}^{K,0,0}_{0,0}$, for $K = 0, 4, 8, 12$, respectively, and of the hyperspherical harmonics $\mathcal{Y}^{6,\pm 6,0}_{0,0}$.

(K, Q)	v_{KQ} (Y-central)	v_{KQ} (Y-string)	$v_{KQ}(\Delta)$	v_{KQ} (Coulomb)	v_{KQ} (Log)
(0,0)	8.18	8.22	16.04	20.04	−6.58
(4,0)	−0.443	−0.398	−0.445	2.93	−1.21
$(6, \pm 6)$	0	−0.027	−0.14	1.88	−0.56
(8,0)	−0.064	−0.064	−0.04	1.41	−0.33
(12,0)	−0.01	−0.01	0	0	−0.17

$$-\frac{1}{2\mu}\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{\mathrm{K}(\mathrm{K}+4)}{R^2}+2\mu E\right]\psi^{\mathrm{K}}_{[\mathrm{m}]}(R) + V_{\mathrm{eff}}(R)\sum_{\mathrm{K}',[\mathrm{m}']} C^{\mathrm{KK}'}_{[\mathrm{m}][\mathrm{m}']}\psi^{\mathrm{K}'}_{[\mathrm{m}']}(R) = 0, \tag{1}$$

with a hyperangular coupling coefficients matrix $C^{\mathrm{KK}'}_{[\mathrm{m}][\mathrm{m}']}$ defined by

$$V_{\mathrm{eff}}(R)C^{\mathrm{K}'\mathrm{K}}_{[\mathrm{m}'][\mathrm{m}]} = \langle \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)|V(R,\alpha,\phi)|\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)\rangle = V(R)\langle \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)|V(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)\rangle. \tag{2}$$

Factorizability of the potential is a simplifying assumption that leads to analytic results in the energy spectrum. It holds for several physically interesting potentials, such as power-law ones, but also other homogeneous ones; see Sec. III. Unfortunately, the sum (and difference) of two factorizable potentials is generally not factorizable itself.

In Eq. (1), we used the factorizability of the potential $V(R,\alpha,\phi) = V(R)V(\alpha,\phi)$ to reduce this set to one (common) hyper-radial Schrödinger equation. The hyper-angular part $V(\alpha,\phi)$ can be expanded in terms of $O(6)$ hyperspherical harmonics with zero angular momenta $L = m = 0$ (due to the rotational invariance of the potential),

$$V(\alpha,\phi) = \sum_{\mathrm{K},Q}^{\infty} v^{3-\mathrm{body}}_{\mathrm{K},Q}\mathcal{Y}^{\mathrm{K}Q\nu}_{00}(\alpha,\phi), \tag{3}$$

where

$$v^{3-\mathrm{body}}_{\mathrm{K},Q} = \int \mathcal{Y}^{\mathrm{K}Q\nu *}_{00}(\Omega_5)V(\alpha,\phi)d\Omega_{(5)}, \tag{4}$$

leading to

$$V_{\mathrm{eff}}(R)C^{\mathrm{K}''\mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']} = V(R)\sum_{\mathrm{K},Q}^{\infty} v^{3-\mathrm{body}}_{\mathrm{K},Q}\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Q\nu}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle \tag{5}$$

There is no summation over the multiplicity index in Eq. (3) because no multiplicity arises for harmonics with $L < 2$. Here, we separate out the $\mathrm{K} = 0$ term and absorb the factor $\frac{v^{3-\mathrm{body}}_{00}}{\pi\sqrt{\pi}}$ into the definition of $V_{\mathrm{eff}}(R) = \frac{v^{3-\mathrm{body}}_{00}}{\pi\sqrt{\pi}}V(R)$ to find

$$C^{\mathrm{K}''\mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']} = \delta_{\mathrm{K}'',\mathrm{K}'}\delta_{[\mathrm{m}''],[\mathrm{m}']} + \pi\sqrt{\pi}\sum_{\mathrm{K}>0,Q}^{\infty}\frac{v^{3-\mathrm{body}}_{\mathrm{K},Q}}{v^{3-\mathrm{body}}_{00}} \times \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)|\mathcal{Y}^{\mathrm{K}Q\nu}_{00}(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle. \tag{6}$$

Homogenous potentials, such as the Δ- and Y-string ones, which are linear in R, and the Coulomb one, see Sec. III for the definition of these potentials, have the first coefficient $v^{3-\mathrm{body}}_{00}$ in the hyperspherical harmonic expansion that is generally (at least) 1 order of magnitude larger than the rest $v^{3-\mathrm{body}}_{\mathrm{K}>0,Q}$; see Table I and Fig. 1. This reflects the fact that, on average, these potentials depend more on the overall size of the system than on its shape, thus justifying the adiabatic (perturbative) approach taken in Ref. [6], with the first term in Eq. (6) taken as the zeroth-order approximation [24].

In such cases, Eq. (1) decouple, leading to zeroth-order solutions for $\psi^{\mathrm{K}}_{0[\mathrm{m}]}(R)$ that are independent of $[m]$ and thus

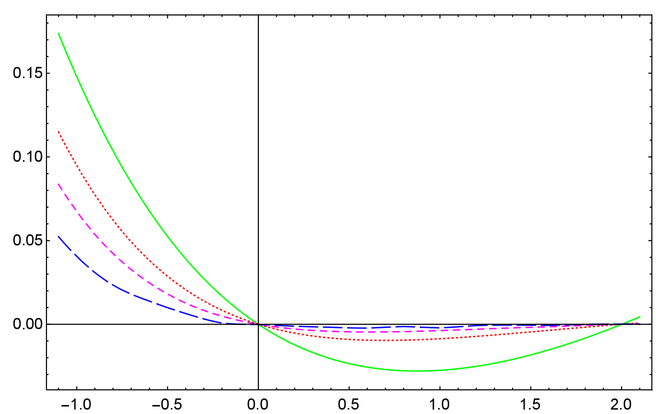

FIG. 1. The graphs of the ratios $v^{\epsilon}_{4,0}/v^{\epsilon}_{00}$ (green, solid), $v^{\epsilon}_{6,6}/v^{\epsilon}_{00}$ (red, dotted), $v^{\epsilon}_{8,0}/v^{\epsilon}_{00}$ (magenta, short dashes), and $v^{\epsilon}_{12,0}/v^{\epsilon}_{00}$ (blue, long dashes) (listed in the decreasing order) as functions of the power ϵ in the potential Eq. (22). One can see the tendency of the higher-order coefficients to diminish with an increasing value of index K.

have equal energies within the same K shell and different energies in different K shells. Two known exceptions are potentials with the homogeneity degree $\mathrm{k} = -1, 2$, which lead to "accidental degeneracies" and have to be treated separately.

The first-order corrections are obtained by diagonalization of the block matrices $C^{\mathrm{KK}}_{[\mathrm{m}][\mathrm{m}']}$, $\mathrm{K} = 1, 2, \ldots$, while the off-diagonal couplings $C^{\mathrm{KK}'}_{[\mathrm{m}][\mathrm{m}']}$, $\mathrm{K} \neq \mathrm{K}'$ appear only in the second-order corrections. Rather than calculating perturbative first-order energy shifts, a better approximation is obtained when the diagonalized block matrices are plugged back into Eq. (1), and equations then decouple into a set of (separate) individual ordinary differential equations in one variable, which differ only in the value of the effective coupling constant,

$$\left[\frac{d^2}{dR^2} + \frac{5}{R}\frac{d}{dR} - \frac{\mathrm{K}(\mathrm{K}+4)}{R^2} + 2\mu(E - V^{\mathrm{K}}_{[\mathrm{m_d}]}(R))\right] \times \psi^{\mathrm{K}}_{[\mathrm{m_d}]}(R) = 0, \tag{7}$$

where $V^{\mathrm{K}}_{[\mathrm{m_d}]}(R) = C^{\mathrm{K}}_{[\mathrm{m_d}]} V_{\mathrm{eff}}(R)$, with $C^{\mathrm{K}}_{[\mathrm{m_d}]}$ being the eigenvalues of matrix $C^{\mathrm{KK}}_{[\mathrm{m}][\mathrm{m}']}$.

The spectrum of three-body systems in homogenous potentials, such as those considered in Refs. [8,9], is now reduced to finding the eigenvalues of a single differential operator, just as in the two-body problem with a radial potential. The matrix elements in Eq. (6) can be readily evaluated using the permutation-symmetric $O(6)$ hyperspherical harmonics and the integrals that are spelled out in Refs. [8,9].

This is the main (algebraic) result of this section: combined with the hyperspherical harmonics recently obtained in Refs. [8,9], it allows one to evaluate the discrete part of the (energy) spectrum of a three-body potential as a function of its shape-sphere harmonic expansion coefficients $v^{3-\mathrm{body}}_{\mathrm{K},Q}$. Generally, these matrix elements obey selection rules: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $\mathrm{m}' = m''$. Moreover, Q is an Abelian (i.e., additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$. That reduces the sum in Eq. (6) to a finite one, which depends on a finite number of coefficients $v^{3-\mathrm{body}}_{\mathrm{K},Q}$; for small values of K, this number is also small.

A matrix such as that in Eq. (6) is generally sparse in the permutation-symmetric basis, so its diagonalization is not a serious problem, and for sufficiently small K values, it can even be accomplished in closed form; for example, for $\mathrm{K} \leq 5$, all results depend only on four coefficients (v_{00}, v_{40}, $v_{6\pm6}$, and v_{80}), and there is at most three-state mixing, so the eigenvalue equations are at most cubic ones, with well-known solutions. As there is only a small probability that many states from the $\mathrm{K} \geq 6$ shells will be observed in the foreseeable future, we limit ourselves to $\mathrm{K} \leq 5$ shells here.

III. THREE-BODY SPIN-INDEPENDENT POTENTIALS

A. Lattice QCD three static quarks potential

Lattice QCD calculations indicate that the confining interactions among quarks do not depend on the quarks' spin and flavor degrees of freedom.

There have been several attempts at extracting the three-quark potential from lattice QCD over the years; see Refs. [21–23]. They were based on lattices of different sizes, $12^3 \times 24$ at $\beta = 5.7$ and $16^3 \times 32$ at $\beta = 5.8, 6.0$ in Ref. [21], $16^3 \times 32$ at $\beta = 5.8, 6.0$ in Ref. [22], and 24^4 at $\beta = 5.7, 5.8, 6.0$ in Ref. [23]. Moreover, Refs. [21,22] use the Wilson loop techniques, whereas Ref. [23] uses the Polyakov loop. Their conclusions also differ markedly: Ref. [21] "supports the Y *Ansatz*," Ref. [22] "finds support for the Δ *Ansatz*," and the most recent Ref. [23] finds that the "potentials of triangle geometries are clearly different from the half of the sum of the two-body quark-antiquark potential," i.e., suggesting that is not the Δ *Ansatz*. All of these indicate that the lattice QCD potential is neither a pure Y *Ansatz* nor a pure Δ *Ansatz*.

A detailed analysis [25] of the Ref. [21] and Ref. [23] published data in terms of hyperspherical coordinates has shown that these two groups have calculated the potential (mostly) in very different geometric configurations, the overlap of which is small so that neither calculation is conclusive.

It stands to reason that the definitive QCD prediction is a linear superposition of the two *Ansätze* and the QCD Coulomb term, but at this stage, it is impossible to evaluate the lattice QCD potential's $O(6)$ expansion coefficients due to the dearth of evaluated points on the hypersphere.

For this reason, we shall analyze both *Ansätze*, separately, in addition to the QCD Coulomb potential, which is a must. Finally, we shall also consider the logarithmic potential, which can be thought of as the best homogeneous-potential approximation to the sum of the Coulomb and the linearly rising potential.

As stated in Sec. II above, any spin-independent three-body potential must be invariant under overall (ordinary) rotations, as it is a scalar; i.e., it contains only the zero-angular momentum hyperspherical components, which significantly simplifies the expansion of the potential in $O(6)$ hyperspherical harmonics. Below, we shall calculate these expansion coefficients in several homogeneous potentials.

B. Y-string and other area-dependent potentials

The complexity of the full Y-string potential, defined by

$$V_{\text{Y-string}} = \sigma_Y \min_{\mathbf{x_0}} \sum_{i=1}^{3} |\mathbf{x_i} - \mathbf{x_0}|, \tag{8}$$

can best be seen when expressed in terms of three-body Jacobi (relative) coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, as follows. The full Y-string potential, Eq. (8), consists of the so-called central Y-string, or "Mercedes Benz-string," term,

$$V_{\text{Y-central}} = \sigma_{\text{Y}}\sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho}\times\boldsymbol{\lambda}|)}, \tag{9}$$

which is valid when

$$\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}, \end{cases} \tag{10}$$

and three other angle-dependent two-body string, also called V-string terms; see Eqs. (A1a)–(A1c).

Because of the complexity of conditions in Eqs. (10) and (A1a)–(A1c) and of the difficulties related to their implementation in calculations, there was a widespread lack of use of the full Y-string potential (8) in comparison to its dominant part, the central Y-string potential $V_{\text{Y-central}}$. In our hyperspherical harmonics approach, however, both the full Y-string potential and its central part are treated in the same manner (just as the rest of the potentials) and present no significant mathematical obstacles. Both the central and the full Y-string potentials are decomposed into hyperspherical harmonics, and the resulting decomposition coefficients turn out close to each other, which renders $V_{\text{Y-central}}$ a good approximation to the full Y-string potential.

However, there is a physical reason that favors retaining only the central part of the Y-string potential over taking account of the full potential: namely, the central Y-string potential $V_{\text{Y-central}}$, Eq. (9), has an exact dynamical $O(2)$ symmetry, unlike the full potential, Eq. (8). To demonstrate this, we first show that the $V_{\text{Y-central}}$ is a function of both Delves-Simonov hyperangles (χ, θ),

$$V_{\text{Y-central}}(R,\chi,\theta) = \sigma_{\text{Y}}R\sqrt{\frac{3}{2}(1 + \sin 2\chi|\sin\theta|)}, \tag{11}$$

but a function of only one Smith-Iwai hyperangle—the "polar angle" α,

$$V_{\text{Y-central}}(R,\alpha,\phi) = \sigma_{\text{Y}}R\sqrt{\frac{3}{2}(1 + |\cos\alpha|)}. \tag{12}$$

This independence of the "azimuthal" Smith-Iwai hyperangle ϕ means that the associated component Q of the hyperangular momentum (as in Ref. [8]) is a constant of the motion. As this is actually a feature of the $|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$ term that is proportional to the area of the triangle subtended by the three quarks, the property is thus shared by all area-dependent potentials, such as the central part of the Y string, Refs. [19].

The expansion (3) of the central Y-string potential (12) in hyperspherical harmonics

$$\begin{aligned} V_{\text{Y-central}}&(R,\alpha,\phi) \\ &= \sigma_{\text{Y}}R\sqrt{\frac{3}{2}}\sum_{K=0,4,\ldots}^{\infty} v^{\text{Y}}_{K0}\mathcal{Y}^{K0\nu}_{00}(\alpha,\phi) \\ &\equiv V^{\text{Y}}_{\text{eff}}(R)\left(1 + \frac{v^{\text{Y}}_{40}}{v^{\text{Y}}_{00}}\pi\sqrt{\pi}\mathcal{Y}^{40}_{000}(\alpha,\phi) + \cdots\right), \end{aligned} \tag{13}$$

where v^{Y}_{KQ} are defined in Eq. (4), runs over $O(6)$ hyperspherical harmonics with $K = 0, 4, 8, \ldots$ and zero value of the democracy quantum number $Q = 0$, as well as vanishing angular momentum $L = m = 0$ [26]. The numerical values are tabulated in Table I.

On the contrary, the expansion of the full Y-string potential (8) has additional terms with $\text{K} = 0(\text{mod } 6)$, $Q = 0(\text{mod } 6)$ that spoil the dynamical $O(2)$ symmetry of the potential in Eq. (9). These terms are much smaller than the corresponding terms in the Δ-string, QCD Coulomb, and logarithmic potentials, see Table I, and may therefore be neglected, in leading approximation, with impunity. In Appendix A, we illustrate how to evaluate the coefficient $v^{\text{Y-string}}_{\text{K}=6,Q=\pm 6}$ and show its value in Table I.

C. QCD Coulomb potential

The QCD Coulomb potential Eq. (14) is attractive in all three pairs, unlike the electromagnetic one; in terms of Jacobi vectors, it reads

$$V_{\text{Coulomb}} = -\alpha_{\text{C}}\sum_{i>j=1}^{3}|\mathbf{x}_i - \mathbf{x}_j|^{-1}. \tag{14}$$

$$\begin{aligned} V_{\text{Coulomb}} = -\alpha_{\text{C}}\Bigg(&\frac{1}{\sqrt{2\boldsymbol{\rho}^2}} + \frac{1}{\sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}} \\ &+ \frac{1}{\sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}}\Bigg). \end{aligned} \tag{15}$$

The Coulomb potential's hyperspherical expansion is

$$\begin{aligned} V_{\text{Coulomb}}(R,\alpha,\phi) &= V_{\text{Coulomb}}(R)V_{\text{Coulomb}}(\alpha,\phi) \\ &= V_{\text{Coulomb}}(R)\sum_{K,Q}^{\infty} v^{\text{Coulomb}}_{K,Q}\mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi), \end{aligned} \tag{16}$$

where $V_{\text{Coulomb}}(R) = -\alpha_{\text{C}}/R$ and the expansion coefficients $v^{\text{Coulomb}}_{K,Q}$ are defined by the Coulomb analog of Eq. (4) and are tabulated in Table I.

We note that this and any other permutation-symmetric sum of two-body potentials (with the sole exception of the harmonic oscillator) has a specific "triple-periodic" azimuthal ϕ hyperangular dependence with the angular period of $\frac{2}{3}\pi$. That provides additional selection rules for the "democracy quantum number" Q-dependent terms in this expansion, besides the $K = 0, 4, \ldots$ rule for $Q = 0$ terms discussed above:

$$\sum_{KQ}^{\infty} v^{\Delta}_{KQ} \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi) = \sum_{K=0,4,\ldots}^{\infty} v^{\Delta}_{K0} \mathcal{Y}^{K0\nu}_{00}(\alpha,\phi) + \sum_{K,Q=\pm 6}^{\infty} v^{\Delta}_{KQ} \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi) + \sum_{K,Q=\pm 12}^{\infty} v^{\Delta}_{KQ} \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi) + \cdots \tag{17}$$

Note that the values of all quantum numbers here are double those in two spatial dimensions (D = 2), [20]. This has to do with the different integration measures for D = 2 and D = 3 hyperspherical harmonics.

D. Δ-string potential

The Δ-string potential

$$V_{\Delta} = \sigma_{\Delta} \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|, \tag{18}$$

written out in terms of Jacobi vectors reads

$$V_{\Delta} = \sigma_{\Delta}\left(\sqrt{2\boldsymbol{\rho}^2} + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right). \tag{19}$$

The Δ-string potential (19) in terms of Iwai-Smith angles reads

$$V_{\Delta}(R,\alpha,\phi) = \sigma_{\Delta} R\left(\sqrt{1+\sin(\alpha)\sin\left(\frac{\pi}{6}-\phi\right)} + \sqrt{1+\sin(\alpha)\sin\left(\phi+\frac{\pi}{6}\right)} + \sqrt{1-\sin(\alpha)\cos(\phi)}\right). \tag{20}$$

To find the general hyperspherical harmonic expansion of the Δ-string potential, we note that it factors into the hyperradial $V_{\Delta}(R) = \sigma_{\Delta} R$ and the hyperangular part $V_{\Delta}(\alpha,\phi)$,

$$V_{\Delta}(R,\alpha,\phi) = V_{\Delta}(R) V_{\Delta}(\alpha,\phi) = V_{\Delta}(R) \sum_{K,Q}^{\infty} v^{\Delta}_{K,Q} \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi), \tag{21}$$

where the expansion coefficients $v^{\Delta}_{K,Q}$ are defined by the Δ analog of Eq. (4) and are tabulated in Table I.

E. General pairwise power-law potential

Infinitely many permutation-symmetric sums of two-body power-law potentials have the generic form of Eq. (18) with different exponents ϵ; i.e., both the Coulomb and the Δ-string potentials are two special cases of the more general attractive homogeneous potential,

$$V_{\epsilon} = \mathrm{sgn}(\epsilon)\sigma_{\epsilon} \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|^{\epsilon} = \mathrm{sgn}(\epsilon)\sigma_{\epsilon}\left((2\boldsymbol{\rho}^2)^{\epsilon/2} + \left(\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})\right)^{\epsilon/2} + \left(\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})\right)^{\epsilon/2}\right), \tag{22}$$

where $\mathrm{sgn}(\epsilon) = \epsilon/|\epsilon|$. Note that in the special case of the harmonic oscillator potential ($\epsilon = 2$) the above form degenerates into an expression proportional to $\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 = R^2$.

In Fig. 1, we display the graphs of four ratios of hyperspherical expansion coefficients as functions of the exponent ϵ. There, one can see that these coefficients depend smoothly on the exponent ϵ and that they uniformly decrease with the increasing value of index K, in this class of potentials. Numerical values of five expansion coefficients of potentials V_{Y}, V_{Δ}, V_{Coulomb}, and V_{Log} are shown in Table I.

F. Logarithmic potential

The logarithmic potential

$$V_{\mathrm{Log}} = \sigma_{\mathrm{Log}} \sum_{i>j=1}^{3} \log(|\mathbf{x}_i - \mathbf{x}_j|) \tag{23}$$

has a divergent short-distance and a steadily rising long-distance part; thence, it can be thought of as a linear combination of the QCD Coulomb (with a homogeneity index $\alpha = -1$) and a linear confining potential (with a homogeneity index $\alpha = 1$), with a common homogeneity index equal to 0: $\alpha = 0$. Note that this homogeneity condition boils down to an additive, rather than multiplicative, factorization of the potential:

$$V_{\mathrm{Log}}(R,\alpha,\phi) = V_{\mathrm{Log}}(R) + V_{\mathrm{Log}}(\alpha,\phi) = V_{\mathrm{Log}}(R) + \sum_{K,Q}^{\infty} v^{\mathrm{Log}}_{K,Q} \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi). \tag{24}$$

The logarithmic potential has been used with great success in the heavy quark-antiquark two-body problem; it reproduces the remarkable mass independence of the $c\bar{c} - J/\Psi$ and $b\bar{b} - \Upsilon$ spectra. It has not been used in the three-quark problem at all, to our knowledge.

IV. RESULTS

In the following, we present the $\mathrm{K} = 0, \ldots, 5$ shells' energy spectra, for two reasons: (1) both as an example of the kind of results that one may expect as K increases and in order to settle some long-standing issues regarding the $\mathrm{K} = 3$ shell [6,7,27] and (2) as an illustration of the methods, see Appendix B, that were used in their calculation. With regard to 2, we note that these examples are all purely algebraic, in the sense that no numerical calculations were necessary, but that ceases to be the case as K increases beyond $\mathrm{K} > 8$, at first only for certain subsets of states and, ultimately, for all states.

We note that we have already reported at a conference [28] some of the $\mathrm{K} = 4$ shell results, albeit without derivation.

A. K = 0, 1, 2 shells

The $\mathrm{K} = 0$, 1 bands are affected only by the v_{00} coefficient, so they need not be treated separately here, whereas the $\mathrm{K} = 2$ band is affected by the v_{00} and v_{40} coefficients. The calculated energy splittings of $\mathrm{K} = 2$ shell states depend only on the SU(6) multiplets,

$$\begin{aligned}
&[20, 1^+] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{3}} v_{40}\right) \\
&[70, 0^+] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{\sqrt{3}} v_{40}\right) \\
&[70, 2^+] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{5\sqrt{3}} v_{40}\right) \\
&[56, 2^+] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{\sqrt{3}}{5} v_{40}\right),
\end{aligned} \tag{25}$$

and the resulting spectrum is shown in Fig. 2. Our main concern is the energy splitting pattern among the states within the $\mathrm{K} = 2$ hyperspherical $O(6)$ multiplet. The hyper-radial matrix elements of the linear hyper-radial potential are identical for all the (hyper-radial ground) states in one K band. Therefore, as is well known, the energy differences among various substates of a particular K band multiplet are integer multiples of the energy splitting "unit" $\Delta_K = \frac{1}{\pi\sqrt{\pi}}\left(\frac{1}{5\sqrt{3}} - \frac{1}{\sqrt{3}}\right)v_{40} = -\frac{1}{\pi\sqrt{\pi}}\frac{4}{5\sqrt{3}}v_{40}$. Note, however, that this kind of spectrum is subject to the condition $v_{00} \neq 0$.

B. K = 3 shell

With the area-dependent (i.e., ϕ-independent) central Y-string potential $V_{\text{Y-central}}$, Eq. (9), in three dimensions, we find that the each $SU(6)$, or S_3 multiplet in the $\mathrm{K} = 3$ band has one of four possible energies shown in Eqs. (26) with $v_{6\pm6}^{\text{Y-central}} = 0$.

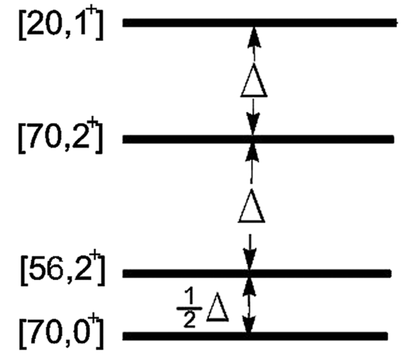

FIG. 2. The $\mathrm{K} = 2$ spectrum of both the Y and Δ strings in three dimensions.

Upon introduction of the ϕ-dependent two-body "V-string" potentials $V_{\text{V-string}}$, Eqs. (A1a)–(A1c) into the full Y string, the $v_{6\pm6}^{\text{Y-string}}$ coefficient becomes $\neq 0$. After diagonalization of the $C_{[K'],[K]}$ matrix, one finds further splittings among the previously degenerate states $[70, 1^-]$, $[56, 3^-]$, and $[20, 3^-]$ as well as among $[70, 3^-]$, $[56, 1^-]$, and $[20, 1^-]$,

$$\begin{aligned}
&[20, 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{\sqrt{3}} v_{40} - \frac{2}{7} v_{66}\right) \\
&[56, 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{\sqrt{3}} v_{40} + \frac{2}{7} v_{66}\right) \\
&[70, 1^-] \frac{1}{\pi\sqrt{\pi}}(v_{00}) \\
&[70, 2^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{3}} v_{40}\right) \\
&[70, 3^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{3}} v_{40}\right) \\
&[20, 3^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{\sqrt{3}}{7} v_{40} - v_{66}\right) \\
&[56, 3^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{\sqrt{3}}{7} v_{40} + v_{66}\right),
\end{aligned} \tag{26}$$

where our $v_{66} < 0$ is negative and Richard and Taxil's is positive. These results are displayed in Fig. 3.

For the $\mathrm{K} = 3$ band in three dimensions, the energy splittings have been calculated by Bowler *et al.* [7,27] for two-body anharmonic potentials perturbing the harmonic oscillator and confirmed and clarified by Richard and Taxil, Ref. [6], in the hyperspherical formalism with linear two-body potentials (the Δ string).

In hindsight, Richard and Taxil's Ref. [6] separation of $V_4(R)$ and $V_6(R)$ potentials' contributions is particularly illuminating: the former corresponds precisely to our

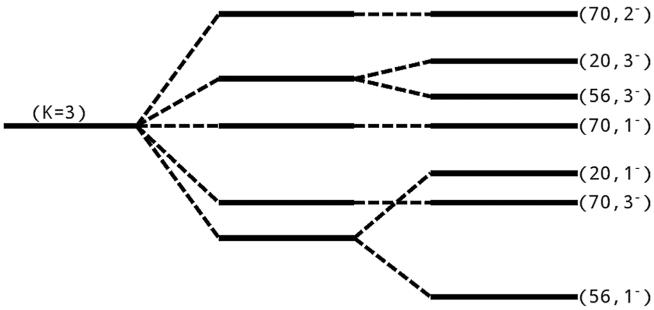

FIG. 3. Schematic representation of the $\mathrm{K}=3$ band in the energy spectrum of the Δ-string potential in three dimensions, following Ref. [6]. The sizes of the two splittings (the v_{40}^{Δ}-induced Δ and the subsequent $v_{6\pm6}^{\Delta}$-induced splitting) are not on the same scale, the latter having been increased, so as to be clearly visible. The Δ here is the same as the Δ in the $\mathrm{K}=2$ band.

"ϕ-independent" term v_{40}, and the latter corresponds to the "ϕ-dependent" potential's contribution to v_{66}.

As both the central Y string and the Δ string contain the former, whereas only the Δ string contains the latter, we see that the latter is the source of different degeneracies/splittings in the spectra of these two types of potentials [29].

C. K = 4 shell

The $SU(6)$, or S_3 multiplets in the $K=4$ band have one of the following 12 values of the diagonalized C matrix $C^{\mathrm{K}}_{[\mathrm{m_d}]} \times \frac{v_{00}}{\pi\sqrt{\pi}}$, from which one can evaluate the eigenenergies. We use the baryon-spectroscopic notation $[\mathrm{dim}, L^P]$, where dim is the dimension of the $SU_{\mathrm{FS}}(6)$ representation and the correspondence with the representations of the permutation group S_3 is given as $70 \leftrightarrow M$, $20 \leftrightarrow A$, $56 \leftrightarrow S$,

$$[70,0^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{2}v_{40}+\frac{1}{2\sqrt{5}}v_{80}\right)$$
$$[56,0^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{2}{\sqrt{5}}v_{80}\right)$$
$$[70,1^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{5}}v_{80}\right)$$
$$[70,2^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{35}(7\sqrt{3}v_{40}+2\sqrt{5}v_{80} - 3\sqrt{3v_{40}^2-2\sqrt{15}v_{40}v_{80}+5v_{80}^2+120v_{6\pm6}^2})\right)$$
$$[70',2^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{35}(7\sqrt{3}v_{40}+2\sqrt{5}v_{80} + 3\sqrt{3v_{40}^2-2\sqrt{15}v_{40}v_{80}+5v_{80}^2+120v_{6\pm6}^2})\right)$$
$$[56,2^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{12\sqrt{3}}{35}v_{40}+\frac{\sqrt{5}}{7}v_{80}\right)$$
$$[20,2^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{\sqrt{5}}v_{80}\right)$$
$$[20,3^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{3\sqrt{3}}{14}v_{40}-\frac{\sqrt{5}}{14}v_{80}\right)$$
$$[70,3^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{5\sqrt{3}}{14}v_{40}+\frac{1}{14\sqrt{5}}v_{80}\right)$$
$$[56,4^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{5\sqrt{3}}{14}v_{40}+\frac{3}{14\sqrt{5}}v_{80}\right)$$
$$[70,4^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{42\sqrt{5}}(-2v_{80} - \sqrt{1215v_{40}^2-54\sqrt{15}v_{40}v_{80}+9v_{80}^2+1280v_{6\pm6}^2})\right)$$
$$[70',4^+]:\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{42\sqrt{5}}(-2v_{80} + \sqrt{1215v_{40}^2-54\sqrt{15}v_{40}v_{80}+9v_{80}^2+1280v_{6\pm6}^2})\right). \tag{27}$$

The Δ-string results are shown in Fig. 4. Again, as the third coefficient $v_{6\pm6}$ vanishes in the central Y-string potential $V_{\text{Y-central}}$ (which is without two-body terms), or as it is roughly ten times smaller than usual, in the full Y-string potential $V_{\text{Y-string}}$, the (second) observable difference between Y-string and Δ-string potentials shows up in the magnitude of splitting between the pairs of $[70,2^+]$, $[70',2^+]$ and $[70,4^+]$, $[70',4^+]$ levels: the Y-string states are ordered as shown in the third ($v_{6\pm6}=0$) column in Fig. 4. As explained earlier, the vanishing of $v_{6\pm6}$ follows from the central Y-string potential's independence of the Iwai angle ϕ, i.e., from the dynamical "kinematic rotations/democracy transformations" $O(2)$ symmetry [19,20] associated with it.

Numerical results for other potentials are shown in Table II. Table II shows that the ordering of $\mathrm{K}=4$ states is not universally valid even for the (convex) potentials considered here; note that, although the three highest-lying multiplets always come from the same set ($[70,3^+]$, $[56,2^+]$, $[20,3^+]$, $[70',4^+]$; see Fig. 4), their orderings are different in these potentials. That, of course, is a consequence of different ratios v_{40}/v_{00}, $v_{6\pm6}/v_{00}$, and v_{80}/v_{00}. This goes to show that one cannot expect strongly restrictive ordering theorems to hold for three-body systems, as they hold in the two-body problem, Ref. [2]. Nevertheless, even the present results are useful, as they indicate that certain sets of multiplets are jointly lifted, or depressed, as a single group in the spectrum, with ordering within the group being subject to the detailed structure of the potential.

Of course, similar conclusions hold also for $\mathrm{K}=3$ spectrum splitting but are less pronounced, as that shell

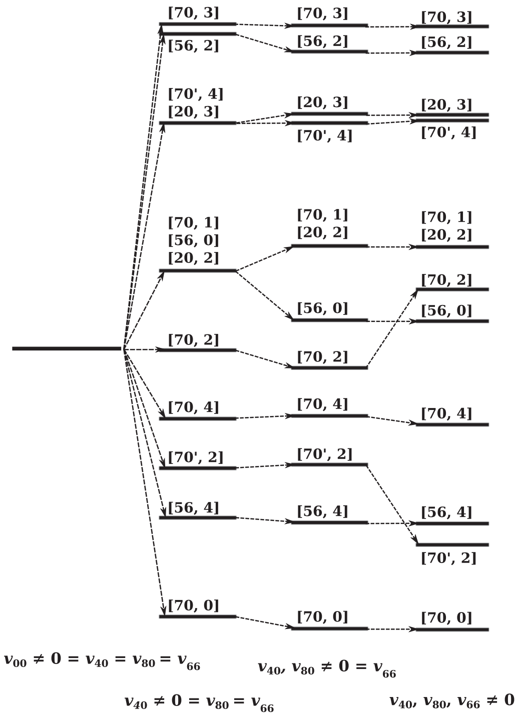

FIG. 4. Schematic representation of the K = 4 band in the energy spectrum of three quarks in the Δ-string potential.

depends only on two numbers: the ratios v_{40}/v_{00} and $v_{6\pm6}/v_{00}$. As the difference between Δ and Y-string potentials is most pronounced in the value of $v_{6\pm6}$, that is the case in which the distinction between these two potentials is most clearly seen.

On the phenomenological side, some eigenenergies of three quarks in the K = 4 shell have been calculated in Ref. [30] using a variational method based on harmonic oscillator wave functions. These calculations included the Δ-string, Y-string, and Coulomb potentials, all at once, as well as a relativistic kinetic energy [this kinetic energy violates the $O(6)$ symmetry]. Each one of these three terms in the potential is homogenous, but their sum is not—therefore, the individual contributions of these terms to the total/potential energy cannot be compared directly with the results of their separate calculations. Moreover, each term in the Hamiltonian breaks the $O(6)$ symmetry differently, thus inducing different splittings of energy spectra. These facts prevent us from directly comparing our results with Ref. [30], but the overall trend for groups of states seem to be in agreement with our results; see Table II for comparison.

D. K = 5 shell

With a ϕ-independent central Y-string potential in three dimensions, we find that each $SU(6)$, or S_3 multiplet in the K = 5 band has one of four of 15 different energies. Upon introduction of a ϕ-dependent ("two-body") component of

TABLE II. The values of effective potentials in the Y-, Δ-string, and (strong) Coulomb potentials for various K = 4 states (for all allowed orbital waves L).

K	$[SU(6), L^P]$	$\langle V_Y/\sigma_Y \rangle$	$\langle V_\Delta/\sigma_\Delta \rangle$	$-\langle V_C/\alpha_C \rangle$
4	$[56, 0^+]$	1.45921	2.87122	3.82554
4	$[70, 0^+]$	1.39729	2.80996	4.11043
4	$[70, 1^+]$	1.47483	2.88587	3.48449
4	$[56, 2^+]$	1.51372	2.92453	3.36709
4	$[20, 2^+]$	1.47483	2.88587	3.48449
4	$[70, 2^+]$	1.44997	2.87749	4.13184
4	$[70', 2^+]$	1.43052	2.82683	3.49379
4	$[70, 3^+]$	1.51906	2.92963	3.281
4	$[20, 3^+]$	1.50137	2.91213	3.36239
4	$[56, 4^+]$	1.4187	2.83095	3.94783
4	$[70, 4^+]$	1.44036	2.85066	3.3656
4	$[70', 4^+]$	1.49938	2.91178	3.81992

the potential, proportional to v_{66}, and upon diagonalization of the $C_{[K'],[K]}$ matrix, one finds four new splittings between previously degenerate states, (1) $[56, 2^-], [20, 2^-]$; (2) $[56', 4^-], [20', 4^-]$; (3) $[70, 1^-], [70', 1^-]$; and (4) $[70, 5^-]$, $[70', 5^-]$, as well as three nondegenerate states of which the energies are shifted by v_{66}. These algebraic results are summarized in

$$[70, 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{5\sqrt{3}v_{40} + 3\sqrt{5}v_{80}}{20} - \frac{\sqrt{75v_{40}^2 - 10\sqrt{15}v_{40}v_{80} + 5v_{80}^2 + 96v_{6\pm6}^2}}{20}\right)$$

$$[70', 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{5\sqrt{3}v_{40} + 3\sqrt{5}v_{80}}{20} + \frac{\sqrt{75v_{40}^2 - 10\sqrt{15}v_{40}v_{80} + 5v_{80}^2 + 96v_{6\pm6}^2}}{20}\right)$$

$$[56, 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}}v_{40} - \frac{2}{5}v_{66} - \frac{1}{2\sqrt{5}}v_{80}\right)$$

$$[20, 1^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}}v_{40} + \frac{2}{5}v_{66} - \frac{1}{2\sqrt{5}}v_{80}\right)$$

$$[70, 2^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{2\sqrt{3}}v_{40} - \frac{1}{2\sqrt{5}}v_{80}\right)$$

$$[56, 2^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{3}{5}v_{66} - \frac{1}{\sqrt{5}}v_{80}\right)$$

$$[20, 2^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{3}{5}v_{66} - \frac{1}{\sqrt{5}}v_{80}\right)$$

$$[56, 3^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}}v_{40} + \frac{14}{15}v_{66} + \frac{7}{6\sqrt{5}}v_{80}\right)$$

$$[20, 3^-] \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{2\sqrt{3}}v_{40} - \frac{14}{15}v_{66} + \frac{7}{6\sqrt{5}}v_{80}\right)$$

$$
\begin{aligned}
&[70,4^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{1}{2\sqrt{3}}v_{40}-\frac{1}{2\sqrt{5}}v_{80}\right)\\
&[56,4^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{7}{6\sqrt{3}}v_{40}-\frac{2}{15}v_{66}+\frac{1}{6\sqrt{5}}v_{80}\right)\\
&[20,4^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{7}{6\sqrt{3}}v_{40}+\frac{2}{15}v_{66}+\frac{1}{6\sqrt{5}}v_{80}\right)\\
&[70,5^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{18}v_{40}+\frac{\sqrt{5}}{30}v_{80}\right.\\
&\qquad\left.+\frac{\sqrt{5}}{165}\sqrt{1815v_{40}^2+66\sqrt{15}v_{40}v_{80}+9v_{80}^2+968v_{6\pm6}^2}\right)\\
&[70',5^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{18}v_{40}+\frac{\sqrt{5}}{30}v_{80}\right.\\
&\qquad\left.-\frac{\sqrt{5}}{165}\sqrt{1815v_{40}^2+66\sqrt{15}v_{40}v_{80}+9v_{80}^2+968v_{6\pm6}^2}\right)\\
&[56,5^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{2\sqrt{3}}v_{40}+\frac{8}{15}v_{66}-\frac{19}{66\sqrt{5}}v_{80}\right)\\
&[20,5^-]\,\frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{2\sqrt{3}}v_{40}-\frac{8}{15}v_{66}-\frac{19}{66\sqrt{5}}v_{80}\right).
\end{aligned}
\tag{28}
$$

The numerical results for three different potentials are displayed in Table III, whereas in Table IV we show the results for the Δ-string potential, with this potential's different multipole contributions separated and graphically displayed in Fig. 5.

TABLE III. The values of the effective potential matrix elements for the Y-, Δ-string, and (strong) Coulomb potentials and various K = 5 states (for all allowed orbital waves L).

K	$[SU(6), L^P]$	$\langle V_Y/\sigma_Y\rangle$	$\langle V_\Delta/\sigma_\Delta\rangle$	$-\langle V_C/\alpha_C\rangle$
5	$[70, 1^-]$	1.39729	2.80778	2.55667
5	$[70', 1^-]$	1.46442	2.87829	2.85542
5	$[56, 1^-]$	1.44898	2.87055	2.46858
5	$[20, 1^-]$	1.44898	2.85059	2.5953
5	$[70, 2^-]$	1.49547	2.90629	2.32887
5	$[20, 2^-]$	1.47483	2.87091	2.47611
5	$[56, 2^-]$	1.47483	2.90084	2.28602
5	$[70, 3^-]$	1.46682	2.84167	2.41462
5	$[70', 3^-]$	1.44037	2.8887	2.30016
5	$[70'', 3^-]$	1.5103	2.92104	2.67414
5	$[56, 3^-]$	1.44031	2.82915	2.84424
5	$[20, 3^-]$	1.44031	2.87571	2.54855
5	$[70, 4^-]$	1.49547	2.90629	2.32887
5	$[56, 4^-]$	1.52299	2.93685	2.23815
5	$[20, 4^-]$	1.52299	2.93020	2.28039
5	$[70, 5^-]$	1.50797	2.91991	2.75234
5	$[70', 5^-]$	1.41405	2.82520	2.30772
5	$[56, 5^-]$	1.44788	2.84623	2.63735
5	$[20, 5^-]$	1.44788	2.87283	2.46839

TABLE IV. The values of the effective three-body Δ-string potential divided by the string tension σ_Δ, $\langle V_\Delta(v_{0,0}, v_{4,0}, v_{6,6}, v_{8,0})/\sigma_\Delta\rangle$, as a function of the expansion coefficients $(v_{0,0}, v_{4,0}, v_{6,6}, v_{8,0})$, for various K = 5 states (for all allowed orbital waves L). Here, $\langle V_\Delta(A)/\sigma_\Delta\rangle = \langle V_\Delta(v_{0,0}, v_{4,0} \neq 0 = v_{6,6} = v_{8,0})/\sigma_\Delta\rangle$, and $\langle V_\Delta(B)/\sigma_\Delta\rangle = \langle V_\Delta(v_{0,0}, v_{4,0}, v_{8,0} \neq 0 = v_{6,6})/\sigma_\Delta\rangle$.

K	$[SU(6), L^P]$	$\langle V_\Delta(A)/\sigma_\Delta\rangle$	$\langle V_\Delta(B)/\sigma_\Delta\rangle$	$\langle V_\Delta/\sigma_\Delta\rangle$
5	$[70, 1^-]$	2.8124	2.80996	2.80778
5	$[70', 1^-]$	2.88099	2.87611	2.87829
5	$[56, 1^-]$	2.85813	2.86057	2.87055
5	$[20, 1^-]$	2.85813	2.86057	2.85059
5	$[70, 2^-]$	2.90385	2.90629	2.90629
5	$[56, 2^-]$	2.88099	2.88587	2.87091
5	$[20, 2^-]$	2.88099	2.88587	2.90084
5	$[70, 3^-]$	2.85051	2.85214	2.84167
5	$[70', 3^-]$	2.87918	2.87827	2.8887
5	$[70'', 3^-]$	2.92091	2.921	2.92104
5	$[56, 3^-]$	2.85813	2.85243	2.82915
5	$[20, 3^-]$	2.85813	2.85243	2.87571
5	$[70, 4^-]$	2.90385	2.90629	2.90629
5	$[56, 4^-]$	2.93434	2.93352	2.93685
5	$[20, 4^-]$	2.93434	2.93352	2.93020
5	$[70, 5^-]$	2.91909	2.91872	2.91991
5	$[70', 5^-]$	2.82764	2.82639	2.82520
5	$[56, 5^-]$	2.85813	2.85953	2.84623
5	$[20, 5^-]$	2.85813	2.85953	2.87283

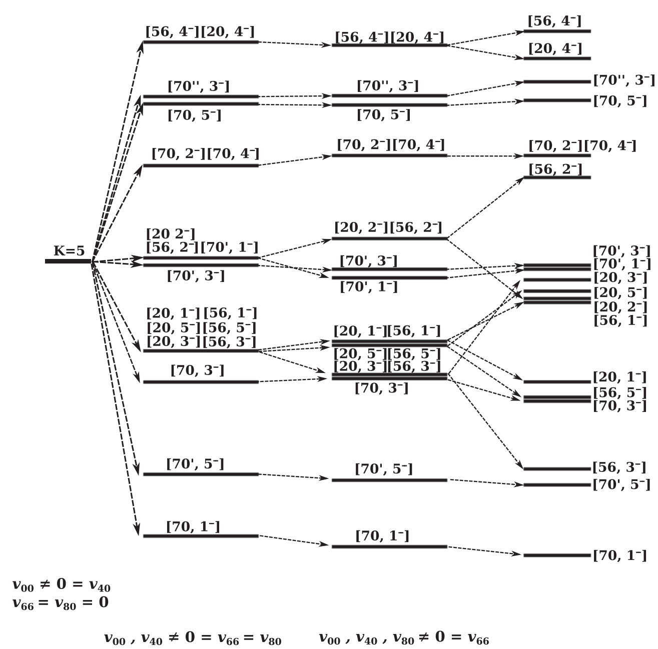

FIG. 5. Schematic representation of the K = 5 band in the energy spectrum of the Δ- and Y-string potentials in three dimensions. The sizes of the two splittings (the v_{40}^Δ-induced Δ and the subsequent v_{80}^Δ-induced splitting) are not on the same scale. The Δ here is not the same as the Δ in the K = 2 band.

V. DISCUSSION AND COMPARISON WITH PREVIOUS CALCULATIONS

The following points ought to be made:

(1) The present results are meant (only) as examples of what can be done; these calculations can be extended with K increasing *ad infinitum*, with the help of $O(6)$ matrix elements that are functions of $O(6)$ Clebsch-Gordan coefficients, which can be found in Ref. [8]. This is subject to the proviso that at some value of K the calculations must become numerical.

(2) The algebraic results shown in Sec. IV do not hold for the QCD Coulomb potential, as the QCD Coulomb hyper-radial potential $-\alpha_C/R$ Eq. (16) has a dynamical $O(7)$ symmetry and therefore accidental degeneracies are expected to appear. That symmetry is broken by the hyperangular part of the Coulomb three-body potential in a manner that still remains to be explored.

(3) In the $\mathrm{K}=2$ band/shell of the three-body energy spectrum, the eigenenergies depend on two coefficients (v_{00}, v_{40}), and the splittings among various levels depend only on the (generally small) ratio v_{40}/v_{00}. This means that the eigenenergies form a fixed pattern ("ordering") that does not depend on the shape of the three-body potential. The actual size of the $\mathrm{K}=2$ shell energy splitting depends on the small parameter v_{40}/v_{00}, provided that the potential is permutation symmetric. This fact was noticed almost 40 years ago, Refs. [31,32], and it suggests that similar patterns might exist in higher-K shells.

The practical advantage of permutation-adapted hyperspherical harmonics over the conventional ones is perhaps best illustrated here: the $\mathrm{K}=2$ shell splittings in the Y- and Δ-string potentials were obtained, after some complicated calculations using conventional hyperspherical harmonics in Ref. [33], whereas here they follow from the calculation of four (simple) hyperangular matrix elements.

(4) Historically, extensions of this kind of calculations to higher ($\mathrm{K}\geq 3$) bands, for general three-body potentials, turned out more problematic than expected; Bowler *et al.*, Ref. [7], published a set of predictions for the $\mathrm{K}=3$, 4 bands, which were later questioned by Richard and Taxil's, Ref. [6], $\mathrm{K}=3$ hyperspherical harmonic calculation; see also Refs. [30,34]. This controversy has not been resolved up to the present day, to our knowledge, so we address that problem first. In the $\mathrm{K}=3$ case, the energies depend on three coefficients (v_{00}, v_{40}, and $v_{6\pm6}$), and there is no mixing of multiplets, so all eigenenergies can be expressed in a simple closed form that agrees with Ref. [6] and depends on two small parameters v_{40}/v_{00} and $v_{6\pm6}/v_{00}$.

Note that the coefficient $v_{6\pm6}$ vanishes in the (simplified) central Y-string potential (without two-body terms) and thus causes the first potentially observable difference between Y- and Δ-string potentials: the splittings between $[20,1^-]$ and $[56,1^-]$ as well as between $[20,3^-]$ and $[56,3^-]$. The actual value of $v_{6\pm6}$ in the exact Y-string potential is so small, so as to be negligible compared with the other two coefficients, v_{00} and v_{40}, in its expansion.

(5) Note that from Eq. (7) it follows that there must exist an upper limit on the values of the ratios $|v_{40}/v_{00}|\leq\sqrt{3}$, and from Eq. (8), it follows that $|v_{6\pm6}/v_{00}|\leq 7/2$. If these limits are exceeded, the overall sign of the effective potential flips, and the solution (motion) becomes unbound. This example clearly shows the limitations of the present method. However, the physically interesting potentials considered in Sec. III all satisfy inequalities $v_{00}\gg|v_{40}|$ and $v_{00}\gg|v_{6\pm6}|$, as can be seen in Table I and Fig. 1, which shows that this method may be applied here.

(6) The above points (2) and (3) display possible "fault line(s)" in the predictions of the ordering of shells with different values of K: in case $v_{00}=0$, the $\mathrm{K}=0$, 1 shells become unbound to leading (adiabatic) order, and their binding becomes a question of higher-order (nonadiabatic) effects.

(7) We shall not attempt a numerical prediction of triple-heavy hyperon masses here, for the following reasons: (a) the mass of heavy quark(s) m_Q is not precisely known in the three-quark environment; (b) the QCD coupling constant α_S is not known in this environment; (c) the value of the effective string tension σ is not known in this environment; and (d) the spin-dependent interactions, which are not included here, may significantly influence the results. Nevertheless, nothing prevents the interested reader from inserting his/her favorite values of m_Q, α_S, and σ into our formulas to obtain some definite predictions.

(8) There are several possible straightforward extensions of the present work: (a) to equal mass systems with a relativistic kinematic energy and (b) to two identical and one distinct quark systems. Both extensions break the $O(6)$ symmetry further still but can be treated within the present approach, with certain *caveats*.

(9) Note that we have kept the full $SU_F(3)$, $SU_{FS}(6)$ notation for the three-quark states, even though there can be only one flavor, with three identical

heavy quarks. This is in order to keep maximum generality and to allow potential future extension to relativistic light-quark systems (cf. Refs. [30,34]).

(10) The present formalism allows a (mathematically proper) extension of the Regge theory/trajectories [35–37] to three-quark systems as well as an extension of Birman-Schwinger's results [38,39] about the number of bound states of a Schrödinger equation in a given potential.

(11) The present formalism allows an extension to atomic and molecular physics, as well, albeit with significant modifications: (a) atomic systems are subject to Coulomb potential, which leads to a higher dynamical symmetry, that needs to be taken into account, and (b) molecular systems are bound by inhomogenous potentials, such as the Lennard-Jones one, which must be treated differently.

VI. SUMMARY AND CONCLUSIONS

In summary, we have reduced the nonrelativistic (quantum) three-identical body problem to a single ordinary differential equation for the hyper-radial wave function with coefficients multiplying the homogenous hyper-radial potential that are determined entirely by $O(6)$ group-theoretical arguments; see Refs. [8,9]. That equation can be solved in the same way as the radial Schrödinger equation in three dimensions. The breaking of the $O(6)$ symmetry by the three-quark potential determines the ordering of states within different shells in the energy spectrum.

The dynamical $O(2)$ symmetry of the Y-string potential was discovered in Ref. [19], with the permutation group $S_3 \subset O(2)$ as the subgroup of the dynamical $O(2)$ symmetry. The existence of an additional dynamical symmetry strongly suggested an algebraic approach, such as that used in two-dimensional space, in Ref. [20]. In three dimensions, the hyperspherical symmetry group is $O(6)$, and the residual dynamical symmetry of the potential is $S_3 \otimes SO(3)_{\rm rot} \subset O(2) \otimes SO(3)_{\rm rot} \subset O(6)$, where $SO(3)_{\rm rot}$ is the rotational symmetry associated with the (total orbital) angular momentum L. We showed how the energy eigenvalues can be calculated as functions of the three-body potential's (hyper)spherical harmonics expansion coefficients $v^{3-\rm body}_{\rm K,\mathit{Q}}$ and $O(6)$ Clebsch-Gordan coefficients that are evaluated in Ref. [8].

We have used these results to calculate the energy splittings of various states [or $SU_{FS}(6)$ and S_3 multiplets] in the $\rm K \le 5$ shells of the Y-, Δ-string, and Coulomb potential spectra. The ordering of bound states has its most immediate application in the physics of three confined quarks, for which the question was raised originally, Refs. [6,7,31,32]. We have shown that in the $\rm K \ge 3$ shells a clear difference appears between the spectra of the Y- and Δ-string models of confinement. That is also the first explicit consequence of the dynamical $O(2)$ symmetry of the Y-string potential.

We stress the algebraic nature of our results, as this method can be used to obtain predictions for arbitrarily large K values, the calculations of which must be numerical, however, as soon as the number of states that are mixed exceeds 5.

The results presented here do not represent the outer boundaries of applicability of our method but are rather just illustrative examples, with a view to its application to atomic, molecular, and nuclear physics.

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APPENDIX A: EVALUATION OF OBTUSE-ANGLED TWO-BODY CONTRIBUTIONS TO THE Y STRING

As stated in Sec. II, at obtuse angles ($\ge 120^0$), there are two-body contributions to the Y-string potential that break the dynamical $O(2)$ symmetry of Eq. (12). Therefore, the expansion coefficient $v^{\text{Y-string}}_{\mathrm{K}=6,Q=\pm 6}$ of the full potential is not zero $v^{\text{Y-string}}_{6,\pm 6} \neq 0$.

Three angle-dependent two-body string in terms of Jacobi vectors $\boldsymbol{\rho}$, $\boldsymbol{\lambda}$ are, see Ref. [33],

$$V_{\text{V-string}} = \sigma\left(\sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})} + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)$$
$$\text{when}\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \ge -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \ge -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \le -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases} \tag{A1a}$$

$$V_{\text{V-string}} = \sigma\left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)$$

$$\text{when}\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases} \tag{A1b}$$

$$V_{\text{V-string}} = \sigma\left(\sqrt{2}\rho + \sqrt{\frac{1}{2}(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)$$

$$\text{when}\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}} \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2} \end{cases}. \tag{A1c}$$

The $O(6)$ $v_{6\pm6}$ coefficient is defined in Eq. (4),

$$v_{6,\pm6}^{\text{Y-string}} = \int \mathcal{Y}_{00}^{6\pm6}(\Omega_5) V_{\text{Y-string}}(\alpha,\phi) d\Omega_{(5)}, \tag{A2}$$

where the integration over $d\Omega_{(5)}$ is constrained by inequalities (A1a)–(A1c) and

$$\mathcal{Y}_{00}^{6,\pm6}(\alpha,\phi) = \frac{2}{\pi^{3/2}} R^{-6}(\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \pm 2i\boldsymbol{\lambda}\cdot\boldsymbol{\rho})^3$$
$$= \frac{\mp 2i}{\pi^{3/2}} \sin^3\alpha \exp(\mp 3i\phi), \tag{A3}$$

which is equivalent, up to the normalization constant, to the $O(3)$ spherical harmonics $Y_{3,\pm3}(\alpha,\phi)$. Numerical evaluation yields $v_{6,\pm6}^{\text{Y-string}} = -0.027$, the value of which is smaller than the subsequent coefficients in the expansion of this potential; see Table I.

APPENDIX B: DETAILS OF CALCULATIONS

1. K = 2 shell

The calculated coefficients entering the effective potentials for states with K = 2 can be found in Table V.

TABLE V. The values of the three-body potential hyperangular matrix elements $\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{4,0}\rangle_{\text{ang}}$, for various K = 2 states (for all allowed orbital waves L). The correspondence between the irreducible representations (S,A,M) of the S_3 permutation group and $SU(6)_{FS}$ symmetry multiplets (20,56,70) of the three-quark system is as follows: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$.

K	(K,Q,L,M,ν)	$[SU(6),L^P]$	$\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{4,0}\rangle_{\text{ang}}$
2	$(2,-2,0,0,0)$	$[70,0^+]$	$\frac{1}{\sqrt{3}}$
2	(2,0,2,2,0)	$[56,2^+]$	$\frac{\sqrt{3}}{5}$
2	$(2,\mp2,2,2,\pm3)$	$[70,2^+]$	$-\frac{1}{5\sqrt{3}}$
2	(2,0,1,1,0)	$[20,1^+]$	$-\frac{1}{\sqrt{3}}$

2. K = 3 shell

The calculated effective potentials in states with of K = 3 and various values L are listed in Tables VI and VII.

3. K = 4 shell

The calculated effective potentials for states with K = 4 and various values of L are listed in Table VIII.

TABLE VI. The values of the three-body potential hyperangular diagonal matrix elements $\langle\mathcal{Y}_{00}^{4,0}\rangle_{\text{ang}}$, for various K = 3 states (for all allowed orbital waves L).

K	(K,Q,L,M,ν)	$[SU(6),L^P]$	$\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{4,0}\rangle_{\text{ang}}$
3	$(3,\mp3,1,1,\pm1)$	$[20,1^-]$	$\frac{1}{\sqrt{3}}$
3	$(3,\mp3,1,1,\pm1)$	$[56,1^-]$	$\frac{1}{\sqrt{3}}$
3	$(3,\pm1,1,1,\pm3)$	$[70,1^-]$	0
3	$(3,\mp1,2,2,\pm5)$	$[70,2^-]$	$-\frac{1}{\sqrt{3}}$
3	$(3,\mp1,3,3,\pm2)$	$[70,3^-]$	$\frac{5}{7\sqrt{3}}$
3	$(3,\pm3,3,3,\mp6)$	$[56,3^-]$	$-\frac{\sqrt{3}}{7}$
3	$(3,\pm3,3,3,\mp6)$	$[20,3^-]$	$-\frac{\sqrt{3}}{7}$

TABLE VII. The values of the off-diagonal matrix elements of the hyperangular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|2\Re e\mathcal{Y}_{0,0}^{6,\pm6,0}|[SU(6)_i, L_i^P]\rangle_{\text{ang}}$, for various K = 3 states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}_{0,0}^{6,\pm6,0}\rangle_{\text{ang}}$
3	$[20,1^-]$	$[20,1^-]$	-1
3	$[56,1^-]$	$[56,1^-]$	1
3	$[20,3^-]$	$[20,3^-]$	$-\frac{2}{7}$
3	$[56,3^-]$	$[56,3^-]$	$\frac{2}{7}$
3	$[70,L^-]$	$[70,L^-]$	0

TABLE VIII. The values of the three-body potential hyperangular diagonal matrix elements $\langle \mathcal{Y}_{00}^{4,0,0} \rangle_{\mathrm{ang}}$ and $\langle \mathcal{Y}_{00}^{8,0,0} \rangle_{\mathrm{ang}}$, for various K = 4 states (for all allowed orbital waves L).

K	(K, Q, L, M, ν)	$[SU(6), L^P]$	$\pi\sqrt{\pi}\langle \mathcal{Y}_{00}^{4,0,0} \rangle_{\mathrm{ang}}$	$\pi\sqrt{\pi}\langle \mathcal{Y}_{00}^{8,0,0} \rangle_{\mathrm{ang}}$
4	$(4, \pm 4, 0, 0, 0)$	$[70, 0^+]$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2\sqrt{5}}$
4	(4,0,0,0,0)	$[56, 0^+]$	0	$\frac{2}{\sqrt{5}}$
4	$(4, \pm 2, 1, 1, \pm 2)$	$[70, 1^+]$	0	$-\frac{1}{\sqrt{5}}$
4	$(4, 0, 2, 2, \mp \sqrt{105})$	$[56, 2^+]$	$-\frac{12\sqrt{3}}{35}$	$\frac{\sqrt{5}}{7}$
4	$(4, 0, 2, 2, \mp \sqrt{105})$	$[20, 2^+]$	0	$-\frac{1}{\sqrt{5}}$
4	$(4, \pm 2, 2, 2, \pm 2)$	$[70, 2^+]$	$\frac{4\sqrt{3}}{35}$	$\frac{\sqrt{5}}{7}$
4	$(4, \pm 4, 2, 2, \mp 3)$	$[70', 2^+]$	$\frac{2\sqrt{3}}{7}$	$-\frac{1}{7\sqrt{5}}$
4	$(4, \mp 2, 3, 3, \pm 13)$	$[70, 3^+]$	$-\frac{5\sqrt{3}}{14}$	$\frac{1}{14\sqrt{5}}$
4	(4,0,3,3,0)	$[20, 3^+]$	$-\frac{3\sqrt{3}}{14}$	$-\frac{\sqrt{5}}{14}$
4	(4,0,4,4,0)	$[56, 4^+]$	$\frac{5\sqrt{3}}{14}$	$\frac{3}{14\sqrt{5}}$
4	$(4, \mp 2, 4, 4, \pm 5)$	$[70, 4^+]$	$\frac{3\sqrt{3}}{14}$	$-\frac{\sqrt{5}}{42}$
4	$(4, \mp 4, 4, 4, \pm 10)$	$[70', 4^+]$	$-\frac{3\sqrt{3}}{14}$	$\frac{1}{42\sqrt{5}}$

The selection rules that we have not derived fully, as of yet, are as follows:

(1) the three-dimensional expansion of the potentials goes in double-valued steps of K and Q, as compared with the two-dimensional case; viz., $K = 0, 4, 8, 12$ and $K = 6$, $Q = 6$ in three dimensions and $K = 0, 2, 4, 6$ and K = 3, $Q = 3$ in two dimensions. The latter can be understood in terms of $O(3)$ Clebsch-Gordan coefficients and spherical harmonics, whereas the former can be understood in terms of $O(6)$ Clebsch-Gordan coefficients, the properties of which are not (well) known, however.

(2) The selection rules read $Q \equiv 0 (\mathrm{mod}\, 6)$ and $K \equiv 0 (\mathrm{mod}\, 4)$, and the Clebsch-Gordan coefficients demand $Q = |Q_f - Q_i|$.

The ϕ-dependent (two-body) component in the three-body potential, which is proportional to $v_{6\pm6}$, enters the $K = 4$ spectrum, only through the off-diagonal matrix elements of two pairs of mixed-symmetry $[70, L^P]$-plets; the multiplet states $|[70, L^+]\rangle$ and $|[70', L^+]\rangle$ have identical physical quantum numbers (K, L^P), whereas the democracy label Q is generally not a good quantum number in permutation-symmetric three-body potentials, so it may be expected to be broken, and the corresponding eigenstates to mix under the influence of general permutation-symmetric three-body potentials. That is precisely what happens when the expansion coefficients $v_{6\pm6} \neq 0$ do not vanish. In that case, the two multiplets $|[70, L^+]\rangle$ and $|[70', L^+]\rangle$ mix, as determined by the diagonalization of the 2×2 potential matrix.

a. $|[70,L^+]\rangle - |[70',L^+]\rangle$ mixing and the physical states

The three-body potential matrix in the $O(6)$ symmetric states basis is nondiagonal in general; for example, for two multiplets ($|\mathrm{a}\rangle$, $|\mathrm{b}\rangle$) that have identical quantum numbers, such as $|[70, L^+]\rangle$ and $|[70', L^+]\rangle$, the potential matrix is 2×2 and can be written as

$$V_{\mathrm{a,b}} = \frac{1}{\pi\sqrt{\pi}} \begin{pmatrix} v_{00} + [v_{40}\langle \mathcal{Y}_{00}^{4,0,0} \rangle_{\mathrm{a}} + v_{80}\langle \mathcal{Y}_{00}^{8,0,0} \rangle_{\mathrm{a}}] & v_{6\pm6}\langle 2\Re e \mathcal{Y}_{0,0}^{6,\pm6,0} \rangle_{\mathrm{a,b}} \\ v_{6\pm6}\langle 2\Re e \mathcal{Y}_{0,0}^{6,\pm6,0} \rangle_{\mathrm{b,a}} & v_{00} + [v_{40}\langle \mathcal{Y}_{00}^{4,0,0} \rangle_{\mathrm{b}} + v_{80}\langle \mathcal{Y}_{00}^{8,0,0} \rangle_{\mathrm{b}}] \end{pmatrix}, \tag{B1}$$

where v_{00}, v_{40}, v_{80}, and $v_{6\pm6}$ are the hyperspherical expansion coefficients of the potential in question; $\langle \mathcal{Y}_{00}^{K,0,0} \rangle_{\mathrm{a}}$ and $\langle \mathcal{Y}_{00}^{K,0,0} \rangle_{\mathrm{b}}$ are the Kth diagonal hyperangular matrix elements for $SO(6)$ states $|\mathrm{a}\rangle$ and $|\mathrm{b}\rangle$, respectively, that can be read off from Table VIII; and $\langle 2\Re e \mathcal{Y}_{0,0}^{6,\pm6,0} \rangle_{\mathrm{a,b}}$ is the off-diagonal matrix element, from Table IX. Diagonalization is accomplished by way of mixing the $|[70, L^+]_{\mathrm{a}}\rangle$ and $|[70, L^+]_{\mathrm{b}}\rangle$ states,

$$\begin{aligned} |[70, L^+]\rangle &= \cos\theta |[70, L^+]_{\mathrm{a}}\rangle + \sin\theta |[70', L^+]_{\mathrm{b}}\rangle, \\ |[70', L^+]\rangle &= -\sin\theta |[70, L^+]_{\mathrm{a}}\rangle + \cos\theta |[70', L^+]_{\mathrm{b}}\rangle, \end{aligned} \tag{B2}$$

the mixing angle θ being determined by

TABLE IX. The values of the off-diagonal matrix elements of the hyperangular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|2\Re e\mathcal{Y}_{0,0}^{6,\pm6,0}|[SU(6)_i, L_i^P]\rangle_{\rm ang}$, for various K = 4 states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}_{0,0}^{6,\pm6,0}\rangle_{\rm ang}$
4	$[70, 2^+]$	$[70', 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70', 2^+]$	$[70, 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70, 4^+]$	$[70', 4^+]$	$\frac{8}{21}$
4	$[70', 4^+]$	$[70, 4^+]$	$\frac{8}{21}$
4	$[20, L^+]$	$[20, L^+]$	0
4	$[56, L^+]$	$[56, L^+]$	0
4	$[20, L^+]$	$[56, L^+]$	0

$$\tan 2\theta = \frac{2v_{6\pm6}\langle 2\Re e\mathcal{Y}_{0,0}^{6,\pm6,0}\rangle_{\rm a,b}}{[v_{40}\langle\mathcal{Y}_{00}^{4,0,0}\rangle_{\rm a} + v_{80}\langle\mathcal{Y}_{00}^{8,0,0}\rangle_{\rm a}] - [v_{40}\langle\mathcal{Y}_{00}^{4,0,0}\rangle_{\rm b} + v_{80}\langle\mathcal{Y}_{00}^{8,0,0}\rangle_{\rm b}]}. \tag{B3}$$

The (diagonal) eigenvalues of the potential matrix

$$V_{\rm a,b} = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \tag{B4}$$

can also be expressed in terms of the matrix elements (a, c, d) as

$$V_\pm = \frac{1}{2}\left(a + d \pm \sqrt{a^2 - 2ad + 4c^2 + d^2}\right),$$

and that leads to, for the [70, 4]-plets,

$$V_\pm([70,4]) = \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{42\sqrt{5}}(-2v_{80} \pm \sqrt{1215v_{40}^2 - 54\sqrt{15}v_{40}v_{80} + 9v_{80}^2 + 1280v_{6\pm6}^2})\right)$$

and, for the [70, 2]-plets,

$$V_\pm([70,2]) = \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{35}(7\sqrt{3}v_{40} + 2\sqrt{5}v_{80} \pm 3\sqrt{3v_{40}^2 - 2\sqrt{15}v_{40}v_{80} + 5v_{80}^2 + 120v_{6\pm6}^2})\right),$$

where $b = v_{40}$, $c = v_{80}$, and $d = v_{6\pm6}$.

TABLE X. The values of the three-body potential hyperangular diagonal matrix elements $\langle\mathcal{Y}_{00}^{4,0,0}\rangle_{\rm ang}$, $\langle\mathcal{Y}_{00}^{8,0,0}\rangle_{\rm ang}$, and $\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}_{00}^{6,\pm6,0}\rangle_{\rm ang}$ for various K = 5 SU(6) multiplets (with orbital angular momentum $L = J$). States containing one or more asterisks $(*)$ are subject to mixing described in the text.

K	(K, Q, L, M, ν)	$[SU(6), L^P]$	$\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{4,0,0}\rangle_{\rm ang}$	$\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{8,0,0}\rangle_{\rm ang}$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}_{00}^{6,\pm6,0}\rangle_{\rm ang}$
5	$(5, -5, 1, 1, 1)$	$[70, 1^-]$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2\sqrt{5}}$	$*$
5	$(5, -1, 1, 1, 3)$	$[70', 1^-]$	0	$\frac{1}{\sqrt{5}}$	$*$
5	$(5, -3, 1, 1, -5)$	$[56, 1^-]$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{2}{5}$
5	$(5, -3, 1, 1, -5)$	$[20, 1^-]$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{5}}$	$\frac{2}{5}$
5	$(5, -1, 2, 2, -13)$	$[70, 2^-]$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{5}}$	0
5	$(5, -3, 2, 2, 3)$	$[56, 2^-]$	0	$-\frac{1}{\sqrt{5}}$	$-\frac{3}{5}$
5	$(5, -3, 2, 2, 3)$	$[20, 2^-]$	0	$-\frac{1}{\sqrt{5}}$	$\frac{3}{5}$
5	$(5, -5, 3, 3, 6)$	$[70, 3^-]$	$\frac{2}{3\sqrt{3}}$	$-\frac{1}{3\sqrt{5}}$	$**$
5	$(5, -1, 3, 3, 7 - \sqrt{241})$	$[70', 3^-]$	$-\frac{5}{12\sqrt{3}} + \frac{85}{12\sqrt{723}}$	$\frac{241+19\sqrt{241}}{2892\sqrt{5}}$	$**$
5	$(5, -1, 3, 3, 7 + \sqrt{241})$	$[70'', 3^-]$	$-\frac{5(241+17\sqrt{241})}{2892\sqrt{3}}$	$\frac{241-19\sqrt{241}}{2892\sqrt{5}}$	$**$
5	$(5, -3, 3, 3, 0)$	$[56, 3^-]$	$\frac{1}{2\sqrt{3}}$	$\frac{7}{6\sqrt{5}}$	$\frac{14}{15}$
5	$(5, -3, 3, 3, 0)$	$[20, 3^-]$	$\frac{1}{2\sqrt{3}}$	$\frac{7}{6\sqrt{5}}$	$-\frac{14}{15}$
5	$(5, -1, 4, 4, 8)$	$[70, 4^-]$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{5}}$	0
5	$(5, -3, 4, 4, 24)$	$[56, 4^-]$	$-\frac{7}{6\sqrt{3}}$	$\frac{1}{6\sqrt{5}}$	$-\frac{2}{15}$
5	$(5, -3, 4, 4, 24)$	$[20, 4^-]$	$-\frac{7}{6\sqrt{3}}$	$\frac{1}{6\sqrt{5}}$	$\frac{2}{15}$
5	$(5, -5, 5, 5, 15)$	$[70, 5^-]$	$-\frac{5}{6\sqrt{3}}$	$\frac{\sqrt{5}}{66}$	$***$
5	$(5, -1, 5, 5, 3)$	$[70', 5^-]$	$\frac{7}{6\sqrt{3}}$	$\frac{17}{66\sqrt{5}}$	$***$
5	$(5, -3, 5, 5, 9)$	$[56, 5^-]$	$\frac{1}{2\sqrt{3}}$	$-\frac{19}{66\sqrt{5}}$	$\frac{8}{15}$
5	$(5, -3, 5, 5, 9)$	$[20, 5^-]$	$\frac{1}{2\sqrt{3}}$	$-\frac{19}{66\sqrt{5}}$	$-\frac{8}{15}$

4. K = 5 shell

The calculated effective potentials of states with K = 5 and various values of L are listed in Tables X,XI, and XII.

The ϕ-dependent (two-body) potential component proportional to $v_{6\pm6}$ enters these effective potentials in two ways: (1) through diagonal matrix elements in Table XI, causing the splitting of symmetric $[56, L^P]$ and antisymmetric $[20, L^P]$ multiplets, as in the $K = 3$ case, and (2) through off-diagonal matrix elements in Table XII, causing further splitting of two mixed-symmetry $[70, L^P]$-plets, as in the $K = 4$ case. Just as in Appendix B 3 a, the three-body potential matrix in the $O(6)$ symmetric states basis is nondiagonal in general and can be diagonalized in the same manner.

TABLE XI. The values of the diagonal matrix elements of the hyperangular part of the three-body potential $\langle\mathcal{Y}(K, Q_f, L, M, \nu_f)|2\Re e\mathcal{Y}^{6,\pm6,0}_{0,0}|\mathcal{Y}(K, Q_i, L, M, \nu_i)\rangle_{\rm ang}$, for various K = 5 states (for all allowed orbital waves L).

K	$[SU(6)_f, L^P_f]$	$[SU(6)_i, L^P_i]$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}^{6,\pm6,0}_{0,0}\rangle_{\rm ang}$
5	$[56, 1^-]$	$[56, 1^-]$	$-\frac{2}{5}$
5	$[20, 1^-]$	$[20, 1^-]$	$\frac{2}{5}$
5	$[56, 2^-]$	$[56, 2^-]$	$\frac{3}{5}$
5	$[20, 2^-]$	$[20, 2^-]$	$-\frac{3}{5}$
5	$[56, 3^-]$	$[56, 3^-]$	$\frac{14}{15}$
5	$[20, 3^-]$	$[20, 3^-]$	$-\frac{14}{15}$
5	$[56, 4^-]$	$[56, 4^-]$	$-\frac{2}{15}$
5	$[20, 4^-]$	$[20, 4^-]$	$\frac{2}{15}$
5	$[56, 5^-]$	$[56, 5^-]$	$\frac{8}{15}$
5	$[20, 5^-]$	$[20, 5^-]$	$-\frac{8}{15}$

TABLE XII. The values of the off-diagonal matrix elements of the hyperangular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L^P_f]|2\Re e\mathcal{Y}^{6,\pm6,0}_{0,0}|[SU(6)_i, L^P_i]\rangle_{\rm ang}$, for various K = 5 states (for all allowed orbital waves L).

K	$[SU(6)_f, L^P_f]$	$[SU(6)_i, L^P_i]$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}^{6,\pm6,0}_{0,0}\rangle_{\rm ang}$
5	$[70, 1^-]$	$[70', 1^-]$	$\frac{\sqrt{6}}{5}$
5	$[70', 1^-]$	$[70, 1^-]$	$\frac{\sqrt{6}}{5}$
5	$[70, 3^-]$	$[70', 3^-]$	$\sqrt{\frac{139}{450}+\frac{2131}{450\sqrt{241}}}$
5	$[70', 3^-]$	$[70, 3^-]$	$\sqrt{\frac{139}{450}+\frac{2131}{450\sqrt{241}}}$
5	$[70, 3^-]$	$[70'', 3^-]$	$-\frac{1}{15}\sqrt{\frac{1}{482}(33499-2131\sqrt{241})}$
5	$[70'', 3^-]$	$[70, 3^-]$	$-\frac{1}{15}\sqrt{\frac{1}{482}(33499-2131\sqrt{241})}$
5	$[70, 5^-]$	$[70', 5^-]$	$\frac{2}{3}\sqrt{\frac{2}{5}}$
5	$[70', 5^-]$	$[70, 5^-]$	$\frac{2}{3}\sqrt{\frac{2}{5}}$

a. Two-state $|[70,L^P]\rangle - |[70',L^P]\rangle$ mixing

Diagonalization of the 2×2 matrices proceeds by way of mixing of the $|[70, L^+]_{\rm a}\rangle$, and $|[70, L^+]_{\rm b}\rangle$ states, as determined by Eq. (B2), and the mixing angle θ being given by Eq. (B3). The (diagonal) eigenvalues of the potential matrix Eq. (B4) can also be expressed in terms of the matrix elements and that leads to, for the $[70, 5^-]$-plets, see Table XII,

$$\begin{aligned} &V_\pm([70,5]) \\ &= \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00}+\frac{\sqrt{3}}{18}v_{40}+\frac{\sqrt{5}}{30}v_{80}+ \\ &\quad \pm\frac{\sqrt{5}}{165}\sqrt{1815v_{40}^2+66\sqrt{15}v_{40}v_{80}+9v_{80}^2+968v_{6\pm6}^2}\Bigg) \end{aligned}$$

and, for the $[70, 1^-]$-plets, see Table XII,

$$\begin{aligned} &V_\pm([70,1]) \\ &= \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00}+\frac{\sqrt{3}}{4}v_{40}+\frac{3\sqrt{5}}{20}v_{80} \\ &\quad \pm\frac{1}{20}\sqrt{75v_{40}^2-10\sqrt{15}v_{40}v_{80}+5v_{80}^2+96v_{6\pm6}^2}\Bigg), \end{aligned}$$

where $b = v_{40}$, $c = v_{80}$, and $d = v_{6\pm6}$.

b. Three-state $|[70,3^-]\rangle - |[70',3^-]\rangle - |[70'',3^-]\rangle$ mixing

In the $L = 3$ case, the mixing potential matrix is 3×3 (see Table XII)

$$V_{\rm a,b} = \begin{pmatrix} \alpha & \delta & 0 \\ \delta & \beta & \epsilon \\ 0 & \epsilon & \gamma \end{pmatrix}. \tag{B5}$$

Its eigenvalues can also be expressed in terms of the matrix elements $(\alpha, \beta, \gamma, \delta, \epsilon)$ as follows,

$$V([70,3^-]) = \frac{1}{3}(\alpha+\beta+\gamma)+\frac{1}{3\sqrt{32}}A-\frac{\sqrt{32}}{3A}I \tag{B6}$$

$$V([70',3^-]) = \frac{1}{3}(\alpha+\beta+\gamma)-\frac{(1-i\sqrt{3})}{6\sqrt{32}}A+\frac{(1+i\sqrt{3})}{3\sqrt{34}A}I \tag{B7}$$

$$V([70'',3^-]) = \frac{1}{3}(\alpha+\beta+\gamma)-\frac{(1-i\sqrt{3})}{6\sqrt{32}}A+\frac{(1+i\sqrt{3})}{3\sqrt{34}A}I, \tag{B8}$$

where C and D have been separated into the unperturbed ($\epsilon = \delta = 0$) part and the perturbation—collect the $\delta^2+\epsilon^2$ terms together:

$$C = 2\alpha^3 + 2\beta^3 + 2\gamma^3 - 3(\beta+\gamma)(\beta\gamma+\alpha^2) - 3(\beta^2+\gamma^2)\alpha + 12\alpha\beta\gamma + 9[(\delta^2 - 2\epsilon^2)\alpha + \gamma(\epsilon^2 - 2\delta^2) + \beta(\epsilon^2+\delta^2)] \tag{B9}$$

$$D = 4I^3 + C^2 \tag{B10}$$

$$I = (-\alpha^2 - \beta^2 - \gamma^2 + \beta\alpha + \gamma\alpha + \beta\gamma) - 3(\delta^2 + \epsilon^2). \tag{B11}$$

Here,

$$\alpha = \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \left(\frac{5}{12\sqrt{3}} - \frac{85}{12\sqrt{723}}\right)v_{40} + \frac{241 + 19\sqrt{241}}{2892\sqrt{5}}v_{80}\right)$$
$$\beta = \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{2}{3\sqrt{3}}v_{40} + -\frac{1}{3\sqrt{5}}v_{80}\right)$$
$$\gamma = \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{5(241 - 17\sqrt{241})}{2892\sqrt{3}}v_{40} + \frac{241 - 19\sqrt{241}}{2892\sqrt{5}}v_{80}\right)$$
$$\delta = \frac{1}{\pi\sqrt{\pi}}\left(\sqrt{\frac{139}{450} + \frac{2131}{450\sqrt{241}}}v_{66}\right)$$
$$\epsilon = \frac{1}{\pi\sqrt{\pi}}\left(-\frac{1}{15}\sqrt{\frac{1}{482}(33499 - 2131\sqrt{241})}v_{66}\right).$$

These formulas are manifestly rather cumbersome, and they do not offer much new insight into the problem that could not be gained by a (simpler) numerical calculation. Clearly, there is no advantage to having explicit algebraic expressions for this kind of quantity. As K increases to $K \geq 6$, the number of mixing multiplets can only increase, as can the number of states within invariant subspaces.

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Algebraic Bethe ansatz for the XXX chain with triangular boundaries and Gaudin model

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Abstract

We implement fully the algebraic Bethe ansatz for the XXX Heisenberg spin chain in the case when both boundary matrices can be brought to the upper-triangular form. We define the Bethe vectors which yield the strikingly simple expression for the off shell action of the transfer matrix, deriving the spectrum and the relevant Bethe equations. We explore further these results by obtaining the off shell action of the generating function of the Gaudin Hamiltonians on the corresponding Bethe vectors through the so-called quasi-classical limit. Moreover, this action is as simple as it could possibly be, yielding the spectrum and the Bethe equations of the Gaudin model.

1. Introduction

The quantum inverse scattering method (QISM) is an approach to construct and solve quantum integrable systems [1–3]. In the framework of the QISM the algebraic Bethe ansatz (ABA) is

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a powerful algebraic tool, which yields the spectrum and corresponding eigenstates for which highest weight type representations are relevant, like for example quantum spin systems, Gaudin models, etc. In particular, the Heisenberg spin chain [4], with periodic boundary conditions, has been studied by the algebraic Bethe ansatz [1,3], including the question of completeness and simplicity of the spectrum [5].

A way to introduce non-periodic boundary conditions compatible with the integrability of the quantum systems solvable by the quantum inverse scattering method was developed in [6]. The boundary conditions at the left and right sites of the system are expressed in the left and right reflection matrices. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. The compatibility at the right site of the model is expressed by the dual reflection equation. The matrix form of the exchange relations between the entries of the Sklyanin monodromy matrix are analogous to the reflection equation. Together with the dual reflection equation they yield the commutativity of the open transfer matrix [6–8].

There is a renewed interest in applying the algebraic Bethe ansatz to the open XXX chain with non-periodic boundary conditions compatible with the integrability of the systems [9–12]. Other approaches include the ABA based on the functional relation between the eigenvalues of the transfer matrix and the quantum determinant and the associated T–Q relation [13], functional relations for the eigenvalues of the transfer matrix based on fusion hierarchy [14] and the Vertex-IRF correspondence [15]. For a review of the coordinate Bethe ansatz for non-diagonal boundaries see [16]. However, we will focus on the case when system admits the so-called pseudo-vacuum, or the reference state [6,9–12]. In his seminal work on boundary conditions in quantum integrable models Sklyanin has studied the XXZ spin chain with diagonal boundaries [6]. The next relevant step was the study of the $s\ell(n)$ spin chain in the case when reflection matrices can be brought into the diagonal form by a suitable similarity transformation which leaves the R-matrix invariant and it is independent of the spectral parameter [17,18]. These results were then generalized to the case of the spin-s XXX chain when there exists a basis in which one reflection matrix is triangular and the other one is diagonal [9]. Recent studies are focused on the XXX chain when both K-matrices can be simultaneously brought to a triangular form by a single similarity matrix which is independent of the spectral parameter [10] and similarly for the XXZ chain [12]. Although the on shell Bethe ansatz is realized, the proposed Bethe vectors are not suitable for the off shell ABA. The case when the reflection matrix $K^{-}(\lambda)$ is diagonal and $K^{+}(\lambda)$ is a two-by-two matrix with non-zero entries was studied in [11].

This work is centred on the implementation of the algebraic Bethe ansatz which yields the off shell action of the transfer matrix the XXX Heisenberg spin chain when the corresponding K-matrices are triangularizable. The Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ we define here are such that they make the off shell action of the transfer matrix strikingly simple since it almost coincides with the corresponding action in the case when the two boundary matrices are diagonal. The Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, are defined explicitly as some polynomial functions of the creation operators. As expected, the off shell action yields the spectrum of the transfer matrix and the corresponding Bethe equations. To explore further these results we use the so-called quasi-classical limit and obtain the off shell action of the generating function of the Gaudin Hamiltonians, with boundary terms, on the corresponding Bethe vectors.

A model of interacting spins in a chain was first considered by Gaudin [19,20]. In his approach, these models were introduced as a quasi-classical limit of the integrable quantum chains. The Gaudin models were extended to any simple Lie algebra, with arbitrary irreducible representation at each site of the chain [20]. Sklyanin studied the rational $s\ell(2)$ model in the framework of the quantum inverse scattering method using the $s\ell(2)$ invariant classical r-matrix [21]. A generalization of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang–Baxter equation [22] was relatively straightforward [23,24]. Therefore, considerable attention has been devoted to Gaudin models corresponding to the classical r-matrices of simple Lie algebras [25–27] and Lie superalgebras [28–32].

Hikami showed how the quasi-classical expansion of the transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the case of non-periodic boundary conditions [33]. Then the ABA was applied to open Gaudin model in the context of the Vertex-IRF correspondence [34–36]. Also, results were obtained for the open Gaudin models based on Lie superalgebras [37]. An approach to study the open Gaudin models based on the classical reflection equation [38] and the non-unitary r-matrices was developed recently, see [39, 40] and the references therein. For a recent review of the open Gaudin model see [41].

In [42] we have derived the generating function of the Gaudin Hamiltonians with boundary terms following Sklyanin's approach in the periodic case [21]. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX chain and the central element, the so-called Sklyanin determinant. Here we use this result with the objective to derive the off shell action of the generating function of the Gaudin Hamiltonians. As we will show below, the quasi-classical expansion of the Bethe vectors we have defined for he XXX Heisenberg spin chain yields the Bethe vectors of the corresponding Gaudin model. The significance of these Bethe vectors is in the striking simplicity of the formulae of the off shell action of the generating function of the Gaudin Hamiltonians.

This paper is organized as follows. In Section 2 we review the $SL(2)$-invariant Yang R-matrix and provide fundamental tools for the study of the inhomogeneous XXX Heisenberg spin chain. The general solutions of the reflection equation and the dual reflection equation are given in Section 3 as well as the triangularization of these K-matrices, when the corresponding parameters obey an extra identity. In Section 4 we expose the Sklyanin approach to the inhomogeneous XXX Heisenberg spin chain with non-periodic boundary conditions. The implementation of the ABA, as one of the main results of the paper, is presented in Section 5, including the definition of the Bethe vectors and the formulae of the off shell action of the transfer matrix. Corresponding Gaudin model and the respective implementation of the ABA are given in Section 6. Our conclusions are presented in Section 7. Finally, in Appendix A are given some basic definitions for the convenience of the reader and in Appendix B are given commutation relations relevant for the implementation of the ABA in Section 5.

2. Inhomogeneous Heisenberg spin chain

The XXX Heisenberg spin chain is related to the Yangian $\mathcal{Y}(s\ell(2))$ (see [43]) and the $SL(2)$-invariant Yang R-matrix [44]

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{2.1}$$

where λ is a spectral parameter, η is a quasi-classical parameter. We use $\mathbb{1}$ for the identity operator and $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The Yang R-matrix satisfies the Yang–Baxter equation [44,45] in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$R_{12}(\lambda-\mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda-\mu), \tag{2.2}$$

we suppress the dependence on the quasi-classical parameter η and use the standard notation of the QISM to denote spaces V_j, $j = 1, 2, 3$ on which corresponding R-matrices $R_{ij}, ij = 12, 13, 23$ act non-trivially [1–3]. In the present case $V_1 = V_2 = V_3 = \mathbb{C}^2$.

The Yang R-matrix also satisfies other relevant properties such as

unitarity	$R_{12}(\lambda)R_{21}(-\lambda) = (\eta^2 - \lambda^2)\mathbb{1}$;
parity invariance	$R_{21}(\lambda) = R_{12}(\lambda)$;
temporal invariance	$R^t_{12}(\lambda) = R_{12}(\lambda)$;
crossing symmetry	$R(\lambda) = \mathcal{J}_1 R^{t_2}(-\lambda - \eta)\mathcal{J}_1^{-1}$,

where t_2 denotes the transpose in the second space and the entries of the two-by-two matrix $\mathcal{J}$ are $\mathcal{J}_{ab} = (-1)^{a-1}\delta_{a,3-b}$.

Here we study the inhomogeneous XXX spin chain with N sites, characterized by the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. The Hilbert space of the system is

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = \left(\mathbb{C}^{2s+1}\right)^{\otimes N}. \tag{2.3}$$

Following [21] we introduce the Lax operator

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{\eta}{\lambda}(\vec{\sigma}_0 \cdot \vec{S}_m) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S_m^3 & \eta S_m^- \\ \eta S_m^+ & \lambda - \eta S_m^3 \end{pmatrix}. \tag{2.4}$$

Notice that $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$. It obeys

$$\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0m}(\eta - \lambda) = \left(1 + \eta^2 \frac{s_m(s_m + 1)}{\lambda(\eta - \lambda)}\right)\mathbb{1}_0, \tag{2.5}$$

where s_m is the value of spin in the space V_m.

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the R-matrix, $\mathbb{L}_{0m}(\lambda) = \frac{1}{\lambda}R_{0m}(\lambda - \eta/2)$.

Due to the commutation relations (A.1), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda - \mu)\mathbb{L}_{0m}(\lambda - \alpha_m)\mathbb{L}_{0'm}(\mu - \alpha_m) = \mathbb{L}_{0'm}(\mu - \alpha_m)\mathbb{L}_{0m}(\lambda - \alpha_m)R_{00'}(\lambda - \mu). \tag{2.6}$$

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N)\cdots\mathbb{L}_{01}(\lambda - \alpha_1) \tag{2.7}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter η and the inhomogeneous parameters $\{\alpha_j, j = 1, \ldots, N\}$. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.8}$$

From RLL-relations (2.6) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda - \mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda - \mu). \tag{2.9}$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

In every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3\omega_m = s_m\omega_m \quad \text{and} \quad S_m^+\omega_m = 0. \tag{2.10}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{2.11}$$

From the definitions above it is straightforward to obtain the action of the entries of the monodromy matrix (2.8) on the vector Ω_+

$$A(\lambda)\Omega_+ = a(\lambda)\Omega_+, \quad \text{with } a(\lambda) = \prod_{m=1}^{N} \frac{\lambda - \alpha_m + \eta s_m}{\lambda - \alpha_m}, \tag{2.12}$$

$$D(\lambda)\Omega_+ = d(\lambda)\Omega_+, \quad \text{with } d(\lambda) = \prod_{m=1}^{N} \frac{\lambda - \alpha_m - \eta s_m}{\lambda - \alpha_m}, \tag{2.13}$$

$$C(\lambda)\Omega_+ = 0. \tag{2.14}$$

To construct integrable spin chains with non-periodic boundary condition, we will follow Sklyanin's approach [6]. Accordingly, before defining the essential operators and corresponding algebraic structure, in the next section we will introduce the relevant boundary K-matrices.

3. Reflection equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk model, was developed in [6]. Boundary conditions on the left and right sites of the system are encoded in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $\mathbb{C}^2$ at the first site $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda - \mu)K_1^-(\lambda)R_{21}(\lambda + \mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda + \mu)K_1^-(\lambda)R_{21}(\lambda - \mu). \tag{3.1}$$

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$\begin{aligned} &R_{12}(\mu - \lambda)K_1^+(\lambda)R_{21}(-\lambda - \mu - 2\eta)K_2^+(\mu) \\ &\quad = K_2^+(\mu)R_{12}(-\lambda - \mu - 2\eta)K_1^+(\lambda)R_{21}(\mu - \lambda). \end{aligned} \tag{3.2}$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda - \eta) \tag{3.3}$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of (3.3) into the dual reflection equation (3.2) one gets the reflection equation (3.1) with shifted arguments.

The general, spectral parameter dependent, solutions of the reflection equation (3.1) and the dual reflection equation (3.2) can be written as follows [46,47]

$$\widetilde{K}^-(\lambda) = \begin{pmatrix} \xi^- - \lambda & \widetilde{\psi}^-\lambda \\ \widetilde{\phi}^-\lambda & \xi^- + \lambda \end{pmatrix}, \tag{3.4}$$

$$\widetilde{K}^+(\lambda) = \begin{pmatrix} \xi^+ + \lambda + \eta & -\widetilde{\psi}^+(\lambda + \eta) \\ -\widetilde{\phi}^+(\lambda + \eta) & \xi^+ - \lambda - \eta \end{pmatrix}. \tag{3.5}$$

We notice that the matrix $K^-(\lambda)$ (3.4) has at most two distinct eigenvalues

$$\epsilon_\pm = \xi^- \pm \lambda\nu^-, \qquad \nu^- = \sqrt{1+\widetilde{\phi}^-\widetilde{\psi}^-}, \tag{3.6}$$

when $\nu^- \neq 0$. Then, for $\widetilde{\psi}^- \neq 0$, there exists a matrix

$$U = \begin{pmatrix} \frac{\widetilde{\psi}^-}{1-\nu^-} & \frac{\widetilde{\psi}^-}{1+\nu^-} \end{pmatrix} \tag{3.7}$$

such that

$$U^{-1}\widetilde{K}^-(\lambda)U = \begin{pmatrix} \xi^- - \lambda\nu^- & 0 \\ 0 & \xi^- + \lambda\nu^- \end{pmatrix}. \tag{3.8}$$

A similar diagonalization exists when $\widetilde{\phi}^- \neq 0$. However, for $\nu^- = 0$, i.e. $\widetilde{\phi}^-\widetilde{\psi}^- = -1$, the matrix $K^-(\lambda)$ cannot be diagonalized and

$$U^{-1}K^-(\lambda)U = \begin{pmatrix} \xi^- & \lambda\widetilde{\phi}^- \\ 0 & \xi^- \end{pmatrix}, \tag{3.9}$$

where

$$U = \begin{pmatrix} \widetilde{\psi}^- & 0 \\ 1 & -\widetilde{\phi}^- \end{pmatrix}. \tag{3.10}$$

Following [10] we notice the condition

$$\left(\widetilde{\phi}^-\widetilde{\psi}^+ - \widetilde{\phi}^+\widetilde{\psi}^-\right)^2 = 4\left(\widetilde{\phi}^- - \widetilde{\phi}^+\right)\left(\widetilde{\psi}^- - \widetilde{\psi}^+\right) \tag{3.11}$$

has to be imposed on the parameters of $K^\mp$ so that the matrices (3.4) and (3.5) are upper triangularizable by a single similarity matrix M. When the square root with the negative sign is taken on the right-hand-side of (3.11) then one possible choice for M is given by

$$M = \begin{pmatrix} -1-\nu^- & \widetilde{\phi}^- \\ \widetilde{\phi}^- & -1-\nu^- \end{pmatrix}. \tag{3.12}$$

Evidently this matrix does not depend on the spectral parameter λ and it is such that

$$K^-(\lambda) = M^{-1}\widetilde{K}^-(\lambda)M = \begin{pmatrix} \xi^- - \lambda\nu^- & \lambda\psi^- \\ 0 & \xi^- + \lambda\nu^- \end{pmatrix}, \tag{3.13}$$

$$K^+(\lambda) = M^{-1}\widetilde{K}^+(\lambda)M = \begin{pmatrix} \xi^+ + (\lambda+\eta)\nu^+ & -\psi^+(\lambda+\eta) \\ 0 & \xi^+ - (\lambda+\eta)\nu^+ \end{pmatrix}, \tag{3.14}$$

with $\psi^- = \widetilde{\phi}^- + \widetilde{\psi}^-$, $\nu^+ = \sqrt{1+\widetilde{\phi}^+\widetilde{\psi}^+}$ and $\psi^+ = \widetilde{\phi}^+ + \widetilde{\psi}^+$. An analogous choice for M exists for the other sign of the square root in (3.11).

4. Inhomogeneous Heisenberg spin chain with boundary terms

In order to develop the formalism necessary to describe an integrable spin chain with non-periodic boundary condition, we use the Sklyanin approach [6]. The main tool in this framework is the corresponding monodromy matrix

$$\mathcal{T}_0(\lambda) = T_0(\lambda)K_0^-(\lambda)\widetilde{T}_0(\lambda), \tag{4.1}$$

it consists of the matrix $T(\lambda)$ (2.7), a reflection matrix $K^-(\lambda)$ (3.13) and the matrix

$$\widetilde{T}_0(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} = \mathbb{L}_{01}(\lambda + \alpha_1 + \eta) \cdots \mathbb{L}_{0N}(\lambda + \alpha_N + \eta). \tag{4.2}$$

It is important to notice that the identity (2.5) can be rewritten in the form

$$\mathbb{L}_{0m}(\lambda - \alpha_m)\mathbb{L}_{0m}(-\lambda + \alpha_m + \eta) = \left(1 + \frac{\eta^2 s_m(s_m + 1)}{(\lambda - \alpha_m)(-\lambda + \alpha_m + \eta)}\right)\mathbb{1}_0. \tag{4.3}$$

It follows from the equation above and the RLL-relations (2.6) that the RTT-relations (2.9) can be recast as follows

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda + \mu)T_0(\lambda) = T_0(\lambda)R_{00'}(\lambda + \mu)\widetilde{T}_{0'}(\mu), \tag{4.4}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu - \lambda) = R_{00'}(\mu - \lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda). \tag{4.5}$$

Using the RTT-relations (2.9), (4.4), (4.5) and the reflection equation (3.1) it is straightforward to show that the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ in $V_0 \otimes V_{0'}$ are

$$R_{00'}(\lambda - \mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda + \mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda + \mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda - \mu), \tag{4.6}$$

using the notation of [6]. From the equation above we can read off the commutation relations of the entries of the monodromy matrix

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{4.7}$$

Following Sklyanin [6] (see also [10]) we introduce the operator

$$\widehat{\mathcal{D}}(\lambda) = \mathcal{D}(\lambda) - \frac{\eta}{2\lambda + \eta}\mathcal{A}(\lambda). \tag{4.8}$$

The relevant commutation relations are given in Appendix B.

The exchange relations (4.6) admit a central element, the so-called Sklyanin determinant,

$$\Delta\big[\mathcal{T}(\lambda)\big] = \operatorname{tr}_{00'} P^-_{00'}\mathcal{T}_0(\lambda - \eta/2)R_{00'}(2\lambda)\mathcal{T}_{0'}(\lambda + \eta/2). \tag{4.9}$$

The element $\Delta[\mathcal{T}(\lambda)]$ can be expressed in form

$$\Delta\big[\mathcal{T}(\lambda)\big] = 2\lambda\widehat{\mathcal{D}}(\lambda - \eta/2)\mathcal{A}(\lambda + \eta/2) - (2\lambda + \eta)\mathcal{B}(\lambda - \eta/2)\mathcal{C}(\lambda + \eta/2). \tag{4.10}$$

The open chain transfer matrix is given by the trace of the monodromy $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^+(\lambda)$ [6],

$$t(\lambda) = \operatorname{tr}_0\big(K^+(\lambda)\mathcal{T}(\lambda)\big). \tag{4.11}$$

The reflection matrix $K^+(\lambda)$ (3.14) is the corresponding solution of the dual reflection equation (3.2). The commutativity of the transfer matrix for different values of the spectral parameter

$$\big[t(\lambda), t(\mu)\big] = 0, \tag{4.12}$$

is guaranteed by the dual reflection equation (3.2) and the exchange relations (4.6) of the monodromy matrix $\mathcal{T}(\lambda)$ [6].

5. Algebraic Bethe ansatz

In [10] it was shown that the most general case in which the algebraic Bethe ansatz can be fully implemented is when both K-matrices have upper-triangular from (3.13) and (3.14). The main aim of this section is to define the Bethe vectors as to obtain the most simplest formulae for the off shell action of the transfer matrix of the spin chain on these Bethe vectors. The first step in this direction is to get the expressions of the entries of the monodromy matrix $\mathcal{T}(\lambda)$ in terms of the corresponding ones of the monodromies $T(\lambda)$ and $\widetilde{T}(\lambda)$. According to definition of the monodromy matrix (4.1) we have

$$\begin{aligned}\mathcal{T}(\lambda) &= \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \xi^- - \lambda\nu^- & \psi^-\lambda \\ 0 & \xi^- + \lambda\nu^- \end{pmatrix} \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}.\end{aligned} \tag{5.1}$$

From the equation above, using (4.2) and the RTT-relations (4.4), we obtain

$$\mathcal{A}(\lambda) = (\xi^- - \lambda\nu^-)A(\lambda)\widetilde{A}(\lambda) + \big((\psi^-\lambda)A(\lambda) + (\xi^- + \lambda\nu^-)B(\lambda)\big)\widetilde{C}(\lambda) \tag{5.2}$$

$$\begin{aligned}\mathcal{D}(\lambda) &= (\xi^- - \lambda\nu^-)\Big(\widetilde{B}(\lambda)C(\lambda) - \frac{\eta}{2\lambda+\eta}\big(D(\lambda)\widetilde{D}(\lambda) - \widetilde{A}(\lambda)A(\lambda)\big)\Big) \\ &\quad + \big((\psi^-\lambda)C(\lambda) + (\xi^- + \lambda\nu^-)D(\lambda)\big)\widetilde{D}(\lambda)\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathcal{B}(\lambda) &= (\xi^- - \lambda\nu^-)\Big(\frac{2\lambda}{2\lambda+\eta}\widetilde{B}(\lambda)A(\lambda) - \frac{\eta}{2\lambda+\eta}B(\lambda)\widetilde{D}(\lambda)\Big) \\ &\quad + \big((\psi^-\lambda)A(\lambda) + (\xi^- + \lambda\nu^-)B(\lambda)\big)\widetilde{D}(\lambda)\end{aligned} \tag{5.4}$$

$$\mathcal{C}(\lambda) = (\xi^- - \lambda\nu^-)C(\lambda)\widetilde{A}(\lambda) + \big((\psi^-\lambda)C(\lambda) + (\xi^- + \lambda\nu^-)D(\lambda)\big)\widetilde{C}(\lambda). \tag{5.5}$$

With the aim of obtaining the action of the operators $\mathcal{A}(\lambda)$, $\mathcal{D}(\lambda)$ and $\mathcal{C}(\lambda)$ on the vector Ω_+ (2.11) we first observe that the action of the operators $\widetilde{A}(\lambda)$, $\widetilde{D}(\lambda)$ and $\widetilde{C}(\lambda)$ on the vector Ω_+

$$\widetilde{A}(\lambda)\Omega_+ = \widetilde{a}(\lambda)\Omega_+, \quad \text{with } \widetilde{a}(\lambda) = \prod_{m=1}^{N} \frac{\lambda + \alpha_m + \eta + \eta s_m}{\lambda + \alpha_m + \eta}, \tag{5.6}$$

$$\widetilde{D}(\lambda)\Omega_+ = \widetilde{d}(\lambda)\Omega_+, \quad \text{with } \widetilde{d}(\lambda) = \prod_{m=1}^{N} \frac{\lambda + \alpha_m + \eta - \eta s_m}{\lambda + \alpha_m + \eta}, \tag{5.7}$$

$$\widetilde{C}(\lambda)\Omega_+ = 0, \tag{5.8}$$

follows directly from the definition (4.2). Using the relations (5.2)–(5.5) and the formulas (2.12)–(2.14) and (5.6)–(5.8) we derive

$$\mathcal{C}(\lambda)\Omega_+ = 0, \tag{5.9}$$

$$\mathcal{A}(\lambda)\Omega_+ = \alpha(\lambda)\Omega_+, \quad \text{with } \alpha(\lambda) = (\xi^- - \lambda\nu^-)a(\lambda)\widetilde{a}(\lambda), \tag{5.10}$$

$$\begin{aligned}&\mathcal{D}(\lambda)\Omega_+ = \delta(\lambda)\Omega_+, \quad \text{with} \\ &\delta(\lambda) = \Big((\xi^- + \lambda\nu^-) - \frac{\eta}{2\lambda+\eta}(\xi^- - \lambda\nu^-)\Big)d(\lambda)\widetilde{d}(\lambda) + \frac{\eta}{2\lambda+\eta}(\xi^- - \lambda\nu^-)a(\lambda)\widetilde{a}(\lambda).\end{aligned} \tag{5.11}$$

In what follows we will use the fact that Ω_+ is an eigenvector of the operator $\widehat{\mathcal{D}}(\lambda)$ (4.8)

$$\widehat{\mathcal{D}}(\lambda)\Omega_+ = \widehat{\delta}(\lambda)\Omega_+, \quad \text{with } \widehat{\delta}(\lambda) = \delta(\lambda) - \frac{\eta}{2\lambda+\eta}\alpha(\lambda), \tag{5.12}$$

or explicitly

$$\widehat{\delta}(\lambda) = \left(\left(\xi^- + \lambda\nu^-\right) - \frac{\eta}{2\lambda+\eta}\left(\xi^- - \lambda\nu^-\right)\right)d(\lambda)\widetilde{d}(\lambda). \tag{5.13}$$

The transfer matrix of the inhomogeneous XXX chain (4.11) with the triangular K-matrix (3.14) can be expressed using Sklyanin's $\widehat{\mathcal{D}}(\lambda)$ operator (4.8) [10]

$$t(\lambda) = \kappa_1(\lambda)\mathcal{A}(\lambda) + \kappa_2(\lambda)\widehat{\mathcal{D}}(\lambda) + \kappa_{12}(\lambda)\mathcal{C}(\lambda), \tag{5.14}$$

with

$$\kappa_1(\lambda) = 2\left(\xi^+ + \lambda\nu^+\right)\frac{\lambda+\eta}{2\lambda+\eta}, \qquad \kappa_2(\lambda) = \xi^+ - (\lambda+\eta)\nu^+,$$
$$\kappa_{12}(\lambda) = -\psi^+(\lambda+\eta). \tag{5.15}$$

Evidently the vector Ω_+ (2.11) is an eigenvector of the transfer matrix

$$t(\lambda)\Omega_+ = \left(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda)\right)\Omega_+ = \Lambda_0(\lambda)\Omega_+. \tag{5.16}$$

For simplicity we have suppressed the dependence of the eigenvalue $\Lambda_0(\lambda)$ on the boundary parameters ξ^+ and ν^+ as well as the quasi-classical parameter η.

We proceed to define the Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ as to make the off shell action of $t(\lambda)$ on them as simple as possible. Before discussing $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for arbitrary positive integer M, we will give explicitly first two Bethe vectors as well as the corresponding formulae for the off shell action of the transfer matrix. To this end, our next step is to show that

$$\Psi_1(\mu) = \mathcal{B}(\mu)\Omega_+ + b_1(\mu)\Omega_+, \tag{5.17}$$

is a Bethe vector, if $b_1(\mu)$ is chosen to be

$$b_1(\mu) = \frac{\psi^+}{2\nu^+}\left(\frac{2\mu}{2\mu+\eta}\alpha(\mu) - \widehat{\delta}(\mu)\right). \tag{5.18}$$

A straightforward calculation, using the relations (B.2), (B.3) and (B.4), shows that the off shell action of the transfer matrix (5.14) on $\Psi_1(\mu)$ is given by

$$t(\lambda)\Psi_1(\mu) = \Lambda_1(\lambda,\mu)\Psi_1(\mu) + \frac{2\eta(\lambda+\eta)(\xi^+ + \mu\nu^+)}{(\lambda-\mu)(\lambda+\mu+\eta)}F_1(\mu)\Psi_1(\lambda) \tag{5.19}$$

where the eigenvalue $\Lambda_1(\lambda,\mu)$ is given by

$$\Lambda_1(\lambda,\mu) = \kappa_1(\lambda)\frac{(\lambda+\mu)(\lambda-\mu-\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\alpha(\lambda) + \kappa_2(\lambda)\frac{(\lambda-\mu+\eta)(\lambda+\mu+2\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\widehat{\delta}(\lambda). \tag{5.20}$$

Evidently $\Lambda_1(\lambda,\mu)$ depends also on boundary parameters ξ^+, ν^+ and the quasi-classical parameter η, but these parameters are omitted in order to simplify the formulae. The unwanted term on the right hand side (5.19) is annihilated by the Bethe equation

$$F_1(\mu) = \frac{2\mu}{2\mu+\eta}\alpha(\mu) - \frac{\xi^+ - (\mu+\eta)\nu^+}{\xi^+ + \mu\nu^+}\widehat{\delta}(\mu) = 0, \tag{5.21}$$

or equivalently,

$$\frac{\alpha(\mu)}{\widehat{\delta}(\mu)} = \frac{(\mu+\eta)\kappa_2(\mu)}{\mu\kappa_1(\mu)} = \frac{(2\mu+\eta)(\xi^+ - (\mu+\eta)\nu^+)}{2\mu(\xi^+ + \mu\nu^+)}. \tag{5.22}$$

Therefore we have shown that $\Psi_1(\mu)$ (5.17) is the Bethe vector of the transfer matrix (5.14) corresponding to the eigenvalue $\Lambda_1(\lambda,\mu)$ (5.20).

We seek the Bethe vector $\Psi_2(\mu_1,\mu_2)$ in the form

$$\begin{aligned}\Psi_2(\mu_1,\mu_2) &= \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ + b_2^{(1)}(\mu_2;\mu_1)\mathcal{B}(\mu_1)\Omega_+ \\ &\quad + b_2^{(1)}(\mu_1;\mu_2)\mathcal{B}(\mu_2)\Omega_+ + b_2^{(2)}(\mu_1,\mu_2)\Omega_+,\end{aligned} \tag{5.23}$$

where $b_2^{(1)}(\mu_1;\mu_2)$ and $b_2^{(2)}(\mu_1,\mu_2)$ are given by

$$\begin{aligned}b_2^{(1)}(\mu_1;\mu_2) &= \frac{\psi^+}{2\nu^+}\bigg(\frac{2\mu_1}{2\mu_1+\eta}\frac{(\mu_1+\mu_2)(\mu_1-\mu_2-\eta)}{(\mu_1-\mu_2)(\mu_1+\mu_2+\eta)}\alpha(\mu_1) \\ &\quad - \frac{(\mu_1-\mu_2+\eta)(\mu_1+\mu_2+2\eta)}{(\mu_1-\mu_2)(\mu_1+\mu_2+\eta)}\widehat{\delta}(\mu_1)\bigg),\end{aligned} \tag{5.24}$$

$$b_2^{(2)}(\mu_1,\mu_2) = \frac{1}{2}\big(b_2^{(1)}(\mu_1;\mu_2)\,b_1(\mu_2) + b_2^{(1)}(\mu_2;\mu_1)\,b_1(\mu_1)\big). \tag{5.25}$$

Starting from the definitions (5.14) and (5.23), using the relations (B.8), (B.9) and (B.10) to push the operators $\mathcal{A}(\lambda)$, $\widehat{\mathcal{D}}(\lambda)$ and $\mathcal{C}(\lambda)$ to the right and after rearranging some terms, we obtain the off shell action of transfer matrix $t(\lambda)$ on $\Psi_2(\mu_1,\mu_2)$

$$\begin{aligned}t(\lambda)\Psi_2(\mu_1,\mu_2) &= \Lambda_2\big(\lambda,\{\mu_i\}\big)\Psi_2(\mu_1,\mu_2) \\ &\quad + \sum_{i=1}^{2}\frac{2\eta(\lambda+\eta)(\xi^+ + \mu_i\nu^+)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}F_2(\mu_i;\mu_{3-i})\Psi_2(\lambda,\mu_{3-i}),\end{aligned} \tag{5.26}$$

where the eigenvalue is given by

$$\begin{aligned}\Lambda_2\big(\lambda,\{\mu_i\}\big) &= \kappa_1(\lambda)\alpha(\lambda)\prod_{i=1}^{2}\frac{(\lambda+\mu_i)(\lambda-\mu_i-\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)} \\ &\quad + \kappa_2(\lambda)\widehat{\delta}(\lambda)\prod_{i=1}^{2}\frac{(\lambda-\mu_i+\eta)(\lambda+\mu_i+2\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}\end{aligned} \tag{5.27}$$

and the two unwanted terms in (5.26) are canceled by the Bethe equations which follow from $F_2(\mu_i;\mu_{3-i}) = 0$, i.e.

$$\begin{aligned}&\frac{2\mu_i}{2\mu_i+\eta}\frac{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)}{(\mu_i-\mu_{3-i})(\mu_i+\mu_{3-i}+\eta)}\alpha(\mu_i) \\ &\quad - \frac{\xi^+ - (\mu_i+\eta)\nu^+}{\xi^+ + \mu_i\nu^+}\frac{(\mu_i-\mu_{3-i}+\eta)(\mu_i+\mu_{3-i}+2\eta)}{(\mu_i-\mu_{3-i})(\mu_i+\mu_{3-i}+\eta)}\widehat{\delta}(\mu_i) = 0,\end{aligned} \tag{5.28}$$

with $i = \{1, 2\}$. Therefore the Bethe equations are

$$\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{(\mu_i+\eta)\kappa_2(\mu_i)}{\mu_i\kappa_1(\mu_i)}\frac{(\mu_i-\mu_{3-i}+\eta)(\mu_i+\mu_{3-i}+2\eta)}{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)}, \tag{5.29}$$

where $i = \{1, 2\}$. Striking property of the Bethe vectors we have introduced so far is the simplicity of the off shell action of the transfer matrix $t(\lambda)$, Eqs. (5.19) and (5.26). Actually, the action of the transfer matrix almost coincides with the one in the case when the two boundary matrices are diagonal [6,33].

We proceed to define $\Psi_M(\mu_1, \mu_2, \dots, \mu_M)$ as a sum of 2^M terms, for arbitrary positive integer M, and as a symmetric function of its arguments

$$\begin{aligned}
&\Psi_M(\mu_1, \mu_2, \dots, \mu_M) \\
&\quad = \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_M)\Omega_+ \\
&\qquad + b_M^{(1)}(\mu_M; \mu_1, \mu_2, \dots, \mu_{M-1})\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_{M-1})\Omega_+ \\
&\qquad + \cdots + b_M^{(2)}(\mu_{M-1}, \mu_M; \mu_1, \mu_2, \dots, \mu_{M-2})\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_{M-2})\Omega_+ \\
&\qquad \vdots \\
&\qquad + b_M^{(M-1)}(\mu_1, \mu_2, \dots, \mu_{M-1}; \mu_M)\mathcal{B}(\mu_M)\Omega_+ + b_M^{(M)}(\mu_1, \mu_2, \dots, \mu_M)\Omega_+,
\end{aligned} \tag{5.30}$$

where the coefficients are given by

$$\begin{aligned}
b_M^{(1)}(\mu_1; \mu_2, \mu_3, \dots, \mu_M) &= \frac{\psi^+}{2\nu^+}\Bigg(\frac{2\mu_1}{2\mu_1+\eta}\alpha(\mu_1)\prod_{j=2}^{M}\frac{(\mu_1+\mu_j)(\mu_1-\mu_j-\eta)}{(\mu_1-\mu_j)(\mu_1+\mu_j+\eta)} \\
&\quad - \widehat{\delta}(\mu_1)\prod_{j=2}^{M}\frac{(\mu_1-\mu_j+\eta)(\mu_1+\mu_j+2\eta)}{(\mu_1-\mu_j)(\mu_1+\mu_j+\eta)}\Bigg),
\end{aligned} \tag{5.31}$$

$$\begin{aligned}
b_M^{(2)}(\mu_1, \mu_2; \mu_3, \dots, \mu_M) &= \frac{1}{2}\big(b_M^{(1)}(\mu_1; \mu_2, \mu_3, \dots, \mu_M)b_{M-1}^{(1)}(\mu_2; \mu_3, \dots, \mu_M) \\
&\quad + b_M^{(1)}(\mu_2; \mu_1, \mu_3, \dots, \mu_M)b_{M-1}^{(1)}(\mu_1; \mu_3, \dots, \mu_M)\big), \\
&\vdots
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&b_M^{(M-1)}(\mu_1, \mu_2, \dots, \mu_{M-1}; \mu_M) \\
&\quad = \frac{1}{(M-1)!}\sum_{\rho\in S_{M-1}} b_M^{(1)}(\mu_{\rho(1)}; \mu_{\rho(2)}, \dots, \mu_M) \\
&\qquad \times b_{M-1}^{(1)}(\mu_{\rho(2)}; \mu_{\rho(3)}, \dots, \mu_M) \\
&\qquad \times b_{M-2}^{(1)}(\mu_{\rho(3)}; \mu_{\rho(4)}, \dots, \mu_M)\cdots b_2^{(1)}(\mu_{\rho(M-1)}; \mu_M)
\end{aligned} \tag{5.33}$$

$$\begin{aligned}
&b_M^{(M)}(\mu_1, \mu_2, \dots, \mu_M) \\
&\quad = \frac{1}{M!}\sum_{\sigma\in S_M} b_M^{(1)}(\mu_{\sigma(1)}; \mu_{\sigma(2)}, \dots, \mu_{\sigma(M)})b_{M-1}^{(1)}(\mu_{\sigma(2)}; \mu_{\sigma(3)}, \dots, \mu_{\sigma(M)}) \\
&\qquad \times b_{M-2}^{(1)}(\mu_{\sigma(3)}; \mu_{\sigma(4)}, \dots, \mu_{\sigma(M)})\cdots b_2^{(1)}(\mu_{\sigma(M-1)}; \mu_{\sigma(M)})b_1(\mu_{\sigma(M)}),
\end{aligned} \tag{5.34}$$

where S_{M-1} and S_M are the symmetric groups of degree $M-1$ and M, respectively.

A straightforward calculation based on evident generalization of the formulas (B.8), (B.9) and (B.10) and subsequent rearranging of terms, yields the off shell action of the transfer matrix on the Bethe vector $\Psi_M(\mu_1, \mu_2, \dots, \mu_M)$

$$
\begin{aligned}
& t(\lambda)\Psi_M(\mu_1,\mu_2,\ldots,\mu_M) \\
&\quad = \Lambda_M\big(\lambda,\{\mu_i\}\big)\Psi_M(\mu_1,\mu_2,\ldots,\mu_M) \\
&\qquad + \sum_{i=1}^{M} \frac{2\eta(\lambda+\eta)(\xi^+ + \mu_i\nu^+)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)} F_M\big(\mu_i;\{\mu_j\}_{j\neq i}\big)\Psi_M\big(\lambda,\{\mu_j\}_{j\neq i}\big),
\end{aligned}
\tag{5.35}
$$

where the corresponding eigenvalue is given by

$$
\begin{aligned}
\Lambda_M\big(\lambda,\{\mu_i\}\big) = \kappa_1(\lambda)\alpha(\lambda)\prod_{i=1}^{M}\frac{(\lambda+\mu_i)(\lambda-\mu_i-\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)} \\
+ \kappa_2(\lambda)\widehat{\delta}(\lambda)\prod_{i=1}^{M}\frac{(\lambda-\mu_i+\eta)(\lambda+\mu_i+2\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}
\end{aligned}
\tag{5.36}
$$

and the M unwanted terms o the right hand side of (5.35) are canceled by the Bethe equations $F_M(\mu_i;\{\mu_j\}_{j\neq i}) = 0$, explicitly

$$
\begin{aligned}
& \frac{2\mu_i}{2\mu_i+\eta}\alpha(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{(\mu_i+\mu_j)(\mu_i-\mu_j-\eta)}{(\mu_i-\mu_j)(\mu_i+\mu_j+\eta)} \\
&\quad - \frac{\xi^+ - (\mu_i+\eta)\nu^+}{\xi^+ + \mu_i\nu^+}\widehat{\delta}(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{(\mu_i-\mu_j+\eta)(\mu_i+\mu_j+2\eta)}{(\mu_i-\mu_j)(\mu_i+\mu_j+\eta)} = 0,
\end{aligned}
\tag{5.37}
$$

or equivalently

$$
\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{(\mu_i+\eta)\kappa_2(\mu_i)}{\mu_i\kappa_1(\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{(\mu_i-\mu_j+\eta)(\mu_i+\mu_j+2\eta)}{(\mu_i+\mu_j)(\mu_i-\mu_j-\eta)},
\tag{5.38}
$$

with $i = \{1,2,\ldots,M\}$. The Bethe vectors $\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)$ we have defined in (5.30) yield the strikingly simple expression (5.35) for the off shell action of the transfer matrix $t(\lambda)$ (5.14). Actually, the action of the transfer matrix is as simple as it could possible be since it almost coincides with the one in the case when the two boundary matrices are diagonal [6,33]. In this way we have fully implemented the algebraic Bethe ansatz for the XXX spin chain in the case when both boundary matrices have upper-triangular form (3.13) and (3.14).

6. Gaudin model

We explore further the results obtained in the previous section on the XXX Heisenberg spin chain in the case when both boundary matrix are upper-triangular. We combine them together with the quasi-classical limit studied in [42] with the aim of implementing fully the off shell Bethe ansatz for the corresponding Gaudin model by defining its Bethe vectors. The significance of these Bethe vectors is in the striking simplicity of the formulae of the off shell action of the generating function of the Gaudin Hamiltonians, yielding the spectrum and the corresponding Bethe equations.

For the study of the open Gaudin model we impose

$$\lim_{\eta\to 0}\big(K^{+}(\lambda)K^{-}(\lambda)\big)=\big(\xi^{2}-\lambda^{2}\nu^{2}\big)\mathbb{1}. \tag{6.1}$$

In particular, this implies that the parameters of the reflection matrices on the left and on the right end of the chain are the same. In general, this is not the case in the study of the open spin chain. However, this condition is essential for the Gaudin model. Then we will write

$$K^{-}(\lambda)\equiv K(\lambda)=\begin{pmatrix}\xi-\lambda\nu & \lambda\psi\\ 0 & \xi+\lambda\nu\end{pmatrix}, \tag{6.2}$$

so that

$$K^{+}(\lambda)=K(-\lambda-\eta)=\begin{pmatrix}\xi+(\lambda+\eta)\nu & -\psi(\lambda+\eta)\\ 0 & \xi-(\lambda+\eta)\nu\end{pmatrix}. \tag{6.3}$$

In [42] we have derived the generating function of the Gaudin Hamiltonians with boundary terms following Sklyanin's approach in the periodic case [21]. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX chain and the central element, the so-called Sklyanin determinant. Finally, the expansion reads [42]

$$\begin{aligned}2\lambda t(\lambda)-\Delta\big[\mathcal{T}(\lambda)\big]&=2\lambda\big(\xi^{2}-\lambda^{2}\nu^{2}\big)\mathbb{1}+\eta\big(\xi^{2}-3\lambda^{2}\nu^{2}\big)\mathbb{1}\\&\quad+\eta^{2}\lambda\bigg(\big(\xi^{2}-\lambda^{2}\nu^{2}\big)\tau(\lambda)-\frac{\nu^{2}}{2}\mathbb{1}\bigg)+\mathcal{O}\big(\eta^{3}\big),\end{aligned} \tag{6.4}$$

where $\tau(\lambda)$ is the generating function of the Gaudin Hamiltonians, with upper triangular reflection matrix (6.2),

$$\tau(\lambda)=\mathrm{tr}_0\,\mathcal{L}_0^{2}(\lambda), \tag{6.5}$$

and the Lax matrix

$$\mathcal{L}_0(\lambda)=\sum_{m=1}^{N}\bigg(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m}+\frac{\vec{\sigma}_0\cdot(K_m^{-1}(\lambda)\vec{S}_mK_m(\lambda))}{\lambda+\alpha_m}\bigg). \tag{6.6}$$

The Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function (6.5) at poles $\lambda=\pm\alpha_m$:

$$\mathrm{Res}_{\lambda=\alpha_m}\,\tau(\lambda)=4H_m\quad\text{and}\quad\mathrm{Res}_{\lambda=-\alpha_m}\,\tau(\lambda)=4\widetilde{H}_m \tag{6.7}$$

where

$$H_m=\sum_{n\neq m}^{N}\frac{\vec{S}_m\cdot\vec{S}_n}{\alpha_m-\alpha_n}+\sum_{n=1}^{N}\frac{(K_m(\alpha_m)\vec{S}_mK_m^{-1}(\alpha_m))\cdot\vec{S}_n+\vec{S}_n\cdot(K_m(\alpha_m)\vec{S}_mK_m^{-1}(\alpha_m))}{2(\alpha_m+\alpha_n)}, \tag{6.8}$$

and

$$\begin{aligned}\widetilde{H}_m&=\sum_{n\neq m}^{N}\frac{\vec{S}_m\cdot\vec{S}_n}{\alpha_m-\alpha_n}\\&\quad+\sum_{n=1}^{N}\frac{(K_m(-\alpha_m)\vec{S}_nK_m^{-1}(-\alpha_m))\cdot\vec{S}_n+\vec{S}_n\cdot(K_m(-\alpha_m)\vec{S}_mK_m^{-1}(-\alpha_m))}{2(\alpha_m+\alpha_n)}.\end{aligned} \tag{6.9}$$

Since the element $\Delta[\mathcal{T}(\lambda)]$ can be written in form (4.10) it is evident that the vector Ω_+ (2.11) is its eigenvector

$$\Delta\big[\mathcal{T}(\lambda)\big]\Omega_+ = 2\lambda\alpha(\lambda+\eta/2)\widehat{\delta}(\lambda-\eta/2)\Omega_+. \tag{6.10}$$

Moreover, it follows from (5.16) and (6.10) that Ω_+ (2.11) is an eigenvector of the difference

$$\big(2\lambda t(\lambda) - \Delta\big[\mathcal{T}(\lambda)\big]\big)\Omega_+ = 2\lambda\big(\Lambda_0(\lambda) - \alpha(\lambda+\eta/2)\widehat{\delta}(\lambda-\eta/2)\big)\Omega_+. \tag{6.11}$$

We can expand the eigenvalue on the right hand side of the equation above in powers of η

$$\begin{aligned}
&2\lambda\big(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda) - \alpha(\lambda+\eta/2)\widehat{\delta}(\lambda-\eta/2)\big)\\
&\quad = 2\lambda\big(\xi^2-\lambda^2\nu^2\big) + \eta\big(\xi^2 - 3\lambda^2\nu^2\big) + \eta^2\lambda\left(\big(\xi^2-\lambda^2\nu^2\big)\chi_0(\lambda) - \frac{\nu^2}{2}\right) + \mathcal{O}\big(\eta^3\big).
\end{aligned} \tag{6.12}$$

Substituting the expansion above into the right hand side of (6.11) and using (6.4) to expand the left hand side, it follows that the vector Ω_+ (2.11) is an eigenvector of the generating function of the Gaudin Hamiltonians

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+, \tag{6.13}$$

with

$$\begin{aligned}
\chi_0(\lambda) = {}& \frac{4\lambda}{\xi^2-\lambda^2\nu^2}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda-\alpha_m} + \frac{s_m}{\lambda+\alpha_m}\right)\\
& + 2\sum_{m,n=1}^{N}\left(\frac{s_m s_n + s_m\delta_{mn}}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{2(s_m s_n + s_m\delta_{mn})}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{s_m s_n + s_m\delta_{mn}}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right).
\end{aligned} \tag{6.14}$$

As expected, the eigenfunction $\chi_0(\lambda)$ also depends on the boundary parameters ξ, ν. In general, we can obtain the spectrum $\chi_M(\lambda,\mu_1,\ldots,\mu_M)$ of the generating function $\tau(\lambda)$ of the Gaudin Hamiltonians through the expansion

$$\begin{aligned}
&2\lambda\big(\Lambda_M(\lambda,\mu_1,\ldots,\mu_M) - \alpha(\lambda+\eta/2)\widehat{\delta}(\lambda-\eta/2)\big)\\
&\quad = 2\lambda\big(\xi^2-\lambda^2\nu^2\big) + \eta\big(\xi^2-3\lambda^2\nu^2\big) + \eta^2\lambda\left(\big(\xi^2-\lambda^2\nu^2\big)\chi_M(\lambda,\mu_1,\ldots,\mu_M) - \frac{\nu^2}{2}\right)\\
&\qquad + \mathcal{O}\big(\eta^3\big),
\end{aligned} \tag{6.15}$$

or explicitly

$$\begin{aligned}
\chi_M(\lambda,\mu_1,\ldots,\mu_M) = {}& \frac{-4\lambda^2\nu^4}{(\xi^2-\lambda^2\nu^2)^2} + 2\sum_{j,k=1}^{M}\left(\frac{1-\delta_{jk}}{(\lambda-\mu_j)(\lambda-\mu_k)} + \frac{2(1-\delta_{jk})}{(\lambda-\mu_j)(\lambda+\mu_k)}\right.\\
&\left. + \frac{1-\delta_{jk}}{(\lambda+\mu_j)(\lambda+\mu_k)}\right) + 2\sum_{m,n=1}^{N}\left(\frac{s_m s_n + s_m\delta_{mn}}{(\lambda-\alpha_m)(\lambda-\alpha_n)}\right.\\
&\left. + \frac{2(s_m s_n + s_m\delta_{mn})}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{s_m s_n + s_m\delta_{mn}}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right)
\end{aligned}$$

$$
-4\left(\sum_{j=1}^{M}\left(\frac{1}{\lambda-\mu_j}+\frac{1}{\lambda+\mu_j}\right)-\frac{\lambda\nu^2}{\xi^2-\lambda^2\nu^2}\right)
$$
$$
\times\left(\sum_{m=1}^{N}\left(\frac{s_m}{\lambda-\alpha_m}+\frac{s_m}{\lambda+\alpha_m}\right)+\frac{\lambda\nu^2}{\xi^2-\lambda^2\nu^2}\right). \tag{6.16}
$$

As our next important step toward obtaining the formulas of the algebraic Bethe ansatz for the corresponding Gaudin model we observe that the first term in the expansion of the function $F_M(\mu_1;\mu_2,\dots,\mu_M)$ in powers of η is

$$
F_M(\mu_1;\mu_2,\dots,\mu_M)=\eta f_M(\mu_1;\mu_2,\dots,\mu_M)+\mathcal{O}(\eta^2), \tag{6.17}
$$

where

$$
f_M(\mu_1;\mu_2,\dots,\mu_M)=\frac{2\mu_1\nu^2}{\xi+\mu_1\nu}-2(\xi-\mu_1\nu)\sum_{j=2}^{M}\left(\frac{1}{\mu_1-\mu_j}+\frac{1}{\mu_1+\mu_j}\right)
$$
$$
+2(\xi-\mu_1\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu_1-\alpha_m}+\frac{s_m}{\mu_1+\alpha_m}\right). \tag{6.18}
$$

We have used the formulas (5.17) and (5.18) as well as (5.4) and (5.13) in order to expand the Bethe vector $\Psi_1(\mu)$ of the Heisenberg spin chain in powers of η and obtained the Bethe vector $\varphi_1(\mu)$ of the Gaudin model

$$
\Psi_1(\mu)=\eta\varphi_1(\mu)+\mathcal{O}(\eta^2), \tag{6.19}
$$

where

$$
\varphi_1(\mu)=\sum_{m=1}^{N}\left(\frac{\xi+\alpha_m\nu}{\mu-\alpha_m}+\frac{\xi+\alpha_m\nu}{\mu+\alpha_m}\right)\left(\frac{\psi\, s_m}{\nu}+S_m^-\right)\Omega_+. \tag{6.20}
$$

As our final step we observe that using (4.10) and (5.19) we have the off shell action of the difference of the transfer matrix of the XXX chain and the central element, the so-called Sklyanin determinant, on the Bethe vector $\Psi_1(\mu)$

$$
\big(2\lambda t(\lambda)-\Delta\big[\mathcal{T}(\lambda)\big]\big)\Psi_1(\mu)=2\lambda\big(\Lambda_1(\lambda,\mu)-\alpha(\lambda+\eta/2)\widehat{\delta}(\lambda-\eta/2)\big)\Psi_1(\mu)
$$
$$
+(2\lambda)\frac{2\eta(\lambda+\eta)(\xi+\mu\nu)}{(\lambda-\mu)(\lambda+\mu+\eta)}F_1(\mu)\Psi_1(\lambda). \tag{6.21}
$$

Finally, the off shell action of the generating function the Gaudin Hamiltonians on the vector $\varphi_1(\mu)$ can be obtained from the equation above by using the expansion (6.4) and (6.19) on the left hand side as well as the expansion (6.15), (6.17) and (6.19) on the right hand side

$$
\tau(\lambda)\varphi_1(\mu)=\chi_1(\lambda,\mu)\varphi_1(\mu)+\frac{4\lambda(\xi+\mu\nu)}{(\xi^2-\lambda^2\nu^2)(\lambda^2-\mu^2)}f_1(\mu)\varphi_1(\lambda). \tag{6.22}
$$

Therefore $\varphi_1(\mu)$ (6.20) is the Bethe vector of the corresponding Gaudin model, i.e. the eigenvector of the generating function the Gaudin Hamiltonians once the unwanted term is canceled by imposing the corresponding Bethe equation

$$
f_1(\mu)=\frac{2\mu\,\nu^2}{\xi+\mu\nu}+2(\xi-\mu\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu-\alpha_m}+\frac{s_m}{\mu+\alpha_m}\right)=0. \tag{6.23}
$$

To obtain the action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_2(\mu_1,\mu_2)$ of the Gaudin model we follow analogous steps to the ones we have done when studding the action of $\tau(\lambda)$ on $\varphi_1(\mu)$. The first term in the expansion of the Bethe vector $\Psi_2(\mu_1,\mu_2)$ (5.23) in powers of η yields the corresponding Bethe vector of the Gaudin model

$$\Psi_2(\mu_1,\mu_2) = \eta^2\varphi_2(\mu_1,\mu_2) + \mathcal{O}\big(\eta^3\big), \tag{6.24}$$

where

$$\begin{aligned}\varphi_2(\mu_1,\mu_2) = {} & \sum_{m,n=1}^{N}\left(\frac{\xi+\alpha_m\nu}{\mu_1-\alpha_m}+\frac{\xi+\alpha_m\nu}{\mu_1+\alpha_m}\right)\left(\frac{\xi+\alpha_n\nu}{\mu_2-\alpha_n}+\frac{\xi+\alpha_n\nu}{\mu_2+\alpha_n}\right)\\ & \times\left(\left(\frac{\psi\, s_m}{\nu}+S_m^-\right)\left(\frac{\psi\, s_n}{\nu}+S_n^-\right)-\frac{\psi}{\nu}\delta_{mn}\left(\frac{\psi\, s_n}{2\nu}+S_n^-\right)\right)\Omega_+.\end{aligned} \tag{6.25}$$

As in the previous case (6.21), it is of interest to study the action of the difference of the transfer matrix $t(\lambda)$ and the so-called Sklyanin determinant $\Delta[\mathcal{T}(\lambda)]$ on the Bethe vector $\Psi_2(\mu_1,\mu_2)$ using (4.10) and (5.26)

$$\begin{aligned}&\big(2\lambda t(\lambda)-\Delta\big[\mathcal{T}(\lambda)\big]\big)\Psi_2(\mu_1,\mu_2)\\ &\quad = 2\lambda\big(\Lambda_2(\lambda,\mu_1,\mu_2)-\alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\big)\Psi_2(\mu_1,\mu_2)\\ &\qquad + (2\lambda)\frac{2\eta(\lambda+\eta)(\xi+\mu_1\nu)}{(\lambda-\mu_1)(\lambda+\mu_1+\eta)}F_2(\mu_1;\mu_2)\Psi_2(\lambda,\mu_2)\\ &\qquad + (2\lambda)\frac{2\eta(\lambda+\eta)(\xi+\mu_2\nu)}{(\lambda-\mu_2)(\lambda+\mu_2+\eta)}F_2(\mu_2;\mu_1)\Psi_2(\lambda,\mu_1).\end{aligned} \tag{6.26}$$

The off shell action of the generating function of the Gaudin Hamiltonians on the Bethe vector $\varphi_2(\mu_1,\mu_2)$ is obtained from the equation above using the expansions (6.4) and (6.24) on the left hand side and (6.15), (6.24) and (6.17) on the right hand side. Then, by comparing the terms of the fourth power in η on both sides of (6.26) we derive

$$\begin{aligned}\tau(\lambda)\varphi_2(\mu_1,\mu_2) = {} & \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) + \frac{4\lambda(\xi+\mu_1\nu)}{(\xi^2-\lambda^2\nu^2)(\lambda^2-\mu_1^2)}f_2(\mu_1;\mu_2)\varphi_2(\lambda,\mu_2)\\ & + \frac{4\lambda(\xi+\mu_2\nu)}{(\xi^2-\lambda^2\nu^2)(\lambda^2-\mu_2^2)}f_2(\mu_2;\mu_1)\varphi_2(\lambda,\mu_1).\end{aligned} \tag{6.27}$$

The two unwanted terms on the right hand side of the equation above are annihilated by the following Bethe equations

$$\begin{aligned}f_2(\mu_1;\mu_2) = {} & \frac{2\mu_1\nu^2}{\xi+\mu_1\nu} - 2(\xi-\mu_1\nu)\left(\frac{1}{\mu_1-\mu_2}+\frac{1}{\mu_1+\mu_2}\right)\\ & + 2(\xi-\mu_1\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu_1-\alpha_m}+\frac{s_m}{\mu_1+\alpha_m}\right) = 0,\end{aligned} \tag{6.28}$$

$$\begin{aligned}f_2(\mu_2;\mu_1) = {} & \frac{2\mu_2\nu^2}{\xi+\mu_2\nu} - 2(\xi-\mu_2\nu)\left(\frac{1}{\mu_2-\mu_1}+\frac{1}{\mu_2+\mu_1}\right)\\ & + 2(\xi-\mu_2\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu_2-\alpha_m}+\frac{s_m}{\mu_2+\alpha_m}\right) = 0.\end{aligned} \tag{6.29}$$

The off shell action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_2(\mu_1,\mu_2)$ of the Gaudin model is strikingly simple (6.27). Actually, it is as simple as it can be since (6.27) practically coincide with the corresponding formula in the case when the boundary matrix $K(\lambda)$ is diagonal [33].

In general, we have that the first term in the expansion of the Bethe vector $\Psi_M(\mu_1,\mu_2,\dots,\mu_M)$ (5.30), for arbitrary positive integer M, in powers of η is

$$\Psi_M(\mu_1,\mu_2,\dots,\mu_M)=\eta^M\varphi_M(\mu_1,\mu_2,\dots,\mu_M)+\mathcal{O}\big(\eta^{M+1}\big), \tag{6.30}$$

where

$$\varphi_M(\mu_1,\mu_2,\dots,\mu_M)=F(\mu_1)F(\mu_2)\cdots F(\mu_M)\Omega_+ \tag{6.31}$$

and the operator $F(\mu)$ is given by

$$F(\mu)=\sum_{m=1}^{N}\left(\frac{\xi+\mu\nu}{\mu-\alpha_m}+\frac{\xi-\mu\nu}{\mu+\alpha_m}\right)\left(\frac{\psi}{\nu}S_m^3+S_m^--\frac{\psi^2}{4\nu^2}S_m^+\right). \tag{6.32}$$

The Bethe vector of the Gaudin model $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ is a symmetric function of its arguments, since a straightforward calculation shows that the operator $F(\mu)$ commutes at different values of the spectral parameter,

$$\big[F(\lambda),F(\mu)\big]=0. \tag{6.33}$$

The action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ is derived analogously to the previous two cases when $M=1$ (6.22) and $M=2$ (6.27). In the present case we use the expansions (6.15), (6.17) and (6.30) to obtain

$$\begin{aligned}\tau(\lambda)&\varphi_M(\mu_1,\mu_2,\dots,\mu_M)\\ &=\chi_M\big(\lambda,\{\mu_i\}_{i=1}^M\big)\varphi_M(\mu_1,\mu_2,\dots,\mu_M)\\ &\quad+\sum_{i=1}^{M}\frac{4\lambda(\xi+\mu_i\nu)}{(\xi^2-\lambda^2\nu^2)(\lambda^2-\mu_i^2)}f_M\big(\mu_i;\{\mu_j\}_{j\neq i}\big)\varphi_M\big(\lambda,\{\mu_j\}_{j\neq i}\big),\end{aligned} \tag{6.34}$$

where $\chi_M(\lambda,\{\mu_i\}_{i=1}^M)$ is given in (6.16) and the unwanted terms on the right hand side of the equation above are canceled by the following Bethe equations

$$\begin{aligned}f_M\big(\mu_i;\{\mu_j\}_{j\neq i}\big)=\frac{2\mu_i\nu^2}{\xi+\mu_i\nu}-2(\xi-\mu_i\nu)\sum_{\substack{j=1\\ j\neq i}}^{M}\left(\frac{1}{\mu_i-\mu_j}+\frac{1}{\mu_i+\mu_j}\right)&\\ +2(\xi-\mu_i\nu)\sum_{m=1}^{N}\left(\frac{s_m}{\mu_i-\alpha_m}+\frac{s_m}{\mu_i+\alpha_m}\right)=0,&\end{aligned} \tag{6.35}$$

for $i=1,2,\dots,M$. As expected, the above action of the generating function $\tau(\lambda)$ is strikingly simple and this simplicity is due to our definition of the Bethe vector $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ (6.31). These results will be studied further in the framework of an alternative approach to the implementation of the algebraic Bethe ansatz for the Gaudin model, with triangular K-matrix (6.2), based on the classical reflection equation and corresponding linear bracket and will be reported in [42].

7. Conclusions

We have implemented fully the off shell algebraic Bethe ansatz for the XXX Heisenberg spin chain in the case when the boundary parameters satisfy an extra condition guaranteeing that both boundary matrices can be brought to the upper-triangular form by a single similarity matrix which does not depend on the spectral parameter. As it turned out the identity satisfied by the Lax operator enables a convenient realization for the Sklyanin monodromy matrix. This realization led to the action of the entries of the Sklyanin monodromy matrix on the vector Ω_+ and consequently to the observation that Ω_+ is an eigenvector of the transfer matrix of the chain.

We have proceeded then to the essential step of the algebraic Bethe ansatz, to the definition of the Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$. Our objective was to make the off shell action of the transform matrix $t(\lambda)$ on them as simple as possible. Before defining the general Bethe vector $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, we gave a step by step presentation of the first two Bethe vectors, including the formulae for the action of $t(\lambda)$, the corresponding eigenvalues and Bethe equations. In this way we have exposed the striking property of these vectors to make the off shell action of the transform matrix as simple as possible. Consequently, the elaborated definition of $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for arbitrary positive integer M, appeared naturally as a generalization of the first two Bethe vectors. As expected, the action of $t(\lambda)$ on the Bethe vector $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is again very simple. Actually, the action of the transfer matrix is as simple as it could possible be since it almost coincides with the corresponding action in the case when the two boundary matrices are diagonal [6,33].

We explored further these results by obtaining the off shell action of the generating function of the Gaudin Hamiltonians on the corresponding Bethe vectors by means of the so-called quasi-classical limit. To study the open Gaudin model we had to impose the condition so that the parameters of the reflection matrices on the left and on the right end of the chain are the same. This is not the case in the study of the open spin chain, but is essential for the Gaudin model. The generating function of the Gaudin Hamiltonians with boundary terms is derived analogously to the periodic case [42]. Based on this result we showed how the quasi-classical limit yields the off shell action of the generating function of the Gaudin Hamiltonians on the Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ as well as the spectrum and the Bethe equations. The off shell action of the generating function $\tau(\lambda)$ on the Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is strikingly simple. As in the case of the spin chain, it is as simple as it can be since it practically coincide with the corresponding formula in the case when the boundary matrix is diagonal [33]. This simplicity of the action of $\tau(\lambda)$ is due to our definition of the Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$.

An important open problem is to calculate the off shell scalar product of the Bethe vectors we have defined above both for the XXX Heisenberg spin chain and the Gaudin model. These results could lead to the correlations functions for both systems. In the case of Gaudin model it would be of interest to establish a relation between Bethe vectors and solutions of the corresponding Knizhnik–Zamolodchikov equations, along the lines it was done in the case when the boundary matrix is diagonal [33].

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Appendix A. Basic definitions

We consider the spin operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations

$$\left[S^3, S^\pm\right] = \pm S^\pm, \quad \left[S^+, S^-\right] = 2S^3, \tag{A.1}$$

and Casimir operator

$$c_2 = \left(S^3\right)^2 + \frac{1}{2}\left(S^+S^- + S^-S^+\right) = \left(S^3\right)^2 + S^3 + S^-S^+ = \vec{S}\cdot\vec{S}.$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S^\alpha_m = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{A.2}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

Appendix B. Commutation relations

Eq. (4.6) yields the exchange relations between the operators $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$, $\mathcal{C}(\lambda)$ and $\widehat{\mathcal{D}}(\lambda)$. The relevant relations are

$$\mathcal{B}(\lambda)\mathcal{B}(\mu) = \mathcal{B}(\mu)\mathcal{B}(\lambda), \qquad \mathcal{C}(\lambda)\mathcal{C}(\mu) = \mathcal{C}(\mu)\mathcal{C}(\lambda), \tag{B.1}$$

$$\begin{aligned} \mathcal{A}(\lambda)\mathcal{B}(\mu) = {} & \frac{(\lambda+\mu)(\lambda-\mu-\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\mathcal{B}(\mu)\mathcal{A}(\lambda) + \frac{2\eta\mu}{(\lambda-\mu)(2\mu+\eta)}\mathcal{B}(\lambda)\mathcal{A}(\mu) \\ & - \frac{\eta}{\lambda+\mu+\eta}\mathcal{B}(\lambda)\widehat{\mathcal{D}}(\mu), \end{aligned} \tag{B.2}$$

$$\begin{aligned} \widehat{\mathcal{D}}(\lambda)\mathcal{B}(\mu) = {} & \frac{(\lambda-\mu+\eta)(\lambda+\mu+2\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)}\mathcal{B}(\mu)\widehat{\mathcal{D}}(\lambda) - \frac{2\eta(\lambda+\eta)}{(\lambda-\mu)(2\lambda+\eta)}\mathcal{B}(\lambda)\widehat{\mathcal{D}}(\mu) \\ & + \frac{4\eta\mu(\lambda+\eta)}{(2\lambda+\eta)(2\mu+\eta)(\lambda+\mu+\eta)}\mathcal{B}(\lambda)\mathcal{A}(\mu), \end{aligned} \tag{B.3}$$

$$\begin{aligned} \left[\mathcal{C}(\lambda), \mathcal{B}(\mu)\right] = {} & \frac{2\eta\lambda(\lambda-\mu+\eta)}{(\lambda-\mu)(\lambda+\mu+\eta)(2\lambda+\eta)}\mathcal{A}(\mu)\mathcal{A}(\lambda) \\ & - \frac{2\eta^2\lambda}{(\lambda-\mu)(2\lambda+\eta)(2\mu+\eta)}\mathcal{A}(\lambda)\mathcal{A}(\mu) \\ & + \frac{\eta(\lambda+\mu)}{(\lambda-\mu)(\lambda+\mu+\eta)}\mathcal{A}(\mu)\widehat{\mathcal{D}}(\lambda) - \frac{2\eta\lambda}{(\lambda-\mu)(2\lambda+\eta)}\mathcal{A}(\lambda)\widehat{\mathcal{D}}(\mu) \\ & - \frac{\eta^2}{(\lambda+\mu+\eta)(2\mu+\eta)}\widehat{\mathcal{D}}(\lambda)\mathcal{A}(\mu) - \frac{\eta}{\lambda+\mu+\eta}\widehat{\mathcal{D}}(\lambda)\widehat{\mathcal{D}}(\mu). \end{aligned} \tag{B.4}$$

For completeness we include the following commutation relations

$$\left[\mathcal{A}(\lambda), \mathcal{A}(\mu)\right] = \frac{\eta}{\lambda+\mu+\eta}\big(\mathcal{B}(\mu)\mathcal{C}(\lambda) - \mathcal{B}(\lambda)\mathcal{C}(\mu)\big) \tag{B.5}$$

$$\left[\mathcal{A}(\lambda), \widehat{\mathcal{D}}(\mu)\right] = \frac{2\eta(\mu+\eta)}{(\lambda-\mu)(2\mu+\eta)}\big(\mathcal{B}(\lambda)\mathcal{C}(\mu) - \mathcal{B}(\mu)\mathcal{C}(\lambda)\big) \tag{B.6}$$

$$\left[\widehat{\mathcal{D}}(\lambda), \widehat{\mathcal{D}}(\mu)\right] = \frac{4\eta(\lambda+\eta)(\mu+\eta)}{(2\lambda+\eta)(2\mu+\eta)(\lambda+\mu+\eta)}\big(\mathcal{B}(\lambda)\mathcal{C}(\mu) - \mathcal{B}(\mu)\mathcal{C}(\lambda)\big) \tag{B.7}$$

From the relations above it follows that

$$\begin{aligned}
&\mathcal{A}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&\quad= \prod_{i=1}^{2}\frac{(\lambda+\mu_i)(\lambda-\mu_i-\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}\alpha(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&\qquad+ \sum_{i=1}^{2}\frac{2\eta\mu_i}{(2\mu_i+\eta)(\lambda-\mu_i)}\frac{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)}{(\mu_i-\mu_{3-i})(\mu_i+\mu_{3-i}+\eta)}\alpha(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+ \\
&\qquad- \sum_{i=1}^{2}\frac{\eta}{\lambda+\mu_i+\eta}\frac{(\mu_i-\mu_{3-i}+\eta)(\mu_i+\mu_{3-i}+2\eta)}{(\mu_i-\mu_{3-i})(\mu_i+\mu_{3-i}+\eta)}\widehat{\delta}(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+.
\end{aligned} \tag{B.8}$$

Analogously,

$$\begin{aligned}
&\widehat{\mathcal{D}}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&\quad= \prod_{i=1}^{2}\frac{(\lambda-\mu_i+\eta)(\lambda+\mu_i+2\eta)}{(\lambda-\mu_i)(\lambda+\mu_i+\eta)}\widehat{\delta}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&\qquad- \sum_{i=1}^{2}\frac{2\eta(\lambda+\eta)}{(2\lambda+\eta)(\lambda-\mu_i)}\frac{(\mu_i-\mu_{3-i}+\eta)(\mu_i+\mu_{3-i}+2\eta)}{(\mu_i-\mu_{3-i})(\mu_1+\mu_{3-i}+\eta)}\widehat{\delta}(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+ \\
&\qquad+ \sum_{i=1}^{2}\frac{4\eta\mu_i(\lambda+\eta)}{(2\lambda+\eta)(2\mu_i+\eta)(\lambda+\mu_i+\eta)} \\
&\qquad\times \frac{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)}{(\mu_i-\mu_{3-i})(\mu_i+\mu_{3-i}+\eta)}\alpha(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+.
\end{aligned} \tag{B.9}$$

Finally,

$$\begin{aligned}
&\mathcal{C}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&\quad= \sum_{i=1}^{2}\Bigg(\frac{4\mu_i\lambda\eta}{(2\lambda+\eta)(2\mu_i+\eta)(\lambda+\mu_i+\eta)} \\
&\qquad\times \frac{(\lambda+\mu_{3-i})(\lambda-\mu_{3-i}-\eta)}{(\lambda-\mu_{3-i})(\lambda+\mu_{3-i}+\eta)}\frac{(\mu_i+\mu_{3-i})(\mu_i-\mu_{3-i}-\eta)}{(\mu_i-\mu_2)(\mu_i+\mu_{3-i}+\eta)}\alpha(\lambda)\alpha(\mu_i) \\
&\qquad- \frac{2\lambda\eta}{(\lambda-\mu_i)(2\lambda+\eta)} \\
&\qquad\times \frac{(\lambda+\mu_2)(\lambda-\mu_2-\eta)}{(\lambda-\mu_2)(\lambda+\mu_2+\eta)}\frac{(\mu_i-\mu_2+\eta)(\mu_i+\mu_2+2\eta)}{(\mu_i-\mu_2)(\mu_i+\mu_2+\eta)}\alpha(\lambda)\widehat{\delta}(\mu_i)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\mu_i\eta}{(\lambda-\mu_i)(2\mu_i+\eta)}\\
&\times\frac{(\lambda-\mu_2+\eta)(\lambda+\mu_2+2\eta)}{(\lambda-\mu_2)(\lambda+\mu_2+\eta)}\frac{(\mu_i+\mu_2)(\mu_i-\mu_2-\eta)}{(\mu_i-\mu_2)(\mu_i+\mu_2+\eta)}\alpha(\mu_i)\widehat{\delta}(\lambda)-\frac{\eta}{\lambda+\mu_i+\eta}\\
&\times\frac{(\lambda-\mu_2+\eta)(\lambda+\mu_2+2\eta)}{(\lambda-\mu_2)(\lambda+\mu_2+\eta)}\frac{(\mu_i-\mu_2+\eta)(\mu_i+\mu_2+2\eta)}{(\mu_i-\mu_2)(\mu_i+\mu_2+\eta)}\widehat{\delta}(\lambda)\widehat{\delta}(\mu_1)\Bigg)\\
&\times\mathcal{B}(\mu_{3-i})\Omega_+\\
&+\Bigg(\frac{8\eta^2\mu_1\mu_2(\mu_1+\mu_2)(\lambda(\lambda+\eta)-\mu_1\mu_2)}{(\lambda-\mu_1)(\lambda-\mu_2)(2\mu_1+\eta)(2\mu_2+\eta)(\lambda+\mu_1+\eta)(\lambda+\mu_2+\eta)(\mu_1+\mu_2+\eta)}\\
&\times\alpha(\mu_1)\alpha(\mu_2)\\
&-\frac{4\eta^2\mu_1(\mu_2-\mu_1+\eta)(\lambda(\lambda+\eta)+\mu_1(\mu_2+\eta))}{(\lambda-\mu_1)(\lambda-\mu_2)(2\mu_1+\eta)(\mu_2-\mu_1)(\lambda+\mu_1+\eta)(\lambda+\mu_2+\eta)}\alpha(\mu_1)\widehat{\delta}(\mu_2)\\
&-\frac{4\eta^2\mu_2(\mu_1-\mu_2+\eta)(\lambda(\lambda+\eta)+\mu_2(\mu_1+\eta))}{(\lambda-\mu_1)(\lambda-\mu_2)(2\mu_2+\eta)(\mu_1-\mu_2)(\lambda+\mu_1+\eta)(\lambda+\mu_2+\eta)}\alpha(\mu_2)\widehat{\delta}(\mu_1)\\
&-\frac{2\eta^2(\mu_1+\mu_2+2\eta)(\eta^2-\lambda^2+\mu_1\mu_2+\eta(\mu_1+\mu_2-\lambda))}{(\lambda-\mu_1)(\lambda-\mu_2)(\lambda+\mu_1+\eta)(\lambda+\mu_2+\eta)(\mu_1+\mu_2+\eta)}\widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)\Bigg)\mathcal{B}(\lambda)\Omega_+
\end{aligned}
\tag{B.10}
$$

The relations (B.8), (B.9) and (B.10) are readily generalized [10].

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Algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with boundary

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Abstract

Following Sklyanin's proposal in the periodic case, we derive the generating function of the Gaudin Hamiltonians with boundary terms. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function. By defining the appropriate Bethe vectors which yield strikingly simple off shell action of the generating function, we fully implement the algebraic Bethe ansatz, obtaining the spectrum of the generating function and the corresponding Bethe equations.

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1. Introduction

A model of interacting spins in a chain was first considered by Gaudin [1,2]. Gaudin derived these models as a quasi-classical limit of the quantum chains. Sklyanin studied the rational $s\ell(2)$ model in the framework of the quantum inverse scattering method using the $s\ell(2)$ invariant classical r-matrix [3]. A generalisation of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang–Baxter equation [4] was relatively straightforward [5,6]. Therefore, considerable attention has been devoted to Gaudin models corresponding to the classical r-matrices of simple Lie algebras [7–12] and Lie superalgebras [13–17].

Hikami, Kulish and Wadati showed that the quasi-classical expansion of the transfer matrix of the periodic chain, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians [18,19]. Hikami showed how the quasi-classical expansion of the transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the case of non-periodic boundary conditions [20]. Then the ABA was applied to open Gaudin model in the context of the Vertex-IRF correspondence [21–23]. Also, results were obtained for the open Gaudin models based on Lie superalgebras [24]. An approach to study the open Gaudin models based on the classical reflection equation [25–27] and the non-unitary r-matrices was developed recently, see [28,29] and the references therein. For a review of the open Gaudin model see [30]. Progress in applying Bethe ansatz to the Heisenberg spin chain with non-periodic boundary conditions compatible with the integrability of the quantum systems [31–41] had recent impact on the study of the corresponding Gaudin model [41,42]. The so-called $T-Q$ approach to implementation of Bethe ansatz [35,36] was used to obtain the eigenvalues of the associated Gaudin Hamiltonians and the corresponding Bethe ansatz equations [42]. In [41] the off shell action of the generating function of the Gaudin Hamiltonians on the Bethe vectors was obtained through the so-called quasi-classical limit.

Here we derive the generating function of the Gaudin Hamiltonians with boundary terms following Sklyanin's approach in the periodic case [3]. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The essential step in this derivation is the expansion of the monodromy matrix in powers of the quasi-classical parameter. Moreover, we show how the representation of the relevant Lax matrix in terms of local spin operators yields the partial fraction decomposition of the generating function. Consequently, the Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function at poles. We derive the relevant linear bracket for the Gaudin Lax operator and certain classical r-matrix, obtained form the $s\ell(2)$ invariant classical r-matrix and the corresponding K-matrix. The local realisation of the Lax matrix together with the linear bracket provide the necessary structure for the implementation of the algebraic Bethe ansatz. In this framework, the Bethe vectors, defined as the symmetric functions of its arguments, have a remarkable property that the off shell action of the generating function on them is strikingly simple. Actually, it is as simple as it can be since it practically coincide with the corresponding formula in the case when the boundary matrix is diagonal [20]. The off shell action of the generating function of the Gaudin Hamiltonians with the boundary terms yields the spectrum of the system and the corresponding Bethe equations. As usual, when the Bethe equations are imposed on the parameters of the Bethe vectors, the unwanted terms in the action of the generating function are annihilated.

However, more compact form of the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, requires further studies. As it is evident form the formulas for the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ given in Appendix B, the problem lies in the definition the scalar coefficients

$c_M^{(m)}(\mu_1, \dots \mu_m; \mu_{m+1}, \dots, \mu_M)$, with $m = 1, 2, \dots, M$. Some of them are straightforward to obtain but, in particular, the coefficient $c_M^{(M)}(\mu_1, \mu_2 \dots, \mu_M)$ still represents a challenge, at least in the present form.

This paper is organised as follows. In Section 2 we review the $SL(2)$-invariant Yang R-matrix and provide fundamental tools for the study of the inhomogeneous XXX Heisenberg spin chain and the corresponding Gaudin model. Moreover, we outline Sklyanin's derivation of the rational $s\ell(2)$ Gaudin model. The general solutions of the reflection equation and the dual reflection equation are given in Section 3. As one of the main results of the paper, the generating function of the Gaudin Hamiltonians with boundary terms is derived in Section 4, using the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXX spin chain and the so-called Sklyanin determinant. The relevant algebraic structure, including the classical reflection equation, is given in Section 5. The implementation of the algebraic Bethe ansatz is presented in Section 6, including the definition of the Bethe vectors and the formulae of the off shell action of the generating function of the Gaudin Hamiltonians. Our conclusions are presented in Section 7. Finally, in Appendix A are given some basic definitions for the convenience of the reader.

2. $s\ell(2)$ Gaudin model

The XXX Heisenberg spin chain is related to the $SL(2)$-invariant Yang R-matrix [43]

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{2.1}$$

where λ is a spectral parameter, η is a quasi-classical parameter, $\mathbb{1}$ is the identity operator and we use $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The Yang R-matrix satisfies the Yang–Baxter equation [43–46] in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu), \tag{2.2}$$

we use the standard notation of the quantum inverse scattering method to denote spaces on which corresponding R-matrices R_{ij}, $ij = 12, 13, 23$ act nontrivially and suppress the dependence on the quasi-classical parameter η [45,46].

The Yang R-matrix also satisfies other relevant properties such as

unitarity	$R_{12}(\lambda) R_{21}(-\lambda) = (\eta^2 - \lambda^2)\mathbb{1}$;
parity invariance	$R_{21}(\lambda) = R_{12}(\lambda)$;
temporal invariance	$R_{12}^t(\lambda) = R_{12}(\lambda)$;
crossing symmetry	$R(\lambda) = \mathcal{J}_1 R^{t_2}(-\lambda - \eta) \mathcal{J}_1$,

where t_2 denotes the transpose in the second space and the entries of the two-by-two matrix $\mathcal{J}$ are $\mathcal{J}_{ab} = (-1)^{a-1} \delta_{a,3-b}$.

Here we study the inhomogeneous XXX spin chain with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. For simplicity, we start by considering the periodic boundary conditions. The Hilbert space of the system is

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{2.3}$$

Following [3] we introduce the Lax operator [41]

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{\eta}{\lambda}\left(\vec{\sigma}_0 \cdot \vec{S}_m\right) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S_m^3 & \eta S_m^- \\ \eta S_m^+ & \lambda - \eta S_m^3 \end{pmatrix}. \tag{2.4}$$

Notice that $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$. It obeys

$$\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0m}(\eta - \lambda) = \left(1 + \frac{\eta^2 c_{2,m}}{\lambda(\eta - \lambda)}\right)\mathbb{1}_0, \tag{2.5}$$

where $c_{2,m}$ is the value of the Casimir operator on the space V_m [41].

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the R-matrix, $\mathbb{L}_{0m}(\lambda) = \frac{1}{\lambda} R_{0m}(\lambda - \eta/2)$.

Due to the commutation relations (A.1), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda - \mu)\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0'm}(\mu) = \mathbb{L}_{0'm}(\mu)\mathbb{L}_{0m}(\lambda)R_{00'}(\lambda - \mu). \tag{2.6}$$

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N)\cdots\mathbb{L}_{01}(\lambda - \alpha_1) \tag{2.7}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter η and the inhomogeneous parameters $\{\alpha_j, j = 1, \ldots, N\}$. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$. From RLL-relations (2.6) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda - \mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda - \mu). \tag{2.8}$$

The periodic boundary conditions and the RTT-relations (2.8) imply that the transfer matrix

$$t(\lambda) = \mathrm{tr}_0 T(\lambda), \tag{2.9}$$

commute at different values of the spectral parameter,

$$[t(\mu), t(\nu)] = 0, \tag{2.10}$$

here we have omitted the nonessential arguments.

The RTT-relations admit a central element

$$\Delta\left[T(\lambda)\right] = \mathrm{tr}_{00'} P^-_{00'} T_0\left(\lambda - \eta/2\right) T_{0'}\left(\lambda + \eta/2\right), \tag{2.11}$$

where

$$P^-_{00'} = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2} = -\frac{1}{2\eta} R_{00'}(-\eta). \tag{2.12}$$

A straightforward calculation shows that

$$\left[\Delta\left[T(\mu)\right], T(\nu)\right] = 0. \tag{2.13}$$

As the first step toward the study of the Gaudin model we consider the expansion of the monodromy matrix (2.7) with respect to the quasi-classical parameter η

$$T(\lambda) = \mathbb{1} + \eta \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{\eta^2}{2} \sum_{\substack{n,m=1 \\ n \neq m}}^{N} \frac{\mathbb{1}_0 \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)}$$
$$+ \frac{\eta^2}{2} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right) + \mathcal{O}(\eta^3). \tag{2.14}$$

If the Gaudin Lax matrix is defined by [3]

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} \tag{2.15}$$

and the quasi-classical property of the Yang R-matrix [3]

$$\frac{1}{\lambda} R(\lambda) = \mathbb{1} - \eta r(\lambda), \qquad \text{where} \qquad r(\lambda) = -\frac{\mathcal{P}}{\lambda} \tag{2.16}$$

is taken into account, then substitution of the expansion (2.14) into the RTT-relations (2.8) yields the so-called Sklyanin linear bracket [3]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]. \tag{2.17}$$

Using the expansion (2.14) it is evident that

$$t(\lambda) = 2 + \eta^2 \sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \mathcal{O}(\eta^3). \tag{2.18}$$

The same expansion (2.14) leads to

$$\Delta[T(\lambda)] = \mathbb{1} + \eta \, \mathrm{tr} L(\lambda) + \frac{\eta^2}{2} \mathrm{tr}_{00'} P_{00'}^{-} \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} - \frac{\vec{\sigma}_{0'} \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P_{00'}^{-} \sum_{m=1}^{N} \sum_{n \neq m}^{N} \left(\frac{\mathbb{1}_0 \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \frac{\mathbb{1}_{0'} \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P_{00'}^{-} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P_{00'}^{-} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_{0'} \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_{0'} \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \eta^2 \mathrm{tr}_{00'} P_{00'}^{-} L_0(\lambda) L_{0'}(\lambda) + \mathcal{O}(\eta^3), \tag{2.19}$$

where $L(\lambda)$ is given in (2.15). The final expression for the expansion of $\Delta[T(\lambda)]$ is obtained after taking all the traces

$$\Delta[T(\lambda)] = \mathbb{1} + \eta^2 \left(\sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} - \frac{1}{2} \mathrm{tr} \, L^2(\lambda) \right) + \mathcal{O}(\eta^3). \tag{2.20}$$

To obtain the generation function of the Gaudin Hamiltonians notice that (2.18) and (2.20) yield

$$t(\lambda) - \Delta[T(\lambda)] = \mathbb{1} + \frac{\eta^2}{2}\operatorname{tr} L^2(\lambda) + \mathcal{O}(\eta^3). \tag{2.21}$$

Therefore

$$\tau(\lambda) = \frac{1}{2}\operatorname{tr} L^2(\lambda) \tag{2.22}$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{2.23}$$

Moreover, from (2.15) it is straightforward to obtain the expansion

$$\tau(\lambda) = \sum_{m=1}^{N} \frac{2H_m}{\lambda - \alpha_m} + \sum_{m=1}^{N} \frac{\vec{S}_m \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} = \sum_{m=1}^{N} \frac{2H_m}{\lambda - \alpha_m} + \sum_{m=1}^{N} \frac{s_m(s_m + 1)}{(\lambda - \alpha_m)^2}, \tag{2.24}$$

and the Gaudin Hamiltonians, in the periodic case, are

$$H_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n}. \tag{2.25}$$

This shows that $\tau(\lambda)$ is the generating function of Gaudin Hamiltonians when the periodic boundary conditions are imposed [3].

3. Reflection equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk model, was developed in [27]. Boundary conditions on the left and right sites of the system are encoded in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $\mathbb{C}^2$ at the first site $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda - \mu) K_1^-(\lambda) R_{21}(\lambda + \mu) K_2^-(\mu) = K_2^-(\mu) R_{12}(\lambda + \mu) K_1^-(\lambda) R_{21}(\lambda - \mu). \tag{3.1}$$

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$\begin{aligned} &R_{12}(\mu - \lambda) K_1^+(\lambda) R_{21}(-\lambda - \mu - 2\eta) K_2^+(\mu) \\ &\quad = K_2^+(\mu) R_{12}(-\lambda - \mu - 2\eta) K_1^+(\lambda) R_{21}(\mu - \lambda). \end{aligned} \tag{3.2}$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda - \eta) \tag{3.3}$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of (3.3) into the dual reflection equation (3.2) one gets the reflection equation (3.1) with shifted arguments.

The general, spectral parameter dependent solutions of the reflection equation (3.1) can be written as follows [47]

$$K^{-}(\lambda) = \begin{pmatrix} \xi - \lambda & \psi\lambda \\ \phi\lambda & \xi + \lambda \end{pmatrix}. \tag{3.4}$$

It is straightforward to check the following useful identities

$$K^{-}(-\lambda)K^{-}(\lambda) = \left(\xi^2 - \lambda^2\,(1 + \phi\psi)\right)\mathbb{1} = \det\left(K^{-}(\lambda)\right)\mathbb{1}, \tag{3.5}$$

$$K^{-}(-\lambda) = \operatorname{tr} K^{-}(\lambda) - K^{-}(\lambda). \tag{3.6}$$

4. $s\ell(2)$ Gaudin model with boundary terms

With the aim of describing the inhomogeneous XXX spin chain with non-periodic boundary condition it is instructive to recall some properties of the Lax operator (2.4). The identity (2.5) can be rewritten in the form [41]

$$\mathbb{L}_{0m}(\lambda - \alpha_m)\mathbb{L}_{0m}(-\lambda + \alpha_m + \eta) = \left(1 + \frac{\eta^2\, s_m(s_m+1)}{(\lambda - \alpha_m)(-\lambda + \alpha_m + \eta)}\right)\mathbb{1}_0. \tag{4.1}$$

It follows from the equation above and the RLL-relations (2.6) that the RTT-relations (2.8) can be recast as follows

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda+\mu)T_0(\lambda) = T_0(\lambda)R_{00'}(\lambda+\mu)\widetilde{T}_{0'}(\mu), \tag{4.2}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu-\lambda) = R_{00'}(\mu-\lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda), \tag{4.3}$$

where

$$\widetilde{T}(\lambda) = \mathbb{L}_{01}(\lambda + \alpha_1 + \eta)\cdots\mathbb{L}_{0N}(\lambda + \alpha_N + \eta). \tag{4.4}$$

The Sklyanin monodromy matrix $\mathcal{T}(\lambda)$ of the inhomogeneous XXX spin chain with non-periodic boundary consists of the two matrices $T(\lambda)$ (2.7) and $\widetilde{T}_0(\lambda)$ (4.4) and a reflection matrix $K^{-}(\lambda)$ (3.4),

$$\mathcal{T}_0(\lambda) = T_0(\lambda)K_0^{-}(\lambda)\widetilde{T}_0(\lambda). \tag{4.5}$$

Using the RTT-relations (2.8), (4.2), (4.3) and the reflection equation (3.1) it is straightforward to show that the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ in $V_0 \otimes V_{0'}$ are [41]

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu). \tag{4.6}$$

The open chain transfer matrix is given by the trace of $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^{+}(\lambda)$ [27],

$$t(\lambda) = \operatorname{tr}_0\left(K^{+}(\lambda)\mathcal{T}(\lambda)\right). \tag{4.7}$$

The reflection matrix $K^{+}(\lambda)$ (3.3) is the corresponding solution of the dual reflection equation (3.2). The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0, \tag{4.8}$$

is guaranteed by the dual reflection equation (3.2) and the exchange relations (4.6) of the monodromy matrix $\mathcal{T}(\lambda)$.

The exchange relations (4.6) admit a central element

$$\Delta[\mathcal{T}(\lambda)] = \mathrm{tr}_{00'} P^{-}_{00'} \mathcal{T}_0(\lambda-\eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda+\eta/2). \tag{4.9}$$

For the study of the open Gaudin model we impose

$$\lim_{\eta\to 0}\Big(K^{+}(\lambda)K^{-}(\lambda)\Big) = \Big(\xi^2 - \lambda^2(1+\phi\psi)\Big)\mathbb{1}. \tag{4.10}$$

In particular, this implies that the parameters of the reflection matrices on the left and on the right end of the chain are the same. In general this not the case in the study of the open spin chain. However, this condition is essential for the Gaudin model. Then we will write

$$K^{-}(\lambda) \equiv K(\lambda), \tag{4.11}$$

so that

$$K^{+}(\lambda) = K(-\lambda-\eta) = K(-\lambda) - \eta M \quad \text{with} \quad M = \begin{pmatrix} -1 & \psi \\ \phi & 1 \end{pmatrix}. \tag{4.12}$$

Remark that the matrix M obeys $M^2 = (1+\psi\phi)\mathbb{1}$.

The expansion of $T(\lambda)$ is given in (2.14). It is easy to get the expansion for $\widetilde{T}(\lambda)$ as introduced in (4.4) and then, the one for $\mathcal{T}(\lambda)$. Using the relation (4.12), we deduce the expansion of $t(\lambda)$ (6.24) in powers of η:

$$\begin{aligned} t(\lambda) = {} & 2\Big(\xi^2 - \lambda^2(1+\phi\psi)\Big) - 2\eta\lambda(1+\phi\psi) \\ & - \eta^2\,\mathrm{tr}_0\left(M_0\left(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\right)\right) \\ & + \eta^2\Big(\xi^2 - \lambda^2(1+\phi\psi)\Big)\sum_{\substack{m,n=1\\ n\neq m}}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right) \\ & - \eta^2\mathrm{tr}_0\, L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda) + \mathcal{O}(\eta^3). \end{aligned} \tag{4.13}$$

Our next step is to obtain the expansion of $\Delta[\mathcal{T}(\lambda)]$ (4.9) in powers of η. We follow the analogous steps as for the periodic case, and after some tedious but straightforward calculations we get

$$\begin{aligned} \Delta[\mathcal{T}(\lambda)] = {} & \lambda\Big(\mathrm{tr}_0^2 K_0(\lambda) - \mathrm{tr}_0 K_0^2(\lambda)\Big) + 2\eta\lambda\mathrm{tr}_0 K_0(\lambda)\,\mathrm{tr}_0\left(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\right) \\ & - 2\eta\lambda\Big(\mathrm{tr}_0\Big\{L_0(\lambda)K_0^2(\lambda)\Big\} - \mathrm{tr}_0\Big\{L_0(-\lambda)K_0^2(\lambda)\Big\}\Big) - \frac{\eta}{2}\mathrm{tr}_0\, K_0^2(\lambda) \\ & + \eta^2\lambda\sum_{\substack{m,n=1\\ n\neq m}}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right)\mathrm{tr}_0\, K_0(-\lambda)K_0(\lambda) \\ & - 2\eta^2\lambda\,\mathrm{tr}_0\, L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda) \\ & - \eta^2\,\mathrm{tr}_0\Big\{\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)K_0(\lambda)\Big\} \\ & + \eta^2\lambda\Big(\mathrm{tr}_0\left\{L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\right\}\Big)^2 \\ & - \eta^2\lambda\,\mathrm{tr}_0\Big\{\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)\Big\} \\ & + \frac{\eta^2\lambda}{4}\mathrm{tr}_0\, M_0^2 + \mathcal{O}(\eta^3). \end{aligned} \tag{4.14}$$

Using the relations (3.5) and (3.6) the first term of the expansion above simplifies and the second and third term together turn out to be propositional to the trace of $L(\lambda)$ (2.15) and therefore vanish,

$$\begin{aligned}\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)-\eta\left(\xi^2+\lambda^2\left(1+\phi\psi\right)\right)+2\eta^2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\\ & \times\sum_{\substack{m,n=1\\ n\neq m}}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)}+\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right)\\ & -2\eta^2\lambda\,\mathrm{tr}_0\,L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda)\\ & -\eta^2\,\mathrm{tr}_0((L_0(\lambda)K_0(\lambda)-K_0(\lambda)L_0(-\lambda))K_0(\lambda))\\ & +\eta^2\lambda\,\mathrm{tr}_0\Big\{\Big(\mathrm{tr}_{0'}\{L_{0'}(\lambda)K_{0'}(\lambda)-K_{0'}(\lambda)L_{0'}(-\lambda)\}-L_0(\lambda)K_0(\lambda)\\ & +K_0(\lambda)L_0(-\lambda)\Big)\Big(L_0(\lambda)K_0(\lambda)-K_0(\lambda)L_0(-\lambda)\Big)\Big\}+\frac{\eta^2\lambda}{2}(1+\phi\psi)\\ & +\mathcal{O}\big(\eta^3\big).\end{aligned}\tag{4.15}$$

In order to simplify some formulae we introduce the following notation

$$\mathcal{L}_0(\lambda)=L_0(\lambda)-K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda).\tag{4.16}$$

Using the formulas (4.13) and (4.15) we calculate the expansion in powers of η of the difference

$$\begin{aligned}2\lambda t(\lambda)-\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)+\eta\left(\xi^2-3\lambda^2\left(1+\phi\psi\right)\right)\\ & -2\eta^2\lambda\,\mathrm{tr}_0\left(M_0\mathcal{L}_0(\lambda)K_0(\lambda)\right)+\eta^2\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0^2(\lambda)\right)\\ & -\eta^2\lambda\,\mathrm{tr}_0\left(\left(\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)\right)\mathcal{L}_0(\lambda)K_0(\lambda)\right)\\ & -\frac{\eta^2\lambda}{2}\left(1+\phi\psi\right)+\mathcal{O}(\eta^3).\end{aligned}\tag{4.17}$$

Actually the third and the fourth term in the expression above vanish

$$\begin{aligned}&\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0^2(\lambda)\right)-2\lambda\,\mathrm{tr}_0\left(M_0\mathcal{L}_0(\lambda)K_0(\lambda)\right)=\mathrm{tr}_0\left(\left(\mathcal{L}_0(\lambda)K_0(\lambda)\right)\left(K_0(\lambda)-2\lambda M_0\right)\right)\\ &\quad=\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0(\lambda)K_0(-\lambda)\right)=\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\mathrm{tr}_0\mathcal{L}_0=0,\end{aligned}\tag{4.18}$$

due to the fact that the $\mathrm{tr}_0\mathcal{L}_0$ is equal to zero. Therefore the expansion (4.17) reads

$$\begin{aligned}2\lambda t(\lambda)-\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)+\eta\left(\xi^2-3\lambda^2\left(1+\phi\psi\right)\right)\\ & -\eta^2\lambda\,\mathrm{tr}_0\left(\left(\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)\right)\mathcal{L}_0(\lambda)K_0(\lambda)\right)\\ & -\frac{\eta^2\lambda}{2}\left(1+\phi\psi\right)+\mathcal{O}(\eta^3).\end{aligned}\tag{4.19}$$

It is important to notice that using the following identity

$$\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)=-K_0(-\lambda)\mathcal{L}_0(\lambda),\tag{4.20}$$

the third term in (4.19) can be simplified

$$\mathrm{tr}_0\,K_0(-\lambda)\mathcal{L}_0(\lambda)\mathcal{L}_0(\lambda)K_0(\lambda)=\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\mathrm{tr}_0\,\mathcal{L}_0^2(\lambda).\tag{4.21}$$

Finally, the expansion (4.19) reads

$$2\lambda t(\lambda) - \Delta[\mathcal{T}(\lambda)] = 2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right) + \eta\left(\xi^2 - 3\lambda^2(1+\phi\psi)\right) + \eta^2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right)\mathrm{tr}_0\,\mathcal{L}_0^2(\lambda) - \frac{\eta^2\lambda}{2}(1+\phi\psi) + \mathcal{O}(\eta^3). \tag{4.22}$$

This shows that

$$\tau(\lambda) = \mathrm{tr}_0\,\mathcal{L}_0^2(\lambda) \tag{4.23}$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{4.24}$$

and therefore can be considered to be the generating function of Gaudin Hamiltonians with boundary terms. The multiplicative factor in (6.31), which is equal to the determinant of $K(\lambda)$, will be useful in the partial fraction decomposition of the generating function.

With the aim of obtaining the Gaudin Hamiltonians with the boundary terms from the generating function (6.31), it is instructive to study the representation of $\mathcal{L}_0(\lambda)$ (4.16) in terms of the local spin operators

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N}\left(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m} + \frac{\left(K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda)\right)\cdot\vec{S}_m}{\lambda+\alpha_m}\right), \tag{4.25}$$

noticing that

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N}\left(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m} + \frac{\vec{\sigma}_0\cdot\left(K_m^{-1}(\lambda)\vec{S}_m K_m(\lambda)\right)}{\lambda+\alpha_m}\right). \tag{4.26}$$

Now it is straightforward to obtain the expression for the generating function (6.31) in terms of the local operators

$$\tau(\lambda) = 2\sum_{m,n=1}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right) + \left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)\cdot\vec{S}_m}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{\left(K_m^{-1}(\lambda)\vec{S}_m K_m(\lambda)\right)\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right). \tag{4.27}$$

It is important to notice that (4.27) simplifies further

$$\tau(\lambda) = 2\sum_{m,n=1}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)} + \frac{\vec{S}_m\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right) + \left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)\cdot\vec{S}_m}{(\lambda-\alpha_m)(\lambda+\alpha_n)}\right). \tag{4.28}$$

The Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function (4.28) at poles $\lambda = \pm\alpha_m$:

$$\mathrm{Res}_{\lambda=\alpha_m} \tau(\lambda) = 4\, H_m \quad \text{and} \quad \mathrm{Res}_{\lambda=-\alpha_m} \tau(\lambda) = (-4)\, \widetilde{H}_m \tag{4.29}$$

where

$$H_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\vec{S}_m \cdot \Big(K_n^{-1}(\alpha_m)\vec{S}_n K_n(\alpha_m)\Big) + \Big(K_n^{-1}(\alpha_m)\vec{S}_n K_n(\alpha_m)\Big) \cdot \vec{S}_m}{2(\alpha_m + \alpha_n)}, \tag{4.30}$$

and

$$\begin{aligned}\widetilde{H}_m = & \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} \\ & + \sum_{n=1}^{N} \frac{\vec{S}_m \cdot \Big(K_n^{-1}(-\alpha_m)\vec{S}_n K_n(-\alpha_m)\Big) + \Big(K_n^{-1}(-\alpha_m)\vec{S}_n K_n(-\alpha_m)\Big) \cdot \vec{S}_m}{2(\alpha_m + \alpha_n)}.\end{aligned} \tag{4.31}$$

The above Hamiltonians can be expressed in somewhat a more symmetric form

$$H_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\Big(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\Big) \cdot \vec{S}_n + \vec{S}_n \cdot \Big(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\Big)}{2(\alpha_m + \alpha_n)}, \tag{4.32}$$

and

$$\begin{aligned}\widetilde{H}_m = & \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} \\ & + \sum_{n=1}^{N} \frac{\Big(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\Big) \cdot \vec{S}_n + \vec{S}_n \cdot \Big(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\Big)}{2(\alpha_m + \alpha_n)}.\end{aligned} \tag{4.33}$$

The next step is to study the quasi-classical limit of the exchange relations (4.6) with the aim of deriving relevant algebraic structure for the Lax operator (4.16).

5. Linear bracket relations

The implementation of the algebraic Bethe ansatz requires the commutation relations between the entries of the Lax operator (4.16). Our aim is to derive these relations as the quasi-classical limit of (4.6). As the first step in this direction we observe that using (2.16) the reflection equation (3.1) can be expressed as

$$\begin{aligned}&\big(\mathbb{1} - \eta r_{12}(\lambda - \mu)\big) K_1(\lambda) \big(\mathbb{1} - \eta r_{21}(\lambda + \mu)\big) K_2(\mu) \\ &\quad = K_2(\mu)\big(\mathbb{1} - \eta r_{12}(\lambda + \mu)\big) K_1(\lambda)\big(\mathbb{1} - \eta r_{21}(\lambda - \mu)\big).\end{aligned} \tag{5.1}$$

The conditions obtained from the zero and first order in η are identically satisfied for the matrix $K(\lambda)$. In particular, it obeys the classical reflection equation [25,26]:

$$
\begin{aligned}
&r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu)+K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)\\
&\quad=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)+K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu).
\end{aligned}
\tag{5.2}
$$

The terms of the second order in η in (5.1) are

$$
r_{12}(\lambda-\mu)K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)r_{21}(\lambda-\mu).
\tag{5.3}
$$

This equation is also satisfied by the K-matrix (3.4) and the classical r-matrix (2.16). In addition, the classical r-matrix (2.16) has the unitarity property

$$
r_{21}(-\lambda)=-r_{12}(\lambda),
\tag{5.4}
$$

and satisfies the classical Yang–Baxter equation

$$
[r_{13}(\lambda),r_{23}(\mu)]+[r_{12}(\lambda-\mu),r_{13}(\lambda)+r_{23}(\mu)]=0.
\tag{5.5}
$$

Now we can proceed to the derivation of the relevant linear bracket relations of the Lax operator (4.16). The desired relations can be obtained by writing Eq. (4.6) in the following form, using (2.16),

$$
\begin{aligned}
&(\mathbb{1}-\eta r_{00'}(\lambda-\mu))\,\mathcal{T}_0(\lambda)\,(\mathbb{1}-\eta r_{0'0}(\lambda+\mu))\,\mathcal{T}_{0'}(\mu)\\
&\quad=\mathcal{T}_{0'}(\mu)\,(\mathbb{1}-\eta r_{00'}(\lambda+\mu))\,\mathcal{T}_0(\lambda)\,(\mathbb{1}-\eta r_{0'0}(\lambda-\mu))
\end{aligned}
\tag{5.6}
$$

and substituting the expansion of $\mathcal{T}(\lambda)$ (4.5) in powers of η

$$
\mathcal{T}(\lambda)=K(\lambda)+\eta\,\mathcal{L}(\lambda)K(\lambda)+\frac{\eta^2}{2}\frac{d^2\mathcal{T}(\lambda)}{d\eta^2}|_{\eta=0}+\mathcal{O}(\eta^3).
\tag{5.7}
$$

The zero and first orders in η are identically satisfied for the matrix $K(\lambda)$ defined in (3.4). The relations we seek follow from the terms of the second order in η in (5.6). When the terms containing the second order derivatives of $\mathcal{T}$ are eliminated and Eq. (5.3) is used to eliminate the other two terms, there are ten terms remaining. Using twice the classical reflection equation (5.2) and the unitarity property (5.4) one obtains

$$
\begin{aligned}
&(\mathcal{L}_0(\lambda)\mathcal{L}_{0'}(\mu)-\mathcal{L}_{0'}(\mu)\mathcal{L}_0(\lambda))\,K_0(\lambda)K_{0'}(\mu)\\
&\quad=(r_{00'}(\lambda-\mu)\mathcal{L}_0(\lambda)-\mathcal{L}_0(\lambda)r_{00'}(\lambda-\mu))\,K_0(\lambda)K_{0'}(\mu)+(\mathcal{L}_0(\lambda)K_{0'}(\mu)r_{00'}(\lambda+\mu)\\
&\qquad-K_{0'}(\mu)r_{00'}(\lambda+\mu)\mathcal{L}_0(\lambda))\,K_0(\lambda)-(r_{0'0}(\mu-\lambda)\mathcal{L}_{0'}(\mu)\\
&\qquad-\mathcal{L}_{0'}(\mu)r_{0'0}(\mu-\lambda))\,K_0(\lambda)K_{0'}(\mu)+(K_0(\lambda)r_{0'0}(\mu+\lambda)\mathcal{L}_{0'}(\mu)\\
&\qquad-\mathcal{L}_{0'}(\mu)K_0(\lambda)r_{0'0}(\mu+\lambda))\,K_{0'}(\mu).
\end{aligned}
\tag{5.8}
$$

Multiplying both sides of Eq. (5.8) from the right by $K_0^{-1}(\lambda)K_{0'}^{-1}(\mu)$, (5.8) can be express as

$$
\begin{aligned}
[\mathcal{L}_0(\lambda),\mathcal{L}_{0'}(\mu)]=&\left[r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K_{0'}^{-1}(\mu),\mathcal{L}_0(\lambda)\right]\\
&-\left[r_{0'0}(\mu-\lambda)-K_0(\lambda)r_{0'0}(\mu+\lambda)K_0^{-1}(\lambda),\mathcal{L}_{0'}(\mu)\right].
\end{aligned}
\tag{5.9}
$$

Defining

$$
r^K_{00'}(\lambda,\mu)=r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K_{0'}^{-1}(\mu),
\tag{5.10}
$$

(5.9) can be written as

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]. \tag{5.11}$$

The commutator (5.11) is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix (5.10) satisfies the classical YB equation

$$[r^K_{32}(\lambda_3,\lambda_2), r^K_{13}(\lambda_1,\lambda_3)] + [r^K_{12}(\lambda_1,\lambda_2), r^K_{13}(\lambda_1,\lambda_3) + r^K_{23}(\lambda_2,\lambda_3)] = 0. \tag{5.12}$$

The commutator (5.11) can also be recasted as an $(\mathfrak{r},\mathfrak{s})$ Maillet algebra [48]. In the following we study the algebraic Bethe ansatz based on the linear bracket (5.11).

6. Algebraic Bethe ansatz

Our preliminary step in the implementation of the algebraic Bethe ansatz for the open Gaudin model is to bring the boundary K-matrix to the upper, or lower, triangular form. As it was pointed out in (3.4), the general form of the K-matrix (4.11) is

$$\widetilde{K}(\lambda) = \begin{pmatrix} \xi - \lambda & \widetilde{\psi}\lambda \\ \widetilde{\phi}\lambda & \xi + \lambda \end{pmatrix}. \tag{6.1}$$

It is straightforward to check that the matrix

$$U = \begin{pmatrix} -1-\nu & \widetilde{\phi} \\ \widetilde{\phi} & -1-\nu \end{pmatrix}, \tag{6.2}$$

with $\nu = \sqrt{1+\widetilde{\phi}\widetilde{\psi}}$, which does not depend on the spectral parameter λ, brings the K-matrix to the upper triangular form by the similarity transformation

$$K(\lambda) = U^{-1}\widetilde{K}(\lambda)U = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix}, \tag{6.3}$$

where $\psi = \widetilde{\phi} + \widetilde{\psi}$. Evidently, the inverse matrix is

$$K^{-1}(\lambda) = \frac{1}{\xi^2 - \lambda^2\nu^2} \begin{pmatrix} \xi + \lambda\nu & -\lambda\psi \\ 0 & \xi - \lambda\nu \end{pmatrix}. \tag{6.4}$$

Direct substitution of the formulas above into (4.25),

$$\mathcal{L}_0(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix} = \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda) \cdot \vec{S}_m}{\lambda + \alpha_m} \right), \tag{6.5}$$

yields the following local realisation for the entries of the Lax matrix

$$E(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^+}{\lambda - \alpha_m} + \frac{(\xi + \lambda\nu)S_m^+}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right), \tag{6.6}$$

$$F(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^-}{\lambda - \alpha_m} + \frac{(\xi - \lambda\nu)^2 S_m^- - \lambda^2\psi^2 S_m^+ - 2\lambda\psi(\xi - \lambda\nu)S_m^3}{(\xi + \lambda\nu)(\xi - \lambda\nu)(\lambda + \alpha_m)} \right), \tag{6.7}$$

$$H(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^3}{\lambda - \alpha_m} + \frac{\lambda\psi\, S_m^+ + (\xi - \lambda\nu)S_m^3}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right). \tag{6.8}$$

The linear bracket (5.11) based on the r-matrix $r^K_{00'}(\lambda,\mu)$ (5.10), corresponding to (6.3), (6.4) and the classical r-matrix (2.16), defines the Lie algebra relevant for the open Gaudin model

$$[E(\lambda), E(\mu)] = 0, \tag{6.9}$$

$$[H(\lambda), E(\mu)] = \frac{2}{\lambda^2 - \mu^2}\left(\lambda\, E(\mu) - \frac{\xi - \lambda\nu}{\xi - \mu\nu}\mu\, E(\lambda)\right), \tag{6.10}$$

$$[E(\lambda), F(\mu)] = \frac{2\psi\mu}{(\lambda + \mu)(\xi + \mu\nu)}\, E(\lambda) + \frac{4}{\lambda^2 - \mu^2}\left(\frac{\xi - \mu\nu}{\xi - \lambda\nu}\lambda\, H(\mu) - \frac{\xi + \lambda\nu}{\xi + \mu\nu}\mu\, H(\lambda)\right), \tag{6.11}$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\lambda + \mu}\left(\frac{\lambda}{\xi - \lambda\nu}\, E(\mu) - \frac{\mu}{\xi - \mu\nu}\, E(\lambda)\right), \tag{6.12}$$

$$[H(\lambda), F(\mu)] = \frac{\psi}{\lambda + \mu}\left(\frac{2\lambda}{\xi - \lambda\nu}\, H(\mu) - \frac{\psi\mu^2}{\xi^2 - \mu^2\nu^2}\, E(\lambda)\right) - \frac{2}{\lambda^2 - \mu^2}\left(\lambda\, F(\mu) - \frac{\xi + \lambda\nu}{\xi + \mu\nu}\mu\, F(\lambda)\right), \tag{6.13}$$

$$[F(\lambda), F(\mu)] = \frac{2\psi}{\lambda + \mu}\left(\frac{\lambda}{\xi + \lambda\nu}\, F(\mu) - \frac{\mu}{\xi + \mu\nu}\, F(\lambda)\right) - \frac{2\psi^2}{\lambda + \mu}\left(\frac{\lambda^2}{\xi^2 - \lambda^2\nu^2}H(\mu) - \frac{\mu^2}{\xi^2 - \mu^2\nu^2}H(\lambda)\right). \tag{6.14}$$

Our next step is to introduce the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the original generators

$$e(\lambda) = \frac{-\xi + \lambda\nu}{\lambda}E(\lambda), \quad h(\lambda) = \frac{1}{\lambda}\left(H(\lambda) - \frac{\psi\lambda}{2\xi}\, E(\lambda)\right),$$
$$f(\lambda) = \frac{1}{\lambda}\left((\xi + \lambda\nu)F(\lambda) + \psi\lambda H(\lambda)\right). \tag{6.15}$$

The key observation is that in the new basis we have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = \left[f(\lambda), f(\mu)\right] = 0. \tag{6.16}$$

Therefore there are only three nontrivial relations

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2}\left(e(\mu) - e(\lambda)\right), \tag{6.17}$$

$$\left[h(\lambda), f(\mu)\right] = \frac{-2}{\lambda^2 - \mu^2}\left(f(\mu) - f(\lambda)\right) - \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi}\left(\mu^2 h(\mu) - \lambda^2 h(\lambda)\right) - \frac{\psi^2}{(\lambda^2 - \mu^2)\xi^2}\left(\mu^2 e(\mu) - \lambda^2 e(\lambda)\right), \tag{6.18}$$

$$\left[e(\lambda), f(\mu)\right] = \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi}\left(\mu^2 e(\mu) - \lambda^2 e(\lambda)\right) - \frac{4}{\lambda^2 - \mu^2}\left((\xi^2 - \mu^2\nu^2)h(\mu) - (\xi^2 - \lambda^2\nu^2)h(\lambda)\right). \tag{6.19}$$

In the Hilbert space $\mathcal{H}$ (2.3), in every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3\omega_m = s_m\omega_m \quad \text{and} \quad S_m^+\omega_m = 0. \tag{6.20}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{6.21}$$

From the definitions above, the formulas (6.6)–(6.8) and (6.15) it is straightforward to obtain the action of the generators $e(\lambda)$ and $h(\lambda)$ on the vector Ω_+

$$e(\lambda)\Omega_+ = 0, \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \tag{6.22}$$

with

$$\rho(\lambda) = \frac{1}{\lambda}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda-\alpha_m}+\frac{s_m}{\lambda+\alpha_m}\right) = \sum_{m=1}^{N}\frac{2s_m}{\lambda^2-\alpha_m^2}. \tag{6.23}$$

The generating function of the Gaudin Hamiltonians (6.31) in terms of the entries of the Lax matrix is given by

$$\tau(\lambda) = \operatorname{tr}_0 \mathcal{L}_0^2(\lambda) = 2H^2(\lambda) + 2F(\lambda)E(\lambda) + [E(\lambda), F(\lambda)]. \tag{6.24}$$

From (6.11) we have that the last term is

$$[E(\lambda), F(\lambda)] = 2\frac{\xi^2+\lambda^2\nu^2}{(\xi^2-\lambda^2\nu^2)\lambda}H(\lambda) - 2H'(\lambda) + \frac{\psi}{\xi+\lambda\nu}E(\lambda), \tag{6.25}$$

and therefore the final expression is

$$\tau(\lambda) = 2\left(H^2(\lambda) + \frac{\xi^2+\lambda^2\nu^2}{(\xi^2-\lambda^2\nu^2)\lambda}H(\lambda) - H'(\lambda)\right) + \left(2F(\lambda) + \frac{\psi}{\xi+\lambda\nu}\right)E(\lambda). \tag{6.26}$$

Our aim is to implement the algebraic Bethe ansatz based on the Lie algebra (6.16)–(6.19). To this end we need to obtain the expression for the generating function $\tau(\lambda)$ in terms of the generators $e(\lambda), h(\lambda)$ and $f(\lambda)$. The first step is to invert the relations (6.15)

$$E(\lambda) = \frac{-\lambda}{\xi-\lambda\nu}e(\lambda), \tag{6.27}$$

$$H(\lambda) = \lambda\left(h(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}e(\lambda)\right), \tag{6.28}$$

$$F(\lambda) = \frac{\lambda}{\xi+\lambda\nu}\left(f(\lambda) - \psi\lambda\, h(\lambda) + \frac{\psi^2\lambda^2}{2\xi(\xi-\lambda\nu)}e(\lambda)\right). \tag{6.29}$$

In particular, we have

$$\begin{aligned} H^2(\lambda) &= \lambda^2\left(h^2(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}\left(2h(\lambda)e(\lambda) - [h(\lambda), e(\lambda)]\right) + \frac{\psi^2\lambda^2}{4\xi^2(\xi-\lambda\nu)^2}e^2(\lambda)\right) \\ &= \lambda^2\left(h^2(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}\left(2h(\lambda)e(\lambda) + \frac{e'(\lambda)}{\lambda}\right) + \frac{\psi^2\lambda^2}{4\xi^2(\xi-\lambda\nu)^2}e^2(\lambda)\right). \end{aligned} \tag{6.30}$$

Substituting (6.27)–(6.30) into (6.26) we obtain the desired expression for the generating function

$$\begin{aligned} \tau(\lambda) &= 2\lambda^2\left(h^2(\lambda) + \frac{2\nu^2}{\xi^2-\lambda^2\nu^2}h(\lambda) - \frac{h'(\lambda)}{\lambda}\right) \\ &\quad - \frac{2\lambda^2}{\xi^2-\lambda^2\nu^2}\left(f(\lambda) + \frac{\psi\lambda^2\nu}{\xi}h(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2}e(\lambda) - \frac{\psi\nu}{\xi}\right)e(\lambda). \end{aligned} \tag{6.31}$$

An important initial observation in the implementation of the algebraic Bethe ansatz is that the vector Ω_+ (6.21) is an eigenvector of the generating function $\tau(\lambda)$, to show this we use the expression above, (6.22) and (6.23)

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2\left(\rho^2(\lambda) + \frac{2\nu^2\rho(\lambda)}{\xi^2-\lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda}\right)\Omega_+, \tag{6.32}$$

using (6.23) the eigenvalue $\chi_0(\lambda)$ can be expressed as

$$\chi_0(\lambda) = 8\lambda^2\left(\sum_{m=1}^{N}\frac{s_m(s_m+1)}{(\lambda^2-\alpha_m^2)^2} + \sum_{m=1}^{N}\frac{s_m}{\lambda^2-\alpha_m^2}\left(\frac{\nu^2}{\xi^2-\lambda^2\nu^2} + \sum_{n\neq m}^{N}\frac{2s_n}{\alpha_m^2-\alpha_n^2}\right)\right). \tag{6.33}$$

An essential step in the algebraic Bethe ansatz is a definition of the corresponding Bethe vectors. In this case, they are symmetric functions of their arguments and are such that the off shell action of the generating function of the Gaudin Hamiltonians is as simple as possible. With this aim we attempt to show that the Bethe vector $\varphi_1(\mu)$ has the form

$$\varphi_1(\mu) = (f(\mu) + c_1(\mu))\,\Omega_+, \tag{6.34}$$

where $c_1(\mu)$ is given by

$$c_1(\mu) = -\frac{\psi\nu}{\xi}\left(1-\mu^2\rho(\mu)\right). \tag{6.35}$$

Evidently, the action of the generating function of the Gaudin Hamiltonians reads

$$\tau(\lambda)\varphi_1(\mu) = \left[\tau(\lambda), f(\mu)\right]\Omega_+ + \chi_0(\lambda)\varphi_1(\mu). \tag{6.36}$$

A straightforward calculation show that the commutator in the first term of (6.36) is given by

$$\begin{aligned}\left[\tau(\lambda), f(\mu)\right]\Omega_+ = &-\frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\varphi_1(\mu)\\ &+ \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda).\end{aligned} \tag{6.37}$$

Therefore the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is given by

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda), \tag{6.38}$$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right). \tag{6.39}$$

The unwanted term in (6.38) vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} = 0. \tag{6.40}$$

Thus we have shown that $\varphi_1(\mu)$ (6.34) is the desired Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_1(\lambda,\mu)$.

We seek the Bethe vector $\varphi_2(\mu_1, \mu_2)$ as the following symmetric function

$$\varphi_2(\mu_1,\mu_2) = f(\mu_1) f(\mu_2)\Omega_+ + c_2^{(1)}(\mu_2;\mu_1) f(\mu_1)\Omega_+ \\ + c_2^{(1)}(\mu_1;\mu_2) f(\mu_2)\Omega_+ + c_2^{(2)}(\mu_1,\mu_2)\Omega_+, \tag{6.41}$$

where the scalar coefficients $c_2^{(1)}(\mu_1;\mu_2)$ and $c_2^{(2)}(\mu_1,\mu_2)$ are

$$c_2^{(1)}(\mu_1;\mu_2) = -\frac{\psi\nu}{\xi}\left(1 - \mu_1^2\rho(\mu_1) + \frac{2\mu_1^2}{\mu_1^2-\mu_2^2}\right), \tag{6.42}$$

$$c_2^{(2)}(\mu_1,\mu_2) = -\frac{\psi^2}{\nu^2}\left(\frac{(\xi^2 - 3\mu_2^2\nu^2)\rho(\mu_1) - (\xi^2 - 3\mu_1^2\nu^2)\rho(\mu_2)}{\mu_1^2-\mu_2^2} \right. \\ \left. + (\xi^2 - (\mu_1^2+\mu_2^2)\nu^2)\rho(\mu_1)\rho(\mu_2)\right). \tag{6.43}$$

One way to obtain the action of $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is to write

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) \\ = \left[\left[\tau(\lambda), f(\mu_1)\right], f(\mu_2)\right]\Omega_+ + \left(f(\mu_2) + c_2^{(1)}(\mu_2;\mu_1)\right)\left[\tau(\lambda), f(\mu_1)\right]\Omega_+ \\ + \left(f(\mu_1) + c_2^{(1)}(\mu_1;\mu_2)\right)\left[\tau(\lambda), f(\mu_2)\right]\Omega_+ + \chi_0(\lambda)\varphi_2(\mu_1,\mu_2). \tag{6.44}$$

Then to substitute (6.37) in the second and third term above and use the relation

$$\left(f(\mu_1) + c_2^{(1)}(\mu_1;\mu_2)\right)\varphi_1(\mu_2) \\ = \varphi_2(\mu_1,\mu_2) - \frac{\psi\nu}{\xi}\frac{2\mu_2^2}{\mu_1^2-\mu_2^2}\varphi_1(\mu_1) \\ - \left(c_2^{(2)}(\mu_1,\mu_2) - c_1(\mu_1)c_1(\mu_2) + 2\frac{\psi\nu}{\xi}\frac{\mu_1^2 c_1(\mu_2) - \mu_2^2 c_1(\mu_1)}{\mu_1^2-\mu_2^2}\right)\Omega_+, \tag{6.45}$$

which follows from the definition (6.41). A straightforward calculation shows that the off shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is given by

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) = \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) + \sum_{i=1}^{2}\frac{8\lambda^2(\xi^2-\mu_i^2\nu^2)}{(\lambda^2-\mu_i^2)(\xi^2-\lambda^2\nu^2)} \\ \times\left(\rho(\mu_i) + \frac{\nu^2}{\xi^2-\mu_i^2\nu^2} - \frac{2}{\mu_i^2-\mu_{3-i}^2}\right)\varphi_2(\lambda,\mu_{3-i}), \tag{6.46}$$

with the eigenvalue

$$\chi_2(\lambda,\mu_1,\mu_2) = \chi_0(\lambda) - \sum_{i=1}^{2}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2} - \frac{1}{\lambda^2-\mu_{3-i}^2}\right). \tag{6.47}$$

The two unwanted terms in the action above (6.46) vanish when the Bethe equations are imposed on the parameters μ_1 and μ_2,

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2-\mu_i^2\nu^2} - \frac{2}{\mu_i^2-\mu_{3-i}^2} = 0, \tag{6.48}$$

with $i = 1, 2$. Therefore $\varphi_2(\mu_1, \mu_2)$ is the Bethe vector of the generating function of the Gaudin Hamiltonians with the eigenvalue $\chi_2(\lambda, \mu_1, \mu_2)$.

As our next step we propose the Bethe vector $\varphi_3(\mu_1, \mu_2, \mu_3)$ in the form of the following symmetric function of its arguments

$$\begin{aligned}
\varphi_3(\mu_1, \mu_2, \mu_3) = {} & f(\mu_1) f(\mu_2) f(\mu_3)\Omega_+ + c_3^{(1)}(\mu_1; \mu_2, \mu_3) f(\mu_2) f(\mu_3)\Omega_+ \\
& + c_3^{(1)}(\mu_2; \mu_3, \mu_1) f(\mu_3) f(\mu_1)\Omega_+ + c_3^{(1)}(\mu_3; \mu_1, \mu_2) f(\mu_1) f(\mu_2)\Omega_+ \\
& + c_3^{(2)}(\mu_1, \mu_2; \mu_3) f(\mu_3)\Omega_+ + c_3^{(2)}(\mu_2, \mu_3; \mu_1) f(\mu_1)\Omega_+ \\
& + c_3^{(2)}(\mu_3, \mu_1; \mu_2) f(\mu_2)\Omega_+ + c_3^{(3)}(\mu_1, \mu_2, \mu_3)\Omega_+,
\end{aligned} \tag{6.49}$$

where the three scalar coefficients above are given by

$$c_3^{(1)}(\mu_1; \mu_2, \mu_3) = -\frac{\psi\nu}{\xi}\left(1 - \mu_1^2\rho(\mu_1) + \frac{2\mu_1^2}{\mu_1^2 - \mu_2^2} + \frac{2\mu_1^2}{\mu_1^2 - \mu_3^2}\right), \tag{6.50}$$

$$\begin{aligned}
& c_3^{(2)}(\mu_1, \mu_2; \mu_3) \\
& \quad = -\frac{\psi^2}{\nu^2}\left(\frac{\xi^2 - 3\mu_2^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_1) - \frac{2}{\mu_1^2 - \mu_3^2}\right) - \frac{\xi^2 - 3\mu_1^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_2) - \frac{2}{\mu_2^2 - \mu_3^2}\right)\right) \\
& \qquad - \frac{\psi^2}{\nu^2}(\xi^2 - (\mu_1^2 + \mu_2^2)\nu^2)\left(\rho(\mu_1) - \frac{2}{\mu_1^2 - \mu_3^2}\right)\left(\rho(\mu_2) - \frac{2}{\mu_2^2 - \mu_3^2}\right),
\end{aligned} \tag{6.51}$$

$$\begin{aligned}
& c_3^{(3)}(\mu_1, \mu_2, \mu_3) \\
& \quad = -\frac{\psi^3}{\nu^3\xi}\left(\frac{4\xi^4 + \left(\xi^2 + \mu_1^2\nu^2\right)\left(4\mu_1^2 - 5(\mu_2^2 + \mu_3^2)\right)\nu^2}{(\mu_1^2 - \mu_2^2)(\mu_1^2 - \mu_3^2)}\rho(\mu_1)\right. \\
& \qquad + \frac{4\xi^4 + \left(\xi^2 + \mu_2^2\nu^2\right)\left(4\mu_2^2 - 5(\mu_3^2 + \mu_1^2)\right)\nu^2}{(\mu_2^2 - \mu_3^2)(\mu_2^2 - \mu_1^2)}\rho(\mu_2) \\
& \qquad \left. + \frac{4\xi^4 + \left(\xi^2 + \mu_3^2\nu^2\right)\left(4\mu_3^2 - 5(\mu_1^2 + \mu_2^2)\right)\nu^2}{(\mu_3^2 - \mu_1^2)(\mu_3^2 - \mu_2^2)}\rho(\mu_3)\right) \\
& \qquad - \frac{\psi^3}{\nu^3\xi}\left(\frac{\xi^2\nu^2\left(\mu_1^4 + \mu_2^4 - \mu_1^2\mu_2^2 + 2\mu_3^2\left(\mu_1^2 + \mu_2^2\right) - 5\mu_3^4\right) - \left(2\xi^4 - \mu_1^2\mu_2^2\nu^4\right)\left(\mu_1^2 + \mu_2^2 - 2\mu_3^2\right)}{(\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2)}\rho(\mu_1)\rho(\mu_2)\right. \\
& \qquad + \frac{\xi^2\nu^2\left(\mu_2^4 + \mu_3^4 - \mu_2^2\mu_3^2 + 2\mu_1^2\left(\mu_2^2 + \mu_3^2\right) - 5\mu_1^4\right) - \left(2\xi^4 - \mu_2^2\mu_3^2\nu^4\right)\left(\mu_2^2 + \mu_3^2 - 2\mu_1^2\right)}{(\mu_2^2 - \mu_1^2)(\mu_3^2 - \mu_1^2)}\rho(\mu_2)\rho(\mu_3) \\
& \qquad \left. + \frac{\xi^2\nu^2\left(\mu_3^4 + \mu_1^4 - \mu_3^2\mu_1^2 + 2\mu_2^2\left(\mu_3^2 + \mu_1^2\right) - 5\mu_2^4\right) - \left(2\xi^4 - \mu_3^2\mu_1^2\nu^4\right)\left(\mu_3^2 + \mu_1^2 - 2\mu_2^2\right)}{(\mu_3^2 - \mu_2^2)(\mu_1^2 - \mu_2^2)}\rho(\mu_3)\rho(\mu_1)\right) \\
& \qquad - \frac{\psi^3}{\nu^3}\xi\left(2\xi^2 - \left(\mu_1^2 + \mu_2^2 + \mu_3^2\right)\nu^2\right)\rho(\mu_1)\rho(\mu_2)\rho(\mu_3).
\end{aligned} \tag{6.52}$$

A lengthy but straightforward calculation based on appropriate generalisation of (6.44) and (6.45) shows that the action of the generating function $\tau(\lambda)$ on $\varphi_3(\mu_1, \mu_2, \mu_3)$ is given by

$$\tau(\lambda)\varphi_3(\mu_1, \mu_2, \mu_3) = \chi_3(\lambda, \mu_1, \mu_2\mu_3)\varphi_3(\mu_1, \mu_2, \mu_3) + \sum_{i=1}^{3}\frac{8\lambda^2(\xi^2 - \mu_i^2\nu^2)}{(\lambda^2 - \mu_i^2)(\xi^2 - \lambda^2\nu^2)}$$

$$\times\left(\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\sum_{j\neq i}^{3}\frac{2}{\mu_i^2-\mu_j^2}\right)\varphi_3(\lambda,\{\mu_j\}_{j\neq i}), \quad (6.53)$$

where the eigenvalue is

$$\chi_3(\lambda,\mu_1,\mu_2,\mu_3)=\chi_0(\lambda)-\sum_{i=1}^{3}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(\rho(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{3}\frac{1}{\lambda^2-\mu_j^2}\right). \quad (6.54)$$

The three unwanted terms in (6.53) vanish when the Bethe equation are imposed on the parameters μ_i,

$$\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\sum_{j\neq i}^{3}\frac{2}{\mu_i^2-\mu_j^2}=0, \quad (6.55)$$

with $i=1,2,3$.

As a symmetric function of its arguments the Bethe vector $\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4)$ is given explicitly in Appendix B. It is possible to check that the off shell action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4)$ is given by

$$\begin{aligned}&\tau(\lambda)\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4)\\&\quad=\chi_4(\lambda,\mu_1,\mu_2\mu_3)\varphi_4(\mu_1,\mu_2,\mu_3,\mu_4)+\sum_{i=1}^{4}\frac{8\lambda^2(\xi^2-\mu_i^2\nu^2)}{(\lambda^2-\mu_i^2)(\xi^2-\lambda^2\nu^2)}\\&\quad\times\left(\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\sum_{j\neq i}^{4}\frac{2}{\mu_i^2-\mu_j^2}\right)\varphi_4(\lambda,\{\mu_j\}_{j\neq i}),\end{aligned} \quad (6.56)$$

with the eigenvalue

$$\chi_4(\lambda,\mu_1,\mu_2,\mu_3,\mu_4)=\chi_0(\lambda)-\sum_{i=1}^{4}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(\rho(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{4}\frac{1}{\lambda^2-\mu_j^2}\right). \quad (6.57)$$

The four unwanted terms on the right hand side of (6.56) vanish when the Bethe equation are imposed on the parameters μ_i,

$$\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\sum_{j\neq i}^{4}\frac{2}{\mu_i^2-\mu_j^2}=0, \quad (6.58)$$

with $i=1,2,3,4$.

Based on the results presented above we can conclude that the local realisation (6.6)–(6.8) of the Lie algebra (6.15)–(6.19) yields the spectrum $\chi_M(\lambda,\mu_1,\dots,\mu_M)$ of the generating function of the Gaudin Hamiltonians

$$\chi_M(\lambda,\mu_1,\dots,\mu_M)=\chi_0(\lambda)-\sum_{i=1}^{M}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(\rho(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right), \quad (6.59)$$

and the corresponding Bethe equations which should be imposed on the parameters μ_i

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{M} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \tag{6.60}$$

where $i = 1, 2, \dots, M$. Moreover, from (4.29) and (6.59) it follows that the eigenvalues of the Gaudin Hamiltonians (4.32) and (4.33) can be obtained as the residues of $\chi_M(\lambda, \mu_1, \dots, \mu_M)$ at poles $\lambda = \pm\alpha_m$

$$\begin{aligned} \mathcal{E}_m &= \frac{1}{4} \mathrm{Res}_{\lambda=\alpha_m} \chi_M(\lambda, \mu_1, \dots, \mu_M) \\ &= \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2 \nu^2} + \sum_{n \neq m}^{N} \frac{2 s_n}{\alpha_m^2 - \alpha_n^2} \right) - 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2}, \end{aligned} \tag{6.61}$$

and

$$\begin{aligned} \widetilde{\mathcal{E}}_m &= -\frac{1}{4} \mathrm{Res}_{\lambda=-\alpha_m} \chi_M(\lambda, \mu_1, \dots, \mu_M) \\ &= \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2 \nu^2} + \sum_{n \neq m}^{N} \frac{2 s_n}{\alpha_m^2 - \alpha_n^2} \right) - 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2}. \end{aligned} \tag{6.62}$$

Evidently, the respective eigenvalues (6.61) and (6.62) of the Hamiltonians (4.32) and (4.33) coincide. When all the spin s_m are set to one half, these energies coincide with the expressions obtained in [42] (up to normalisation). The Bethe equations are also equivalent, the correspondence between our variables and the one used in [42] being given by (the left hand sides correspond to our variables, the left hand sides to the ones used in [42]):

$$\mu_j = \frac{\lambda_j}{1 - \xi^{(1)}}\ ; \quad \alpha_m = \frac{\theta_m}{1 - \xi^{(1)}}\ ; \quad \frac{\xi}{\nu} = \frac{\xi}{1 - \xi^{(1)}}. \tag{6.63}$$

However, explicit and compact form of the relevant Bethe vector $\varphi_M(\mu_1, \mu_2, \dots, \mu_M)$, for an arbitrary positive integer M, requires further studies and will be reported elsewhere. As it is evident form the formulas for the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ given in Appendix B, the main problem lies in the definition the scalar coefficients $c_M^{(m)}(\mu_1, \dots, \mu_m; \mu_{m+1}, \dots, \mu_M)$, with $m = 1, 2, \dots, M$. Some of them can be obtained straightforwardly, but, in particular, the coefficient $c_M^{(M)}(\mu_1, \mu_2, \dots, \mu_M)$ still represents a challenge, at least in the present form of the Bethe vectors.

7. Conclusion

Following Sklyanin's proposal in the periodic case [3], here we have derived the generating function of the Gaudin Hamiltonians with boundary terms. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function. Then we have studied the appropriate algebraic structure, including the classical reflection equation.

Our approach to the algebraic Bethe ansatz is based on the relevant Lax matrix which satisfies certain linear bracket and simultaneously provides the local realisation for the corresponding Lie algebra. By defining the appropriate Bethe vectors we have obtained the strikingly simple off shell action of the generating function of the Gaudin Hamiltonians. Actually, the action of the generating function is as simple as it could possible be since it almost coincides with the one in the case when the boundary matrix is diagonal [20]. In this way we have implemented the algebraic Bethe ansatz, obtaining the spectrum of the generating function and the corresponding Bethe equations.

Although the off shell action of the generating function which we have established is very simple, it would be important to obtain more compact formula for the Bethe vector $\varphi_M(\mu_1, \mu_2, \dots, \mu_M)$, for an arbitrary positive integer M. In particular, simpler expression for the scalar coefficients $c_M^{(m)}(\mu_1, \dots, \mu_m; \mu_{m+1}, \dots, \mu_M)$, with $m = 1, 2, \dots, M$ would be of utmost importance. Such a formula would be crucial for the off shell scalar product of the Bethe vectors and these results could lead to the correlations functions of Gaudin model with boundary. Moreover, it would be of considerable interest to establish a relation between Bethe vectors and solutions of the corresponding Knizhnik–Zamolodchikov equations, along the lines it was done in the case when the boundary matrix is diagonal [20].

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Appendix A. Basic definitions

We consider the spin operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^3, \tag{A.1}$$

and Casimir operator

$$c_2 = (S^3)^2 + \frac{1}{2}(S^+S^- + S^-S^+) = (S^3)^2 + S^3 + S^-S^+ = \vec{S} \cdot \vec{S}.$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{A.2}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \dots, N$.

Appendix B. Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$

Here we present explicit formulas of the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$. The vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ is a symmetric function of its arguments and is given by

$$\begin{aligned}
&\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)\\
&\quad = f(\mu_1) f(\mu_2) f(\mu_3) f(\mu_4)\Omega_+ + c_4^{(1)}(\mu_4; \mu_1, \mu_2, \mu_3) f(\mu_1) f(\mu_2) f(\mu_3)\Omega_+\\
&\qquad + c_4^{(1)}(\mu_3; \mu_1, \mu_2, \mu_4) f(\mu_1) f(\mu_2) f(\mu_4)\Omega_+\\
&\qquad + c_4^{(1)}(\mu_2; \mu_1, \mu_3, \mu_4) f(\mu_1) f(\mu_3) f(\mu_4)\Omega_+\\
&\qquad + c_4^{(1)}(\mu_1; \mu_2, \mu_3, \mu_4) f(\mu_2) f(\mu_3) f(\mu_4)\Omega_+ + c_4^{(2)}(\mu_3, \mu_4; \mu_1, \mu_2) f(\mu_1) f(\mu_2)\Omega_+\\
&\qquad + c_4^{(2)}(\mu_2, \mu_4; \mu_1, \mu_3) f(\mu_1) f(\mu_3)\Omega_+ + c_4^{(2)}(\mu_2, \mu_3; \mu_1, \mu_4) f(\mu_1) f(\mu_4)\Omega_+\\
&\qquad + c_4^{(2)}(\mu_1, \mu_4; \mu_2, \mu_3) f(\mu_2) f(\mu_3)\Omega_+ + c_4^{(2)}(\mu_1, \mu_3; \mu_2, \mu_4) f(\mu_2) f(\mu_4)\Omega_+\\
&\qquad + c_4^{(2)}(\mu_1, \mu_2; \mu_3, \mu_4) f(\mu_3) f(\mu_4)\Omega_+ + c_4^{(3)}(\mu_2, \mu_3, \mu_4; \mu_1) f(\mu_1)\Omega_+\\
&\qquad + c_4^{(3)}(\mu_1, \mu_2, \mu_4; \mu_2) f(\mu_2)\Omega_+ + c_4^{(3)}(\mu_1, \mu_2, \mu_4; \mu_3) f(\mu_3)\Omega_+\\
&\qquad + c_4^{(3)}(\mu_1, \mu_2, \mu_3; \mu_4) f(\mu_4)\Omega_+ + c_4^{(4)}(\mu_1, \mu_2, \mu_3, \mu_4)\Omega_+,
\end{aligned} \tag{B.1}$$

where the four scalar coefficients are

$$c_4^{(1)}(\mu_1; \mu_2, \mu_3, \mu_4) = -\frac{\psi\nu}{\xi}\left(1 - \mu_1^2\rho(\mu_1) + \sum_{i=2}^{4}\frac{2\mu_1^2}{\mu_1^2 - \mu_i^2}\right), \tag{B.2}$$

$$\begin{aligned}
&c_4^{(2)}(\mu_1, \mu_2; \mu_3, \mu_4)\\
&\quad = -\frac{\psi^2}{\nu^2}\left(\frac{\xi^2 - 3\mu_2^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_1) - \sum_{i=3}^{4}\frac{2}{\mu_1^2 - \mu_i^2}\right)\right.\\
&\qquad \left. - \frac{\xi^2 - 3\mu_1^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_2) - \sum_{j=3}^{4}\frac{2}{\mu_2^2 - \mu_j^2}\right)\right) - \frac{\psi^2}{\nu^2}(\xi^2 - (\mu_1^2 + \mu_2^2)\nu^2)\\
&\qquad \times \left(\rho(\mu_1) - \sum_{i=3}^{4}\frac{2}{\mu_1^2 - \mu_i^2}\right)\left(\rho(\mu_2) - \sum_{j=3}^{4}\frac{2}{\mu_2^2 - \mu_j^2}\right),
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&c_4^{(3)}(\mu_1, \mu_2, \mu_3; \mu_4)\\
&\quad = -\frac{\psi^3}{\nu^3\xi}\left(\frac{4\xi^4 + \left(\xi^2 + \mu_1^2\nu^2\right)\left(4\mu_1^2 - 5(\mu_2^2 + \mu_3^2)\right)\nu^2}{(\mu_1^2 - \mu_2^2)(\mu_1^2 - \mu_3^2)}\left(\rho(\mu_1) - \frac{2}{\mu_1^2 - \mu_4^2}\right)\right.\\
&\qquad + \frac{4\xi^4 + \left(\xi^2 + \mu_2^2\nu^2\right)\left(4\mu_2^2 - 5(\mu_3^2 + \mu_1^2)\right)\nu^2}{(\mu_2^2 - \mu_3^2)(\mu_2^2 - \mu_1^2)}\left(\rho(\mu_2) - \frac{2}{\mu_2^2 - \mu_4^2}\right)\\
&\qquad \left. + \frac{4\xi^4 + \left(\xi^2 + \mu_3^2\nu^2\right)\left(4\mu_3^2 - 5(\mu_1^2 + \mu_2^2)\right)\nu^2}{(\mu_3^2 - \mu_1^2)(\mu_3^2 - \mu_2^2)}\left(\rho(\mu_3) - \frac{2}{\mu_3^2 - \mu_4^2}\right)\right)\\
&\qquad - \frac{\psi^3}{\nu^3\xi}\left(\frac{\xi^2\nu^2\left(\mu_1^4 + \mu_2^4 - \mu_1^2\mu_2^2 + 2\mu_3^2\left(\mu_1^2 + \mu_2^2\right) - 5\mu_3^4\right) - \left(2\xi^4 - \mu_1^2\mu_2^2\nu^4\right)\left(\mu_1^2 + \mu_2^2 - 2\mu_3^2\right)}{(\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2)}\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\rho(\mu_1) - \frac{2}{\mu_1^2-\mu_4^2}\right)\left(\rho(\mu_2) - \frac{2}{\mu_2^2-\mu_4^2}\right) \\
&+ \frac{\xi^2\nu^2\left(\mu_2^4+\mu_3^4-\mu_2^2\mu_3^2+2\mu_1^2\left(\mu_2^2+\mu_3^2\right)-5\mu_1^4\right)-\left(2\xi^4-\mu_2^2\mu_3^2\nu^4\right)\left(\mu_2^2+\mu_3^2-2\mu_1^2\right)}{(\mu_2^2-\mu_1^2)(\mu_3^2-\mu_1^2)} \\
&\times \left(\rho(\mu_2) - \frac{2}{\mu_2^2-\mu_4^2}\right)\left(\rho(\mu_3) - \frac{2}{\mu_3^2-\mu_4^2}\right) \\
&+ \frac{\xi^2\nu^2\left(\mu_3^4+\mu_1^4-\mu_3^2\mu_1^2+2\mu_2^2\left(\mu_3^2+\mu_1^2\right)-5\mu_2^4\right)-\left(2\xi^4-\mu_3^2\mu_1^2\nu^4\right)\left(\mu_3^2+\mu_1^2-2\mu_2^2\right)}{(\mu_3^2-\mu_2^2)(\mu_1^2-\mu_2^2)} \\
&\times \left(\rho(\mu_3) - \frac{2}{\mu_3^2-\mu_4^2}\right)\left(\rho(\mu_1) - \frac{2}{\mu_1^2-\mu_4^2}\right)\Bigg) - \frac{\psi^3}{\nu^3}\xi\left(2\xi^2-\left(\mu_1^2+\mu_2^2+\mu_3^2\right)\nu^2\right) \\
&\times \left(\rho(\mu_1) - \frac{2}{\mu_1^2-\mu_4^2}\right)\left(\rho(\mu_2) - \frac{2}{\mu_2^2-\mu_4^2}\right)\left(\rho(\mu_3) - \frac{2}{\mu_3^2-\mu_4^2}\right).
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
&c_4^{(4)}(\mu_1,\mu_2,\mu_3,\mu_4) \\
&\quad = -\frac{2\psi^4}{\nu^4}\Bigg(\frac{9\xi^4+\xi^2\nu^2\left(27\mu_1^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)+3\mu_1^2\nu^4\left(8\mu_1^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)(\mu_1^2-\mu_4^2)}\rho(\mu_1) \\
&\quad + \frac{9\xi^4+\xi^2\nu^2\left(27\mu_2^2-7(\mu_1^2+\mu_3^2+\mu_4^2)\right)+3\mu_2^2\nu^4\left(8\mu_2^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)(\mu_2^2-\mu_4^2)}\rho(\mu_2) \\
&\quad + \frac{9\xi^4+\xi^2\nu^2\left(27\mu_3^2-7(\mu_1^2+\mu_2^2+\mu_4^2)\right)+3\mu_3^2\nu^4\left(8\mu_3^2-7(\mu_1^2+\mu_2^2+\mu_4^2)\right)}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)(\mu_3^2-\mu_4^2)}\rho(\mu_3) \\
&\quad + \frac{9\xi^4+\xi^2\nu^2\left(27\mu_4^2-7(\mu_1^2+\mu_2^2+\mu_3^2)\right)+3\mu_4^2\nu^4\left(8\mu_4^2-7(\mu_1^2+\mu_2^2+\mu_3^2)\right)}{(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_2^2)(\mu_4^2-\mu_3^2)}\rho(\mu_4)\Bigg) \\
&\quad - \frac{\psi^4}{\nu^4}\bigg(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_2^2+\mu_2^4)-3(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_2^2\mu_3^2+\mu_2^2\mu_4^2-2\mu_3^2\mu_4^2)\right) \\
&\quad - 18\xi^2\nu^2\mu_1^2\mu_2^2(\mu_3^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_2^2-6\mu_1^2\mu_2^4+\mu_2^6)-4(\mu_1^2+\mu_2^2)^2 \\
&\quad \times(\mu_3^2+\mu_4^2)+7(\mu_1^2+\mu_2^2)(\mu_3^2+\mu_4^2)^2+2\mu_3^2\mu_4^2(5(\mu_1^2+\mu_2^2)-7(\mu_3^2+\mu_4^2))\Big) \\
&\quad + \nu^4\left(14\mu_1^2\mu_2^2(\mu_3^4+\mu_4^4)-(4\mu_1^2\mu_2^2+7\mu_3^2\mu_4^2)(\mu_1^2+\mu_2^2)(\mu_3^2+\mu_4^2)\right) \\
&\quad +2\nu^4\left(4\mu_3^2\mu_4^2(\mu_1^2+\mu_2^2)^2-3\mu_1^2\mu_2^2(\mu_1^2-\mu_2^2)^2-\mu_1^2\mu_2^2(3\mu_1^2\mu_2^2+5\mu_3^2\mu_4^2)\right)\bigg) \\
&\quad \times \frac{\rho(\mu_1)\rho(\mu_2)}{(\mu_1^2-\mu_3^2)(\mu_1^2-\mu_4^2)(\mu_2^2-\mu_3^2)(\mu_2^2-\mu_4^2)} \\
&\quad - \frac{\psi^4}{\nu^4}\bigg(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_3^2+\mu_3^4)-3(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_3^2+\mu_3^2\mu_4^2-2\mu_2^2\mu_4^2)\right) \\
&\quad - 18\xi^2\nu^2\mu_1^2\mu_3^2(\mu_2^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_3^2-6\mu_1^2\mu_3^4+\mu_3^6)-4(\mu_1^2+\mu_3^2)^2 \\
&\quad \times(\mu_2^2+\mu_4^2)+7(\mu_1^2+\mu_3^2)(\mu_2^2+\mu_4^2)^2+2\mu_2^2\mu_4^2(5(\mu_1^2+\mu_3^2)-7(\mu_2^2+\mu_4^2))\Big)
\end{aligned}
$$

$$+\nu^4\left(14\mu_1^2\mu_3^2(\mu_2^4+\mu_4^4)-(4\mu_1^2\mu_3^2+7\mu_2^2\mu_4^2)(\mu_1^2+\mu_3^2)(\mu_2^2+\mu_4^2)\right)$$
$$+2\nu^4\left(4\mu_2^2\mu_4^2(\mu_1^2+\mu_3^2)^2-3\mu_1^2\mu_3^2(\mu_1^2-\mu_3^2)^2-\mu_1^2\mu_3^2(3\mu_1^2\mu_3^2+5\mu_2^2\mu_4^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_3)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_4^2)(\mu_3^2-\mu_2^2)(\mu_3^2-\mu_4^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_4^2+\mu_4^4)-3(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2-2\mu_2^2\mu_3^2)\right)$$
$$-18\xi^2\nu^2\mu_1^2\mu_4^2(\mu_2^2+\mu_3^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_4^2-6\mu_1^2\mu_4^4+\mu_4^6)-4(\mu_1^2+\mu_4^2)^2$$
$$\times(\mu_2^2+\mu_3^2)+7(\mu_1^2+\mu_4^2)(\mu_2^2+\mu_3^2)^2+2\mu_2^2\mu_3^2(5(\mu_1^2+\mu_4^2)-7(\mu_2^2+\mu_3^2))\Big)$$
$$+\nu^4\left(14\mu_1^2\mu_4^2(\mu_2^4+\mu_3^4)-(4\mu_1^2\mu_4^2+7\mu_2^2\mu_3^2)(\mu_1^2+\mu_4^2)(\mu_2^2+\mu_3^2)\right)$$
$$+2\nu^4\left(4\mu_2^2\mu_3^2(\mu_1^2+\mu_4^2)^2-3\mu_1^2\mu_4^2(\mu_1^2-\mu_4^2)^2-\mu_1^2\mu_4^2(3\mu_1^2\mu_4^2+5\mu_2^2\mu_3^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_4)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)(\mu_4^2-\mu_2^2)(\mu_4^2-\mu_3^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_2^4+\mu_2^2\mu_3^2+\mu_3^4)-3(\mu_1^2\mu_2^2+\mu_2^2\mu_4^2+\mu_1^2\mu_3^2+\mu_3^2\mu_4^2-2\mu_1^2\mu_4^2)\right)$$
$$-18\xi^2\nu^2\mu_2^2\mu_3^2(\mu_1^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_2^6-6\mu_2^4\mu_3^2-6\mu_2^2\mu_3^4+\mu_3^6)-4(\mu_2^2+\mu_3^2)^2$$
$$\times(\mu_1^2+\mu_4^2)+7(\mu_2^2+\mu_3^2)(\mu_1^2+\mu_4^2)^2+2\mu_1^2\mu_4^2(5(\mu_2^2+\mu_3^2)-7(\mu_1^2+\mu_4^2))\Big)$$
$$+\nu^4\left(14\mu_2^2\mu_3^2(\mu_1^4+\mu_4^4)-(4\mu_2^2\mu_3^2+7\mu_1^2\mu_4^2)(\mu_2^2+\mu_3^2)(\mu_1^2+\mu_4^2)\right)$$
$$+2\nu^4\left(4\mu_1^2\mu_4^2(\mu_2^2+\mu_3^2)^2-3\mu_2^2\mu_3^2(\mu_2^2-\mu_3^2)^2-\mu_2^2\mu_3^2(3\mu_2^2\mu_3^2+5\mu_1^2\mu_4^2)\right)\Big)$$
$$\times\frac{\rho(\mu_2)\rho(\mu_3)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_4^2)(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_4^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_2^4+\mu_2^2\mu_4^2+\mu_4^4)-3(\mu_1^2\mu_2^2+\mu_2^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2-2\mu_1^2\mu_3^2)\right)$$
$$-18\xi^2\nu^2\mu_2^2\mu_4^2(\mu_1^2+\mu_3^2)+\xi^2\nu^2\Big(-2(\mu_2^6-6\mu_2^4\mu_4^2-6\mu_2^2\mu_4^4+\mu_4^6)-4(\mu_2^2+\mu_4^2)^2$$
$$\times(\mu_1^2+\mu_3^2)+7(\mu_2^2+\mu_4^2)(\mu_1^2+\mu_3^2)^2+2\mu_1^2\mu_3^2(5(\mu_2^2+\mu_4^2)-7(\mu_1^2+\mu_3^2))\Big)$$
$$+\nu^4\left(14\mu_2^2\mu_4^2(\mu_1^4+\mu_3^4)-(4\mu_2^2\mu_4^2+7\mu_1^2\mu_3^2)(\mu_2^2+\mu_4^2)(\mu_1^2+\mu_3^2)\right)$$
$$+2\nu^4\left(4\mu_1^2\mu_3^2(\mu_2^2+\mu_4^2)^2-3\mu_2^2\mu_4^2(\mu_2^2-\mu_4^2)^2-\mu_2^2\mu_4^2(3\mu_2^2\mu_4^2+5\mu_1^2\mu_3^2)\right)\Big)$$
$$\times\frac{\rho(\mu_2)\rho(\mu_4)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_3^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_3^4+\mu_3^2\mu_4^2+\mu_4^4)-3(\mu_2^2\mu_3^2+\mu_1^2\mu_3^2+\mu_2^2\mu_4^2+\mu_1^2\mu_4^2-2\mu_1^2\mu_2^2)\right)$$
$$-18\xi^2\nu^2\mu_3^2\mu_4^2(\mu_1^2+\mu_2^2)+\xi^2\nu^2\Big(-2(\mu_3^6-6\mu_3^4\mu_4^2-6\mu_3^2\mu_4^4+\mu_4^6)-4(\mu_3^2+\mu_4^2)^2$$

$$
\begin{aligned}
&\times(\mu_1^2+\mu_2^2)+7(\mu_3^2+\mu_4^2)(\mu_1^2+\mu_2^2)^2+2\mu_1^2\mu_2^2(5(\mu_3^2+\mu_4^2)-7(\mu_1^2+\mu_2^2))\Big)\\
&+\nu^4\left(14\mu_3^2\mu_4^2(\mu_1^4+\mu_2^4)-(4\mu_3^2\mu_4^2+7\mu_1^2\mu_2^2)(\mu_3^2+\mu_4^2)(\mu_1^2+\mu_2^2)\right)\\
&+2\nu^4\left(4\mu_1^2\mu_2^2(\mu_3^2+\mu_4^2)^2-3\mu_3^2\mu_4^2(\mu_3^2-\mu_4^2)^2-\mu_3^2\mu_4^2(3\mu_3^2\mu_4^2+5\mu_1^2\mu_2^2)\right)\Big)\\
&\times\frac{\rho(\mu_3)\rho(\mu_4)}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_2^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_4^4-2\mu_4^2(\mu_1^2+\mu_2^2+\mu_3^2)+\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2\right)\\
&-\xi^2\nu^2\Big(7\mu_4^6-4\mu_4^4(\mu_1^2+\mu_2^2+\mu_3^2)+\mu_4^2\times(3(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)-2(\mu_1^4\\
&+\mu_2^4+\mu_3^4))+\mu_1^4\mu_2^2+\mu_1^4\mu_3^2+\mu_1^2\mu_2^4+\mu_1^2\mu_3^4+\mu_2^4\mu_3^2+\mu_2^2\mu_3^4-4\mu_1^2\mu_2^2\mu_3^2\Big)\\
&-\nu^4\Big(2\mu_4^4(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)-\mu_4^2(\mu_1^4\mu_2^2+\mu_1^2\mu_2^4+\mu_1^4\mu_3^2+\mu_1^2\mu_3^4\\
&+\mu_2^4\mu_3^2+\mu_2^2\mu_3^4+6\mu_1^2\mu_2^2\mu_3^2)\Big)-\nu^4\left(2\mu_1^2\mu_2^2\mu_3^2(\mu_1^2+\mu_2^2+\mu_3^2)\right)\Big)\\
&\times\frac{\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)}{(\mu_1^2-\mu_4^2)(\mu_2^2-\mu_4^2)(\mu_3^2-\mu_4^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_3^4-2\mu_3^2(\mu_1^2+\mu_2^2+\mu_4^2)+\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2\right)\\
&-\xi^2\nu^2\Big(7\mu_3^6-4\mu_3^4(\mu_1^2+\mu_2^2+\mu_4^2)+\mu_3^2(3(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2)-2(\mu_1^4\\
&+\mu_2^4+\mu_4^4))+\mu_1^4\mu_2^2+\mu_1^4\mu_4^2+\mu_1^2\mu_2^4+\mu_1^2\mu_4^4+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4-4\mu_1^2\mu_2^2\mu_4^2\Big)\\
&-\nu^4\Big(2\mu_3^4(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2)-\mu_3^2(\mu_1^4\mu_2^2+\mu_1^2\mu_2^4+\mu_1^4\mu_4^2+\mu_1^2\mu_4^4\\
&+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4+6\mu_1^2\mu_2^2\mu_4^2)\Big)-\nu^4\left(2\mu_1^2\mu_2^2\mu_4^2(\mu_1^2+\mu_2^2+\mu_4^2)\right)\Big)\\
&\times\frac{\rho(\mu_1)\rho(\mu_2)\rho(\mu_4)}{(\mu_1^2-\mu_3^2)(\mu_2^2-\mu_3^2)(\mu_4^2-\mu_3^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_2^4-2\mu_2^2(\mu_1^2+\mu_3^2+\mu_4^2)+\mu_1^2\mu_3^2+\mu_1^3\mu_4^2+\mu_3^2\mu_4^2\right)\\
&-\xi^2\nu^2\Big(7\mu_2^6-4\mu_2^4(\mu_1^2+\mu_3^2+\mu_4^2)+\mu_2^2(3(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2)-2(\mu_1^4\\
&+\mu_3^4+\mu_4^4))+\mu_1^4\mu_3^2+\mu_1^4\mu_4^2+\mu_1^2\mu_3^4+\mu_1^2\mu_4^4+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4-4\mu_1^2\mu_3^2\mu_4^2\Big)\\
&-\nu^4\Big(2\mu_2^4(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2)-\mu_2^2(\mu_1^4\mu_3^2+\mu_1^2\mu_3^4+\mu_1^4\mu_4^2+\mu_1^2\mu_4^4\\
&+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4+6\mu_1^2\mu_3^2\mu_4^2)\Big)-\nu^4\left(2\mu_1^2\mu_3^2\mu_4^2(\mu_1^2+\mu_3^2+\mu_4^2)\right)\Big)\\
&\times\frac{\rho(\mu_1)\rho(\mu_3)\rho(\mu_4)}{(\mu_1^2-\mu_2^2)(\mu_3^2-\mu_2^2)(\mu_4^2-\mu_2^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_1^4-2\mu_1^2(\mu_2^2+\mu_3^2+\mu_4^2)+\mu_2^2\mu_3^2+\mu_2^3\mu_4^2+\mu_3^2\mu_4^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \xi^2\nu^2 \Big(7\mu_1^6 - 4\mu_1^4(\mu_2^2+\mu_3^2+\mu_4^2) + \mu_1^2(3(\mu_2^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2) - 2(\mu_2^4 \\
&+ \mu_3^4+\mu_4^4)) + \mu_2^4\mu_3^2 + \mu_2^4\mu_4^2 + \mu_2^2\mu_3^4 + \mu_2^2\mu_4^4 + \mu_3^4\mu_4^2 + \mu_3^2\mu_4^4 - 4\mu_2^2\mu_3^2\mu_4^2\Big) \\
&- \nu^4 \Big(2\mu_1^4(\mu_2^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2) - \mu_1^2(\mu_2^4\mu_3^2+\mu_2^2\mu_3^4+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4 \\
&+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4+6\mu_2^2\mu_3^2\mu_4^2)\Big) - \nu^4\Big(2\mu_2^2\mu_3^2\mu_4^2(\mu_2^2+\mu_3^2+\mu_4^2)\Big)\Big) \\
&\times \frac{\rho(\mu_2)\rho(\mu_3)\rho(\mu_4)}{(\mu_2^2-\mu_1^2)(\mu_3^2-\mu_1^2)(\mu_4^2-\mu_1^2)} \\
&- \frac{\psi^4}{\nu^4}\xi^2\Big(3\xi^2 - \Big(\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2\Big)\nu^2\Big)\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)\rho(\mu_4)
\end{aligned}
\tag{B.5}
$$

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Generalized $s\ell(2)$ Gaudin algebra and corresponding Knizhnik–Zamolodchikov equation

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Abstract

The Gaudin model has been revisited many times, yet some important issues remained open so far. With this paper we aim to properly address its certain aspects, while clarifying, or at least giving a solid ground to some other. Our main contribution is establishing the relation between the off-shell Bethe vectors with the solutions of the corresponding Knizhnik–Zamolodchikov equations for the non-periodic $s\ell(2)$ Gaudin model, as well as deriving the norm of the eigenvectors of the Gaudin Hamiltonians. Additionally, we provide a closed form expression also for the scalar products of the off-shell Bethe vectors. Finally, we provide explicit closed form of the off-shell Bethe vectors, together with a proof of implementation of the algebraic Bethe ansatz in full generality.

1. Introduction

Historically, Gaudin model was first proposed almost half a century ago [1–3], and has promptly gained attention primarily due to its long-range interactions feature [4,5]. It was shortly

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generalized to different underlining simple Lie algebras, as well as to trigonometric and elliptic types, cf. [6–9] and the references therein. The non-periodic boundary conditions were treated somewhat later [10–17], while in [18,19] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms and obtained the spectrum of the generating function with the corresponding Bethe equations. The very latest developments are taking the field in various new directions, e.g. [20–23] which shows that the topic is still very attractive.

However, in spite of the substantial interest for the topic, certain issues have not yet been, to our knowledge, fully addressed. First and foremost, we note that the relation of the Knizhnik–Zamolodchikov (KZ) equations [24] with the Gaudin $s\ell(2)$ model [25,26] with non-periodic boundary was not yet established for arbitrary spins. Hikami comes close to this goal in his paper [10], but does not tackle the issue in full generality – namely, he constrains his analysis to a special case of equal spins at all nodes, moreover fixing these spins to the value $\frac{1}{2}$. He also does not provide the expression for the norms of the eigenvectors of the Gaudin Hamiltonians, which can be obtained from the KZ approach. One of our goals here is to improve on both of these points: we successfully establish the relation between solutions of the corresponding KZ equations with the off-shell Bethe vectors in the case of arbitrary spins and derive the norm formula.

Superior to the formula for norm of the on-shell Bethe vectors is a formula for scalar product of arbitrary off-shell Bethe vectors. Following an approach laid in [27], we derive such an expression pertinent to the non-periodic $s\ell(2)$ case for arbitrary spins, in a closed form. The expression involves a sum of certain matrix determinants and its significance stems from the fact that it represents the first step towards the correlation functions.

En route to our treatment of the KZ equations, we present a closed form expression for the off-shell Bethe vectors and prove the implementation of the algebraic Bethe ansatz in full generality (for arbitrary reflection matrices and to arbitrary number of excitations). Such a development was a result of a suitable change of generalized Gaudin algebra basis (as compared to the one used in [19]), combined with observation of certain algebraic relations that we came across. The resulting simplifications have also facilitated calculations related to KZ equations.

The paper is structured as follows. In the next section, we introduce some standard notions while nevertheless relying heavily on the notation and conclusions of our previous paper [19], to which we direct the reader as a preliminary. The third section is devoted to the task of deriving the general off-shell form of the Bethe vectors and to proving its validity. As a key step to this end we, within the same section, first present a new basis of the generalized Gaudin algebra [28,29], and point to its advantages. In the fourth section we finally turn to KZ equations, establishing their relation to the previously derived Bethe vectors and obtaining the norm formula. In the same section we also present the novel formula for the scalar product of off-shell Bethe vectors. Finally, we summarize our results in the last section.

2. Preliminaries

The generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms was derived in [19]. Besides, the suitable Lax operator, accompanied by the corresponding linear bracket and an appropriate non-unitary r-matrices, as well as the transfer matrix, were also obtained. In this section we will briefly review only the most relevant of these results, while for the details of the notations and derivation we refer to the [19].

We study the $s\ell(2)$ Gaudin model with N sites, characterised by the local space $V_m = \mathbb{C}^{2s_m+1}$ and inhomogeneous parameter α_m, implying non-periodic boundary conditions. The relevant classical r-matrix was given e.g. in [6], $r(\lambda) = -\frac{\mathcal{P}}{\lambda}$, where $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

In the case of periodic boundary conditions, this structure is essentially sufficient (after proceeding in the standard manner) to obtain the complete solution of the system [6], together with the corresponding correlation functions [30]. However, the non-periodic case which is the subject of our present consideration is substantially more involved. In this case, of relevance is the classical reflection equation [31–33]:

$$
\begin{aligned}
&r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = \\
&= K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu).
\end{aligned}
\tag{2.1}
$$

In [19] we have derived the general form of the K-matrix solution, and have shown that it can be, without any loss of generality, brought into the upper triangular form:

$$
K(\lambda) = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix},
\tag{2.2}
$$

where neither of the parameters ξ, ψ, ν depends on the spectral parameter λ.

In the course of our analysis in [19] we arrived to the generalized $s\ell(2)$ Gaudin algebra [28, 29] with generators $\widetilde{e}(\lambda)$, $\widetilde{h}(\lambda)$ and $\widetilde{f}(\lambda)$. To facilitate later comparison with the new basis, we give the three nontrivial relations:

$$
\left[\widetilde{h}(\lambda), \widetilde{e}(\mu)\right] = \frac{2}{\lambda^2-\mu^2}\left(\widetilde{e}(\mu) - \widetilde{e}(\lambda)\right),
\tag{2.3}
$$

$$
\begin{aligned}
\left[\widetilde{h}(\lambda), \widetilde{f}(\mu)\right] = {} & \frac{-2}{\lambda^2-\mu^2}\left(\widetilde{f}(\mu) - \widetilde{f}(\lambda)\right) - \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\widetilde{h}(\mu) - \lambda^2\widetilde{h}(\lambda)\right) \\
& - \frac{\psi^2}{(\lambda^2-\mu^2)\xi^2}\left(\mu^2\widetilde{e}(\mu) - \lambda^2\widetilde{e}(\lambda)\right),
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\left[\widetilde{e}(\lambda), \widetilde{f}(\mu)\right] = {} & \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2\widetilde{e}(\mu) - \lambda^2\widetilde{e}(\lambda)\right) \\
& - \frac{4}{\lambda^2-\mu^2}\left((\xi^2-\mu^2\nu^2)\widetilde{h}(\mu) - (\xi^2-\lambda^2\nu^2)\widetilde{h}(\lambda)\right),
\end{aligned}
\tag{2.5}
$$

as well as the form of generating function of the Gaudin Hamiltonians in [19]:

$$
\begin{aligned}
\tau(\lambda) = {} & 2\lambda^2\left(\widetilde{h}^2(\lambda) + \frac{2\nu^2}{\xi^2-\lambda^2\nu^2}\widetilde{h}(\lambda) - \frac{\widetilde{h}'(\lambda)}{\lambda}\right) \\
& - \frac{2\lambda^2}{\xi^2-\lambda^2\nu^2}\left(\widetilde{f}(\lambda) + \frac{\psi\lambda^2\nu}{\xi}\widetilde{h}(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2}\widetilde{e}(\lambda) - \frac{\psi\nu}{\xi}\right)\widetilde{e}(\lambda).
\end{aligned}
\tag{2.6}
$$

In [19] we tried to implement the algebraic Bethe ansatz based on these generators. Although the approached looked promising and resulted in the conjecture for the spectra of the generating function $\tau(\lambda)$ and the corresponding Gaudin Hamiltonians, the expression for the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, was missing. It turned out, as we show in the following section, that the full implementation of the algebraic Bethe ansatz in this case requires to define a new set of generators which will enable explicit expressions for the Bethe vectors as well as the algebraic proof of the off shell action of the generating function $\tau(\lambda)$ and its spectrum.

3. New generators and the eigenvectors

In the algebraic Bethe ansatz it is essential to find the commutation relations between the generating function and a product of the creation operators in a closed form. To this end, with the aim to simplify the relations (2.4) and (2.5) as well as the expression (2.6), we introduce new generators $e(\lambda)$, $h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the previous ones:

$$e(\lambda) = \widetilde{e}(\lambda), \quad h(\lambda) = \widetilde{h}(\lambda) + \frac{\psi}{2\xi\nu}\widetilde{e}(\lambda), \quad f(\lambda) = \widetilde{f}(\lambda) + \frac{\psi\xi}{\nu}\widetilde{h}(\lambda) + \frac{\psi^2}{4\nu^2}\widetilde{e}(\lambda). \tag{3.1}$$

It is straightforward to check that in the new basis we still have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = [f(\lambda), f(\mu)] = 0, \tag{3.2}$$

while the key simplification occurs in the three nontrivial relations which are now given by

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2}\left(e(\mu) - e(\lambda)\right), \tag{3.3}$$

$$[h(\lambda), f(\mu)] = \frac{-2}{\lambda^2 - \mu^2}\left(f(\mu) - f(\lambda)\right), \tag{3.4}$$

$$[e(\lambda), f(\mu)] = \frac{-4}{\lambda^2 - \mu^2}\Big((\xi^2 - \mu^2\nu^2)\,h(\mu) - (\xi^2 - \lambda^2\nu^2)\,h(\lambda)\Big). \tag{3.5}$$

By using these generators the expression for the generating function of the Gaudin Hamiltonians with boundary terms (2.6) also simplifies. We invert the relations (3.1) and obtain the expression for the generating function in terms of the new generators

$$\tau(\lambda) = 2\lambda^2\left(h^2(\lambda) + \frac{2\nu^2}{\xi^2 - \lambda^2\nu^2}h(\lambda) - \frac{h'(\lambda)}{\lambda}\right) - \frac{2\lambda^2}{\xi^2 - \lambda^2\nu^2}f(\lambda)e(\lambda). \tag{3.6}$$

Evidently we have achieved our first objective, as the relations (3.3)–(3.5) and the expression (3.6) are much simple than before. Below we will demonstrate how these new results facilitate the study of the Bethe vectors.

As in [19], we define the vacuum Ω_+ which is annihilated by $e(\lambda)$, while being an eigenstate for $h(\lambda)$:

$$h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \quad \text{with} \quad \rho(\lambda) = \frac{1}{\lambda}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda - \alpha_m} + \frac{s_m}{\lambda + \alpha_m}\right) = \sum_{m=1}^{N}\frac{2s_m}{\lambda^2 - \alpha_m^2}. \tag{3.7}$$

The next relevant remark is that the vector Ω_+ is an eigenvector of the generating function $\tau(\lambda)$. To show this we use (3.6) and the action (3.7):

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2\left(\rho^2(\lambda) + \frac{2\nu^2\,\rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda}\right)\Omega_+. \tag{3.8}$$

Our main aim in this section it to prove that the generator $f(\lambda)$ (3.1) defines the Bethe vectors naturally, that is, to show that the Bethe vector in the general case is given by the following symmetric function of its arguments:

$$\varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = f(\mu_1)\cdots f(\mu_M)\Omega_+. \tag{3.9}$$

We stress that this was not possible in the old basis (of tilde operators), and thus the general form of the Bethe vector lacked in [19].

The action of the generating function of the Gaudin Hamiltonians $\tau(\lambda)$ on $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is given by

$$\tau(\lambda)\varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = [\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\,\Omega_+ + \chi_0(\lambda)\varphi_M(\mu_1, \mu_2, \ldots, \mu_M). \tag{3.10}$$

The key part of the proof will be to determine the commutator in the first term of the righthand side. Due to the simplicity of the new commutation relations (3.3)–(3.5) we will show that it is now possible to evaluate this commutator in an algebraically closed form. As the first step we will calculate the commutator between the generating function (3.6) and a single generator $f(\lambda)$. A straightforward calculation yields

$$\begin{aligned}[\tau(\lambda), f(\mu)] = &-\frac{8\lambda^2}{\lambda^2-\mu^2} f(\mu)\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\\ &+\frac{8\lambda^2}{\lambda^2-\mu^2}\frac{\xi^2-\mu^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda)\left(h(\mu)+\frac{\nu^2}{\xi^2-\mu^2\nu^2}\right).\end{aligned} \tag{3.11}$$

For the general case, we assert that the following holds:

$$\begin{aligned}&[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\\ &= f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)\\ &+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda)f(\mu_2)\cdots f(\mu_M)\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\\ &\vdots\\ &+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)\\ &\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right).\end{aligned} \tag{3.12}$$

Our proof of this statement is based on the induction method: we assume that, for some integer $M \geq 1$, the above formula (i.e. the induction hypothesis) is satisfied and proceed to show that this assumption implicates the same relation for the product of $M+1$ operators. To this end we write

$$\begin{aligned}\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)f(\mu_{M+1})\right] = &[\tau(\lambda), f(\mu_1)\cdots f(\mu_M)]\, f(\mu_{M+1})\\ &+ f(\mu_1)\cdots f(\mu_M)\left[\tau(\lambda), f(\mu_{M+1})\right].\end{aligned} \tag{3.13}$$

To evaluate the first term on the right-hand-side of (3.13) we use the induction assumption (3.12), while in the second term we apply (3.11) and obtain

$$\begin{aligned}&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]\\ &= f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right) f(\mu_{M+1})\end{aligned}$$

$$
\begin{aligned}
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_M)\\
&\times\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)f(\mu_{M+1})\\
&\vdots\\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)\\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M}\frac{2}{\mu_M^2-\mu_j^2}\right)f(\mu_{M+1})\\
&+f(\mu_1)\cdots f(\mu_M)\left(\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}f(\mu_{M+1})\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\right.\\
&\left.+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right)\right).
\end{aligned}
\tag{3.14}
$$

Then, using (3.4), we rearrange the terms having $f(\mu_{M+1})$ on the right

$$
\begin{aligned}
&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]\\
&=f(\mu_1)\cdots f(\mu_{M+1})\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)\\
&+f(\mu_1)\cdots f(\mu_M)\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(\frac{-2}{\lambda^2-\mu_{N+1}^2}\left(f(\mu_{M+1})-f(\lambda)\right)\right)\\
&+f(\mu_1)\cdots f(\mu_{M+1})\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_M)\left(\frac{-2}{\mu_1^2-\mu_{M+1}^2}\left(f(\mu_{M+1})-f(\mu_1)\right)\right)\\
&\vdots\\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)f(\mu_{M+1})\\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right)
\end{aligned}
$$

$$+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)\left(\frac{-2}{\mu_M^2-\mu_{M+1}^2}\left(f(\mu_{M+1})-f(\mu_M)\right)\right)$$
$$+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_M)f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right). \tag{3.15}$$

The next step is to add similar terms appropriately

$$\begin{aligned}
&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right]\\
&=f(\mu_1)\cdots f(\mu_{M+1})\sum_{i=1}^{M}\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M+1}\frac{1}{\lambda^2-\mu_j^2}\right)\\
&+f(\mu_1)\cdots f(\mu_{M+1})\frac{-8\lambda^2}{\lambda^2-\mu_{M+1}^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j=1}^{M}\frac{1}{\lambda^2-\mu_j^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\lambda)f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M+1}\frac{2}{\mu_1^2-\mu_j^2}\right)\\
&\vdots\\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_{M-1})f(\lambda)f(\mu_{M+1})\\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M+1}\frac{2}{\mu_M^2-\mu_j^2}\right)\\
&+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2}f(\mu_1)\cdots f(\mu_M)f(\lambda)\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}\right)\\
&+f(\mu_1)\cdots f(\mu_M)f(\lambda)\sum_{i=1}^{M}\left(\frac{-8\lambda^2}{\lambda^2-\mu_i^2}\frac{2}{\lambda^2-\mu_{M+1}^2}+\frac{8\lambda^2}{\lambda^2-\mu_i^2}\frac{2}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_i^2\nu^2}{\xi^2-\lambda^2\nu^2}\right).
\end{aligned}\tag{3.16}$$

Using the following identity

$$\begin{aligned}
&\frac{-\lambda^2}{\lambda^2-\mu_i^2}\frac{1}{\lambda^2-\mu_{M+1}^2}+\frac{\lambda^2}{\lambda^2-\mu_i^2}\frac{1}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_i^2\nu^2}{\xi^2-\lambda^2\nu^2}\\
&\quad=\frac{\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{1}{\mu_i^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2},
\end{aligned}\tag{3.17}$$

for $i=1,\ldots,N$, we can bring together all the terms in the last two lines of (3.16) and obtain the final expression

$$
\begin{aligned}
&\left[\tau(\lambda), f(\mu_1)\cdots f(\mu_{M+1})\right] \\
&= f(\mu_1)\cdots f(\mu_{M+1}) \sum_{i=1}^{M+1} \frac{-8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M+1}\frac{1}{\lambda^2-\mu_j^2}\right) \\
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\lambda) f(\mu_2)\cdots f(\mu_{M+1})\left(h(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M+1}\frac{2}{\mu_1^2-\mu_j^2}\right) \\
&\vdots \\
&+\frac{8\lambda^2}{\lambda^2-\mu_N^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_{M-1}) f(\lambda) f(\mu_{M+1}) \\
&\times\left(h(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j\neq M}^{M+1}\frac{2}{\mu_M^2-\mu_j^2}\right) \\
&+\frac{8\lambda^2}{\lambda^2-\mu_{M+1}^2}\frac{\xi^2-\mu_{M+1}^2\nu^2}{\xi^2-\lambda^2\nu^2} f(\mu_1)\cdots f(\mu_M) f(\lambda) \\
&\times\left(h(\mu_{M+1})+\frac{\nu^2}{\xi^2-\mu_{M+1}^2\nu^2}-\sum_{j=1}^{M}\frac{2}{\mu_{M+1}^2-\mu_j^2}\right).
\end{aligned}
\tag{3.18}
$$

Since we have already explicitly showed that the induction hypothesis is valid for $M=1$ (the (3.11) is a special case of (3.12)), this completes our proof of (3.12) by induction.

Now, using the result (3.12), we finally find the off shell action (3.10) of the generating function $\tau(\lambda)$ on $\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)$ to be:

$$
\tau(\lambda)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M)=\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) \tag{3.19}
$$

$$
\begin{aligned}
&+\frac{8\lambda^2}{\lambda^2-\mu_1^2}\frac{\xi^2-\mu_1^2\nu^2}{\xi^2-\lambda^2\nu^2}\left(\rho(\mu_1)+\frac{\nu^2}{\xi^2-\mu_1^2\nu^2}-\sum_{j\neq 1}^{M}\frac{2}{\mu_1^2-\mu_j^2}\right)\varphi_M(\lambda,\mu_2,\ldots,\mu_M) \\
&\vdots \\
&+\frac{8\lambda^2}{\lambda^2-\mu_M^2}\frac{\xi^2-\mu_M^2\nu^2}{\xi^2-\lambda^2\nu^2}\left(\rho(\mu_M)+\frac{\nu^2}{\xi^2-\mu_M^2\nu^2}-\sum_{j=1}^{M-1}\frac{2}{\mu_M^2-\mu_j^2}\right)\varphi_M(\mu_1,\ldots,\mu_{M-1},\lambda),
\end{aligned}
\tag{3.20}
$$

and the eigenvalue is

$$
\chi_M(\lambda,\mu_1,\mu_2,\ldots,\mu_M)=\chi_0(\lambda)-\sum_{i=1}^{M}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(h(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\sum_{j\neq i}^{M}\frac{1}{\lambda^2-\mu_j^2}\right).
\tag{3.21}
$$

The above off shell action of the generating function also contains the M unwanted terms which vanish when the following Bethe equations are imposed on the parameters $\mu_1, \dots, \mu_M$,

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{M} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \tag{3.22}$$

where $i = 1, 2, \dots, M$.

Hence we have showed that the symmetric function $\varphi_M(\mu_1, \mu_2, \dots, \mu_M)$ defined in (3.9) is the Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_M(\lambda, \mu_1, \mu_2, \dots, \mu_M)$, stated above (3.21). With this proof we close the topic of the implementation of the algebraic Bethe ansatz for this model.

4. Solutions to the Knizhnik–Zamolodchikov equations

Finding the off-shell action on Bethe vectors in the previous section was, in this approach, a necessary prerequisite for solving of the corresponding Knizhnik–Zamolodchikov equations [25,27]. In this context the local realization of Gaudin algebra basis operators is also relevant:

$$e(\lambda) = -2 \sum_{m=1}^{N} \frac{\xi - \alpha_m \nu}{\lambda^2 - \alpha_m^2} S_m^+, \tag{4.1}$$

$$h(\lambda) = 2 \sum_{m=1}^{N} \frac{1}{\lambda^2 - \alpha_m^2} \left(S_m^3 - \frac{\psi}{2\nu} S_m^+ \right), \tag{4.2}$$

$$f(\lambda) = 2 \sum_{m=1}^{N} \frac{\xi + \alpha_m \nu}{\lambda^2 - \alpha_m^2} \left(S_m^- + \frac{\psi}{\nu} S_m^3 - \frac{\psi^2}{4\nu^2} S_m^+ \right), \tag{4.3}$$

where S_m^3, $S_m^\pm$, are the usual spin generators at the local node m (see [19]). In this local realization the vacuum vector Ω_+ has the form

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}, \tag{4.4}$$

where vector ω_m belongs to local node Hilbert space $V_m = \mathbb{C}^{2s+1}$ and:

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \tag{4.5}$$

The Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function $\tau(\lambda)$ at poles $\lambda = \pm\alpha_m$ [19] and in order to make the paper self contained, we state these result also here:

$$\mathrm{Res}_{\lambda=\alpha_m} \tau(\lambda) \; = \; 4\, H_m \quad \text{and} \quad \mathrm{Res}_{\lambda=-\alpha_m} \tau(\lambda) \; = \; (-4)\, \widetilde{H}_m, \tag{4.6}$$

yielding:

$$H_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\left(K_m(\alpha_m) \vec{S}_m K_m^{-1}(\alpha_m) \right) \cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(\alpha_m) \vec{S}_m K_m^{-1}(\alpha_m) \right)}{2(\alpha_m + \alpha_n)}, \tag{4.7}$$

and

$$\widetilde{H}_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\left(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\right)\cdot \vec{S}_n + \vec{S}_n \cdot \left(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\right)}{2(\alpha_m + \alpha_n)}. \tag{4.8}$$

It follows from the above relations and (3.21) that the eigenvalues of the Gaudin Hamiltonians (4.7) and (4.8) can be derived as the residues of $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$, obtained in the previous section, at the poles $\lambda = \pm\alpha_m$ [19]. It turns out that the respective eigenvalues of the Hamiltonians (4.7) and (4.8) coincide:

$$\begin{aligned}\mathcal{E}_{m,M} &= \frac{1}{4}\mathrm{Res}_{\lambda=\alpha_m}\chi_M(\lambda, \mu_1, \ldots, \mu_M) \\ &= \widetilde{\mathcal{E}}_{m,M} = \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2\nu^2} + \sum_{n\neq m}^{N} \frac{2s_n}{\alpha_m^2 - \alpha_n^2}\right) \\ &- 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2}.\end{aligned} \tag{4.9}$$

When all the spin s_m are set to one half, these energies, as well as the Bethe equations, coincide with the expressions obtained in [14] (up to normalisation; for the connection of the corresponding notations, cf. [19]).

The key observation in what follows will be that by taking the residue of both sides of the equation (3.19) at $\lambda = \alpha_n$, using (4.6), (4.7) and (4.9), and dividing both sides of the equation by the factor of four one obtains

$$\begin{aligned}& H_n\varphi_M(\mu_1, \mu_2, \ldots, \mu_M) = \mathcal{E}_{n,M}\, \varphi_M(\mu_1, \mu_2, \ldots, \mu_M) + \sum_{j=1}^{M} \frac{2\alpha_n^2}{\alpha_n^2 - \mu_j^2}\, \frac{\xi^2 - \mu_j^2\nu^2}{\xi^2 - \alpha_n^2\nu^2} \times \\ & \times \left(\rho(\mu_j) + \frac{\nu^2}{\xi^2 - \mu_j^2\nu^2} - \sum_{k\neq j}^{M} \frac{2}{\mu_j^2 - \mu_k^2}\right) \frac{\xi + \alpha_n\nu}{\alpha_n} \\ & \times \left(S_n^- + \frac{\psi}{\nu}S_n^3 - \frac{\psi^2}{4\nu^2}S_n^+\right)\varphi_{M-1}(\mu_1, \ldots, \widehat{\mu_j}, \ldots, \mu_M)\,,\end{aligned} \tag{4.10}$$

here the notation $\widehat{\mu_j}$ means that the argument μ_j is not present.

The solutions to the Knizhnik–Zamolodchikov equations we seek in the form of contour integrals over the variables $\mu_1, \mu_2, \ldots, \mu_M$ [25,27]:

$$\psi(\alpha_1, \alpha_2, \ldots, \alpha_N) = \oint \cdots \oint \phi(\vec{\mu}|\vec{\alpha})\varphi_M(\vec{\mu}|\vec{\alpha})\, d\mu_1 \cdots d\mu_M, \tag{4.11}$$

where the integrating factor $\phi(\vec{\mu}|\vec{\alpha})$ is a scalar function

$$\phi(\vec{\mu}|\vec{\alpha}) = \exp\left(\frac{S(\vec{\mu}|\vec{\alpha})}{\kappa}\right) \tag{4.12}$$

obtained by exponentiating a function $S(\vec{\mu}|\vec{\alpha})$ [34]. As in [10], from now on, the K-matrix parameters take fixed values $\psi = \xi = 0$ and $\nu = 1$. For these values it is straightforward to check that i) the Gaudin Hamiltonians are Hermitian; and ii) Hamiltonians (4.7) and (4.8) coincide.

We find that the proper form of $S(\vec{\mu}|\vec{\alpha})$ in this case is:

$$\begin{aligned} S(\vec{\mu}|\vec{\alpha}) = \sum_{n=1}^{N} \frac{s_n(s_n-1)}{2\ln(\alpha_n)} + \sum_{n<m}^{N} \alpha_n\alpha_m \, \ln(\alpha_n^2-\alpha_m^2) + \sum_{j=1}^{M} \ln(\mu_j) \\ + \sum_{j<k}^{M} \ln(\mu_j^2-\mu_k^2) - \sum_{j=1}^{M}\sum_{n=1}^{N} s_n \ln(\alpha_n^2-\mu_j^2). \end{aligned} \tag{4.13}$$

In order to show this, it is important to notice that the function $\phi(\vec{\mu}|\vec{\alpha})$ as defined above also satisfies the following equations

$$\kappa\, \partial_{\alpha_n}\phi = \mathcal{E}_{n,M}\, \phi, \tag{4.14}$$

$$\kappa\, \partial_{\mu_j}\phi = \beta_M(\mu_j)\, \phi, \tag{4.15}$$

where

$$\beta_M(\mu_j) := -\mu_j \left(\rho(\mu_j) - \frac{1}{\mu_j^2} - \sum_{k\neq j}^{M} \frac{2}{\mu_j^2-\mu_k^2} \right). \tag{4.16}$$

Introducing the notation

$$\widetilde{\varphi}_{M-1}^{(j,n)} := S_n^- \; \varphi_{M-1}(\mu_1,\ldots,\widehat{\mu_j},\ldots,\mu_M) \tag{4.17}$$

the equation (4.10) can be expressed in the following form

$$H_n\varphi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{E}_{n,M}\; \varphi_M(\mu_1,\mu_2,\ldots,\mu_M) + \sum_{j=1}^{M} \frac{(-2)\mu_j}{\alpha_n^2-\mu_j^2}\, \beta_M(\mu_j)\, \widetilde{\varphi}_{M-1}^{(j,n)}. \tag{4.18}$$

Using the definition of φ_M (3.9) and the local realisation of the generator $f(\mu)$ (4.3) it follows that

$$\partial_{\alpha_n}\varphi_M = (-2)\sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{\mu_j\, \widetilde{\varphi}_{M-1}^{(j,n)}}{\mu_j^2-\alpha_n^2} \right). \tag{4.19}$$

Then it is straightforward to show that

$$\kappa\, \partial_{\alpha_n}(\phi\varphi_M) = H_n(\phi\varphi_M) + \kappa \sum_{j=1}^{M} \partial_{\mu_j} \left(\frac{(-2)\mu_j}{\mu_j^2-\alpha_n^2} \phi \widetilde{\varphi}_{M-1}^{(j,n)} \right). \tag{4.20}$$

A closed contour integration of $\phi\varphi_M$ with respect to the variables $\mu_1,\mu_2,\ldots,\mu_M$ will cancel the contribution from the terms under the sum in (4.20) and therefore $\psi(\alpha_1,\alpha_2,\ldots,\alpha_N)$ given by (4.11) satisfies the Knizhnik–Zamolodchikov equations

$$\kappa\, \partial_{\alpha_n}\psi(\alpha_1,\alpha_2,\ldots,\alpha_N) = H_n\psi(\alpha_1,\alpha_2,\ldots,\alpha_N). \tag{4.21}$$

Moreover, the interplay between the Gaudin model and the Knizhnik–Zamolodchikov equations, once the Bethe equations are imposed

$$\frac{\partial S}{\partial \mu_j} = \beta_M(\mu_j) = -\mu_j \left(\sum_{m=1}^{N} \frac{2s_m}{\mu_j^2 - \alpha_m^2} - \frac{1}{\mu_j^2} - \sum_{k \neq j}^{M} \frac{2}{\mu_j^2 - \mu_k^2} \right) = 0, \tag{4.22}$$

enabled us to determine the on-shell norm of the Bethe vectors

$$||\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)||^2 = 2^M \det \left(\frac{\partial^2 S}{\partial \mu_j \partial \mu_k} \right). \tag{4.23}$$

It turns out to be possible to derive also a stronger formula than the one above for the norms [27]. Indeed, we calculate the following expression for the off-shell scalar product of arbitrary two Bethe vectors:

$$\Omega_+^* e(\lambda_1) e(\lambda_2) \cdots e(\lambda_M) f(\mu_M) \cdots f(\mu_2) f(\mu_1) \Omega_+ = 4^M \sum_{\sigma \in S_M} \det \mathcal{M}^\sigma \ , \tag{4.24}$$

where S_M is the symmetric group of degree M and the $M \times M$ matrix $\mathcal{M}^\sigma$ is given by

$$\mathcal{M}_{jj}^\sigma = -\frac{\lambda_j^2 \rho(\lambda_j) - \mu_{\sigma(j)}^2 \rho(\mu_{\sigma(j)})}{\lambda_j^2 - \mu_{\sigma(j)}^2} - \sum_{k \neq j} \frac{\lambda_k^2 + \mu_{\sigma(k)}^2}{(\lambda_j^2 - \lambda_k^2)(\mu_{\sigma(j)}^2 - \mu_{\sigma(k)}^2)} \ , \tag{4.25}$$

$$\mathcal{M}_{jk}^\sigma = -\frac{\lambda_k^2 + \mu_{\sigma(k)}^2}{(\lambda_j^2 - \lambda_{\sigma(k)}^2)(\mu_{\sigma(j)}^2 - \mu_{\sigma(k)}^2)} \ , \quad \text{for} \quad j, k = 1, 2, \ldots, M. \tag{4.26}$$

This formula (that can be proved by commuting $e(\lambda)$ operators to the right and using mathematical induction) has obvious potential applications as the first step towards the general correlation functions. It should be noted that in [13] a related problem was analysed in the trigonometric case and under certain restrictions: local spins were all fixed to the value $\frac{1}{2}$ and it was required that $N = 2M$ (in the notation of that paper). Our formula is more compact and valid for arbitrary spins and arbitrary number of excitations.

5. Conclusion

In this paper we addressed a number of open problems related to Gaudin model with non periodic boundary conditions.

First, we obtained a new basis of the generalized $s\ell(2)$ Gaudin algebra, in which the commutation relations and the generating function are manifestly simpler. This step allowed us to calculate Bethe vectors and off-shell action of the generating function upon them in a closed form, for arbitrary number of excitations. The obtained expressions we have proved by mathematical induction.

Once having the general expressions for the Bethe vectors and for the corresponding eigenvalues, we could proceed to relate KZ equations with the Bethe vectors. Taking residues of the off-shell action at poles $\pm\alpha_m$, we obtained both Gaudin Hamiltonians and their eigenvalues. By finding the appropriate form of the function S in (4.13), we managed to establish and prove relations (4.14) and (4.15) which led to solution to KZ equations. Proceeding in the same framework, we also obtained the expression for norms of Bethe vectors on shell. Moreover, we went a step further and provided a closed form formula for the scalar product of arbitrary two Bethe vectors.

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Dynamical energy loss formalism: from describing suppression patterns to implications for future experiments

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Abstract

We overview our recently developed DREENA-C and DREENA-B frameworks, where DREENA (Dynamical Radiative and Elastic ENergy loss Approach) is a computational implementation of the dynamical energy loss formalism; C stands for constant temperature and B for the medium evolution modeled by Bjorken expansion. At constant temperature our predictions overestimate v_2, in contrast to other models, but consistent with simple analytical estimates. With Bjorken expansion, we obtain good agreement with both R_{AA} and v_2 measurements. We find that introducing medium evolution has a larger effect on v_2 predictions, but for precision predictions it has to be taken into account in R_{AA} predictions as well. We also propose a new observable, which we call *path length sensitive suppression ratio*, for which we argue that the path length dependence can be assessed in a straightforward manner. We also argue that $Pb + Pb$ vs. $Xe + Xe$ collisions make a good system to assess the path length dependence. As an outlook, we expect that introduction of more complex medium evolution (beyond Bjorken expansion) in the dynamical energy loss formalism can provide a basis for a state of the art QGP tomography tool – e.g. to jointly constrain the medium properties from the point of both high-$p_\perp$ and low-$p_\perp$ data.

Keywords: relativistic heavy ion collisions, quark-gluon plasma, energy loss, hard probes, heavy flavor

1. Introduction

Energy loss of high-$p_\perp$ particles traversing QCD medium is considered to be an excellent probe of QGP properties [1, 2, 3]. The theoretical predictions can be generated and compared with a wide range of experimental data, coming from different experiments, collision systems, collision energies, centralities, observables. This comprehensive comparison of theoretical predictions and high $p_\perp$ data, can then be used together with low $p_\perp$ theory and data to study the properties of created QCD medium [4, 5, 6, 7], that is, for precision QGP tomography. However, to implement this idea, it is crucial to have a reliable high $p_\perp$ parton energy loss model. With this goal in mind, during the past several years, we developed the dynamical energy loss formalism [8]. Contrary to the widely used approximation of static scattering centers, this model

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takes into account that QGP consists of dynamical (moving) partons, and that the created medium has finite size. The calculations are based on the finite temperature field theory, and generalized HTL approach. The formalism takes into account both radiative and collisional energy losses, is applicable to both light and heavy flavor, and has been recently generalized to the case of finite magnetic mass and running coupling [9]. Most recently, we also relaxed the soft-gluon approximation within the model [15]. Finally, the formalism is integrated in an up-to-date numerical procedure [9], which contains parton production [10], fragmentation functions [11], path-length [12, 13] and multi-gluon fluctuations [14].

The model up-to-now explained a wide range of R_{AA} data [9, 16, 17, 18], with the same numerical procedure, the same parameter set, and with no fitting parameters, including explaining puzzling data and generating predictions for future experiments. This then strongly suggests that the model provides a realistic description of high $p_\perp$ parton-medium interactions. However, the model did not take into account the medium evolution, so we used it to provide predictions only for those observables that are considered to be weakly sensitive to QGP evolution.

Therefore, our goal, which will be addressed in this proceedings, is to develop a framework which will allow systematic comparison of experimental data and theoretical predictions, obtained by the same formalism and the same parameter set. In particular, we want to develop a framework, which can systematically generate predictions for different observables (both R_{AA} and v_2), different collision systems ($Pb + Pb$ and $Xe+Xe$), different probes (light and heavy), different collision energies and different centralities [19, 20, 21]. Within this, our major goal is to introduce medium evolution in the dynamical energy loss formalism [20], where we start with 1+1D Bjorken expansion [22], and where our developments in this direction, will also be outlined in this proceedings. Finally, we also want to address an important question of how to differentiate between different energy loss models; in particular, what is the appropriate observable, and what are appropriate systems, to assess energy loss path-length dependence [21]. Note that only the main results are presented here; for a more detailed version, see [19, 20, 21], and references therein.

2. Results and discussion

As a first step towards the goals specified above, we developed DREENA-C framework [19], which is a fully optimized computational suppression procedure based on our dynamical energy loss formalism in constant temperature finite size QCD medium. Within this framework we, for the first time, generated joint R_{AA} and v_2 predictions based on our dynamical energy loss formalism. We generated predictions for both light and heavy flavor probes, and different centrality regions in $Pb + Pb$ collisions at the LHC (see [19] for more details). We obtained that, despite the fact that DREENA-C does not contain medium evolution (to which v_2 is largely sensitive), it leads to qualitatively good agreement with this data, though quantitatively, the predictions are visibly above the experimental data.

The theoretical models up-to-now, faced difficulties in jointly explaining R_{AA} and v_2 data, i.e. lead to underprediction of v_2, unless new phenomena are introduced, which is known as v_2 puzzle [23]. Having this in mind, the overestimation of v_2, obtained by DREENA-C, seems surprising. However, by using a simple scaling arguments, where fractional energy loss is proportional to T^a and L^b , and where, within our model a, b are close to 1, we straightforwardly obtain that in constant T medium, $R_{AA} \approx 1 - \xi T L$ and $v_2 \approx \frac{\xi T \Delta L}{2}$, while in evolving medium R_{AA} retains the same expressions and $v_2 \approx \frac{\xi T \Delta L - \xi \Delta T L}{2}$ (see [19] for more details, ξ is a proportionality factor that depends on initial jet $p_\perp$). So, it is our expectation that, within our model, the medium evolution will not significantly affect R_{AA}, while it will notably lower the v_2 predictions.

To check the reliability of these simple estimates, we developed DREENA-B framework [20], which is our most recent development within dynamical energy loss formalism. Here B stands for 1+1D Bjorken expansion [22], i.e. the medium evolution is introduced in dynamical energy loss formalism in a simple analytic way. We provided first joint R_{AA} and v_2 predictions with dynamical energy loss formalism in expanding QCD medium, which are presented in Fig. 1 (for charged hadrons), and we observe very good agreement with both R_{AA} and v_2 data. We equivalently obtained the same good agreement for D mesons, and predicted non-zero v_2 for high $p_\perp$ B mesons.

In Fig. 2, we further present predictions for $Xe + Xe$ data [21], where we note that these predictions were generated before the data became available. In this figure (see also Fig. 1), we compare DREENA-C and

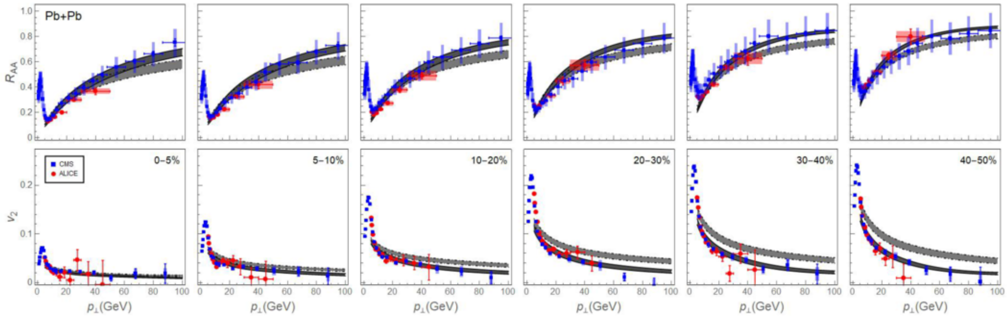

Fig. 1. **Joint R_{AA} and v_2 predictions for charged hadrons in 5.02 TeV $Pb+Pb$ collisions.** *Upper panels:* Predictions for R_{AA} *vs.* $p_\perp$ are compared with ALICE [24] (red circles) and CMS [25] (blue squares) charged hadron experimental data in 5.02 TeV $Pb+Pb$ collisions. *Lower panels:* Predictions for v_2 *vs.* $p_\perp$ are compared with ALICE [26] (red circles) and CMS [27] (blue squares) experimental data in 5.02 TeV $Pb+Pb$ collisions. Full and dashed curves correspond, respectively, to the predictions obtained with DREENA-B and DREENA-C frameworks. In each panel, the upper (lower) boundary of each gray band corresponds to $\mu_M/\mu_E = 0.6$ ($\mu_M/\mu_E = 0.4$). Columns 1-6 correspond, respectively, to $0-5\%$, $5-10\%$, $10-20\%$,..., $40-50\%$ centrality regions. The figure is adapted from [19, 20] and the parameter set is specified there.

DREENA-B frameworks, to assess the importance of including medium evolution on R_{AA} and v_2 observables. We see that introduction of expanding medium affects both R_{AA} and v_2 data. That is, it systematically somewhat increase R_{AA}, while significantly decreasing v_2; this observation is in agreement with our estimate provided above. Consequently, we see that this effect has large influence on v_2 predictions, confirming previous arguments that v_2 observable is quite sensitive to medium evolution. On the other hand, this effect is rather small on R_{AA}, consistent with the notion that R_{AA} is not very sensitive to medium evolution [28, 29]. However, our observation from Figs. 1 and 2 is that medium evolution effect on R_{AA}, though not large, should still not be neglected in precise R_{AA} calculations, especially for high $p_\perp$ and higher centralities.

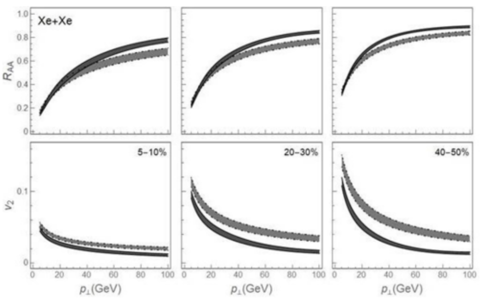

Fig. 2. **Joint R_{AA} and v_2 predictions for charged hadrons in 5.44 TeV $Xe+Xe$ collisions.** Predictions for R_{AA} *vs.* $p_\perp$ and v_2 *vs.* $p_\perp$ are shown on upper and lower panels, respectively. Columns 1-3, respectively, correspond to $5-10\%$, $20-30\%$ and $40-50\%$ centrality regions. Full and dashed curves correspond, respectively, to the predictions obtained with DREENA-B and DREENA-C frameworks. The figure is adapted from [20] and the parameter set is specified there.

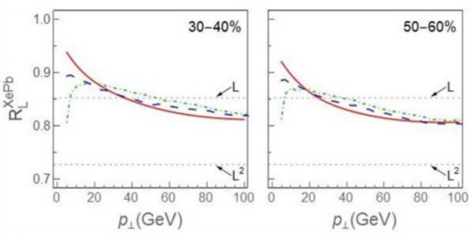

Fig. 3. **Path-length sensitive suppression ratio (R_L^{XePb}) for light and heavy probes.** Predictions for R_L^{XePb} *vs.* $p_\perp$ is shown for charged hadrons (full), D mesons (dashed) and B mesons (dot-dashed). First and second column, respectively, correspond to $30-40\%$ and $50-60\%$ centrality regions. $\mu_M/\mu_E = 0.4$. The figure is adapted from [21] and the parameter set is specified there.

Finally, as the last topic of this proceedings, we address a question on how to differentiate between different energy loss models. With regard to this, note that path length dependence provides an excellent signature differentiating between different energy loss models, and consequently also between the underlying energy loss mechanisms. For example, some energy loss models have linear, some have quadratic, and our dynamical energy loss has the path-length dependence between linear and quadratic, which is due to both collisional and radiative energy loss mechanisms included in the model. To address this question, we first have to answer what is an appropriate system for such a study. We argue that comparison of suppressions in $Pb+Pb$ and $Xe+Xe$ is an excellent way to study the path length dependence: From the suppression calculation perspective, almost all properties of these two systems are the same. That is, we show [21] that these two systems have very similar initial momentum distributions, average temperature for each centrality region and path length distributions (up to rescaling factor $A^{1/3}$). That is, the main property differentiating the two systems is its size, i.e. rescaling factor $A^{1/3}$, which therefore makes comparison of suppressions in $Pb+Pb$ and $Xe+Xe$ collisions an excellent way to study the path length dependence.

The second question is what is appropriate observable? With regards to that, the ratio of the two R_{AA} seems a natural choice, as has been proposed before. However, in this way the path length dependence cannot be naturally extracted, as shown in [21]. For example, this ratio approaches one for high $p_\perp$ and high centralities, suggesting no path length dependence, while the dynamical energy loss has strong path length dependence. Also, the ratio has strong centrality dependence. That is, from this ratio, no useful information can be deduced. The reason for this is that this ratio includes a complicated relationship (see [21] for more details) which depends on the initial jet energy and centrality; so extracting the path-length dependence from this observable would not be possible.

However, based on the derivation presented in [21], we propose to use the ratio of 1-R_{AA} instead. From this estimate, we see that this ratio $R_L^{XePb} \equiv \frac{1-R_{XeXe}}{1-R_{PbPb}} \approx \left(\frac{A_{Xe}}{A_{Pb}}\right)^{b/3}$ has a simple dependence on only the size of the medium ($A^{1/3}$ ratio) and the path length dependence (exponent b). In Fig. 3 we plot this ratio, where we see that the path length dependence can be extracted from this ratio in a simple way, and moreover there is only a weak centrality dependence. Therefore, 1-R_{AA} ratio seems as a natural observable, which we propose to call *path-length sensitive suppression ratio*.

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Permutation-symmetric three-particle hyper-spherical harmonics based on the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ subgroup chain

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Abstract

We construct the three-body permutation symmetric hyperspherical harmonics to be used in the non-relativistic three-body Schrödinger equation in three spatial dimensions (3D). We label the state vectors according to the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ subgroup chain, where S_3 is the three-body permutation group and S_2 is its two element subgroup containing transposition of first two particles, $O(2)$ is the "democracy transformation", or "kinematic rotation" group for three particles; $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. We discuss the good quantum numbers implied by the above chain of algebras, as well as their relation to the S_3 permutation properties of the harmonics, particularly in view of the $SO(3)_{rot} \subset SU(3)$ degeneracy. We provide a definite, practically implementable algorithm for the calculation of harmonics with arbitrary finite integer values of the hyper angular momentum K, and show an explicit example of this construction in a specific case with degeneracy, as well as tables of $\mathrm{K} \leq 6$ harmonics. All harmonics are expressed as homogeneous polynomials in the Jacobi vectors $(\boldsymbol{\lambda}, \boldsymbol{\rho})$ with coefficients given as algebraic numbers unless the "operator method" is chosen for the lifting of the $SO(3)_{rot} \subset SU(3)$ multiplicity and the dimension of the degenerate subspace is greater than four – in which case one must resort to numerical diagonalization; the latter condition is not met by any $\mathrm{K} \leq 15$ harmonic, or by any $L \leq 7$ harmonic with arbitrary K. We also calculate a certain type of matrix elements (the Gaunt integrals of products of three harmonics) in two ways: 1) by explicit evaluation of integrals and 2) by reduction to known $SU(3)$ Clebsch–Gordan coefficients. In this way we complete the

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calculation of the ingredients sufficient for the solution to the quantum-mechanical three-body bound state problem.

1. Introduction

The quantum mechanical three-body bound-state problem is an old one: it has a huge literature in which the hyperspherical harmonics, Refs. [1–23], form one of the most firmly established theoretical tools – for recent reviews, see Refs. [24–29].[1] Classification of wave functions into distinct classes under permutation symmetry is a fundamental property of non-relativistic quantum mechanics with non-trivial consequences in the three-body system. Permutation symmetric three-body hyperspherical harmonics in three dimensions, however, are explicitly known only in a few special cases such as the total orbital angular momentum $L = 0, 1$ ones, cf. Refs. [5,6,8,19]. Hyperspherical harmonics with higher values of L can be constructed by means of a (numerical) recursive procedure that symmetrizes non-permutation-symmetric hyperspherical harmonics, see Refs. [20,21].

In so doing one loses track, however, of a certain dynamical $O(2)$ symmetry that is related to the so-called "kinematic rotation", Ref. [2], or equivalently to the "democracy" transformations, Refs. [6,8,12]. This "kinematic rotation" invariance, or "democracy" symmetry was viewed as mathematical esoterics, until recently Ref. [30] showed it to be the dynamical symmetry of area-dependent potentials, which class includes the so-called Y-string potential in QCD. The Y-string is the leading candidate for the confinement mechanism of three quarks in QCD, Refs. [30–34]. Consequently followed the increased interest in the properties of the "kinematic rotation" invariant, or "democratic" potentials and in their spectra.

In two spatial dimensions (2D) the problem of constructing permutation symmetric hyperspherical harmonics was solved, at first by Smith, Ref. [4], and then in Ref. [35] in a general way that makes the "kinematic rotation" $O(2)$ invariance (or "democracy" symmetry) explicit, following certain fairly abstract internal geometric ("shape space") considerations by Iwai, Refs. [36,37]. These 2D permutation-symmetrized $SO(4)$ hyperspherical harmonics are closely related to the (3D magnetic) monopole harmonics, Ref. [38], and to so-called spin-weighted spherical harmonics, Ref. [39]. In Refs. [40–42] we have used these symmetrized hyperspherical harmonics with the "kinematic rotation" $O(2)$ label to solve the Schrödinger equation for three-body bound states in two spatial dimensions with area-dependent potentials based on the $so(2) \oplus so_L(2) \subset so(3) \oplus so(3) \subset so(4)$ chain of algebras (where $so_L(2)$ is the total angular momentum part and $so(2)$ is the "democracy" transformation, or "kinematic rotation" generator). Those results show explicitly the role of the "kinematic rotation" $O(2)$ invariance in the energy level-degeneracy and/or splitting in area-dependent potentials in 2D.

Similarly to the three-body problem in two dimensions (2D), Refs. [40–42], the knowledge of the three-body permutation symmetric hyperspherical harmonics in three dimensions (3D) with the "kinematic rotation" $O(2)$ label would allow one to calculate the discrete part of the energy spectrum of the three-body problem. A systematic construction of (all) permutation symmetric

[1] It is commonly assumed that Faddeev's work on quantum-mechanical three-body equations has solved the three-body problem – that is only partially true: Faddeev's equations allow one to solve the three-body scattering problem, but do not affect the bound state problem significantly.

hyperspherical harmonics in 3D, based on the $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$ "chain" of subgroups for labeling purposes, is the first basic contribution of the present paper. This construction is complete in the sense that a definite algorithm is provided for the construction of arbitrary-integer-K harmonics (where K is the hyper angular, or "grand angular" momentum), that has been used to construct harmonics up to some finite value of K. We do not have a simple formula for arbitrary-K harmonics, however. As the second basic contribution of this paper, we provide explicit results of the evaluation of an integral over tri-linear products of harmonics (this is the $SO(6)$ analogon of the Gaunt formula), in terms of (known) $SU(3)$ Clebsch–Gordan coefficients.

The basic idea that we used is not new: we started out by constructing certain homogeneous polynomials of the two Jacobi vectors, just as Simonov did in Ref. [5], but without the restriction to scalars under spatial rotations. In so doing we used the $O(6)$ group labels to classify the hyperspherical harmonics. In this way the three-body problem in three dimensions can be effectively reduced to an $O(6)$ group theoretical problem. By a careful study of the labeling of three-body states, we arrive at the subgroup "chain" $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. Here $SO_{rot}(3)$ is the total angular momentum part and $O(2)$ is the subgroup of so-called "democracy" transformations, Ref. [12], or equivalently "kinematic rotations", Ref. [2], and the S_3 and S_2 are the (discrete) three-particle permutation group and the two-particle permutation (of particles 1 and 2) group, respectively, that are also subgroups of these "democracy" transformations.

As the first step, we construct "core polynomials" of order K that have particular predefined transformation properties w.r.t. the $U(1) \otimes SO(3)_{rot}$ subgroup action, where $U(1) \equiv SO(2)$ is the unit determinant subgroup of the $O(2)$ group of "democracy" transformations containing only cyclic, i.e., even particle permutations. In the following step these core polynomials are "filtered out", i.e., projected so as to become harmonic, i.e., to obtain sharp values of the hyperangular momentum K.

The obtained harmonic polynomials, however, are still plagued with what is known as the multiplicity problem in $SU(3)$ group theory, Refs. [43–47]: in general, the labels of the $U(1) \otimes SO(3)_{rot}$ subgroup together with the hyperangular momentum K are insufficient to uniquely specify the $SO(6)$ harmonics. We offer two possible solutions to this "multiplicity problem": i) the traditional approach, Refs. [43–47], based on the introduction of a multiplicity lifting operator, that must be diagonalized, where we discuss several different such operators, primarily in the light of the (so-induced) permutation properties of the harmonics; and ii) a novel (non-traditional) approach, based on a new auxiliary integer label, that is introduced in the process of constructing the harmonics. Both of these choices present definite algorithms for the construction of an arbitrary (positive integer) K-th order $SO(6)$ three-body permutation-symmetric hyperspherical harmonic, albeit with different advantages and drawbacks.

In the first case, the resulting hyperspherical harmonics can be, in general, expressed in closed algebraic form only when K ≤ 15 and/or $L \leq 7$, whereas, beyond K ≥ 16 and $L \geq 8$ some harmonics have to be expressed numerically, due to restrictions imposed by Galois theory. Consequently, such harmonics cannot be used for the study of arbitrary-K, L properties, e.g. the Regge trajectories, of three-body states. We present here the $SO(6)$ three-body permutation-symmetric hyperspherical harmonics, based on the Racah degeneracy-lifting operator, Ref. [43], together with their transformation properties under permutations, i.e., the irreducible representations of the permutation group S_3.

In the second case, the multiplicity labeling procedure does not rely on solving any operator eigenvalue problem, so that *all hyperspherical harmonics can be expressed in a closed algebraic* *form*. Such a significant simplification comes at a price, however, *viz.* the new auxiliary label does not have a clear $SU(3)$ group-theoretical meaning. Consequently, it has not been used to evaluate the corresponding Clebsch–Gordan coefficients in the literature (see below). For this reason, here we shall merely state some of the implications of this choice and proceed with further discussion based exclusively on the first type of solution to the multiplicity problem.

Hyperspherical harmonics obtained through the described procedure are labeled according to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$, plus the multiplicity label (that may, or may not be related to the $so(6)$ enveloping algebra). However, odd particle permutations do not belong to this chain, as they correspond to transformations of the six-dimensional configuration space with determinant equal to -1. In the concluding step of construction of the permutation symmetric harmonics we discuss their action (in a separate section on permutation properties) which then extends the symmetry group/chain to $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. The final result are hyperspherical harmonics with clearly established and manifest S_3 permutation properties, that are simple linear combinations of the previously derived $SO(6)$ harmonic functions.

Then we calculated a certain class of integrals (matrix elements) over trilinear products of hyperspherical harmonics that appear in standard quantum-mechanical three-body problems, Ref. [48–50]. We did so firstly by explicit evaluation, i.e., by a reduction to certain gamma (Γ) function type of integrals, and secondly by group theoretical techniques, i.e., by reduction, at first to a product of two $SO(6)$ Clebsch–Gordan coefficients, which are not well known, and then we used a one-to-one relation between the $SO(6)$ hyperspherical harmonics and the $SU(3)$ irreducible representations, to express the integrals as a product of two $SU(3)$ Clebsch–Gordan coefficients, which are quite well known, [51–57]. We do not foresee further simplifications of our results, at least not in matters of principle, though we cannot exclude potential improvements of numerical algorithms used for their evaluation. In this way, we have reduced these integrals over trilinear products of hyperspherical harmonics to their simplest form that is also amenable to straightforward numerical evaluation.

Our results are not specific to any particular three-body problem, i.e., they can, and we hope they will, find application in many realistic 3D three-body problems, such as in the three-quark problem in hadronic physics, as well as in atomic and molecular physics.

As stated above, symmetrized three-body hyper-spherical harmonics have been pursued before, albeit without emphasis on the "kinematic rotation" $O(2)$ symmetry label. To our knowledge, aside from the special case ($L = 0$) results of Simonov and of Dragt, Refs. [5–7], and the $L = 1$ results of Lévy-Leblond and of Barnea and Mandelzweig, Refs. [8,19], several other attempts, based on the so-called "tree pruning" techniques, exist in the literature, Refs. [18,23,28], beside the recursively symmetrized N-body hyperspherical harmonics of Barnea and Novoselsky, Refs. [20,21]. The latter are based on the $O(3) \otimes S_N \subset O(3N - 3)$ chain of algebras, which does not include the "kinematic rotation"/"democracy" $O(2)$ symmetry. Moreover, no explicit examples of three-body symmetrized hyperspherical harmonics were given in Refs. [20,21], as they were meant primarily for numerical computations, and not for fundamental studies.

In several early papers, Refs. [6–8], and, somewhat later, also in Refs. [14,15], the same $SU(3)$ group was used to label and construct some three-particle continuum states with $\mathrm{K} \leq 12$, but their applications to bound state problems was not considered. Refs. [14,15] are particularly close in spirit to our approach, albeit not in technical detail. For a fuller discussion of these other approaches and their relation to the present work, see Sect. 9.

This paper consists of nine sections. After providing the necessary preliminaries in Sect. 2, we explain our $SO(6)$ algebraic methods for constructing the core polynomials in Sect. 3. Then, we project out the core polynomials to get harmonic three-body hyper-spherical polynomials in Sect. 4. In Sect. 5 we discuss how to resolve the multiplicity of three-body hyper-spherical harmonic polynomials in general, and in Sect. 6 we illustrate the procedure with a few examples. Then in Sect. 7 we discuss the permutation symmetry and define final expressions for the harmonics that possess simple and manifest transformation properties with respect to the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset O(6)$. In Sect. 8 we show the calculation of a certain type of matrix elements, and discuss their group theoretical ramifications. Finally, in Section 9 we present a summary and discussion of the results and of their relation to other papers in the literature. Some useful integrals are shown in Appendix A, the details of $SO(6)$ Clebsch–Gordan coefficients are shown in Appendix B, and a list of h.s. harmonics with $K \leq 6$ is given in Appendix C.

2. Preliminaries

2.1. Three-body hyper-spherical harmonics

Coordinates of three (identical) particles (with equal masses) in the center-of-mass (c.m.) rest frame are given by two Jacobi three-vectors:

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \tag{1}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

The kinetic energy in the rest frame is of the form:

$$T = \frac{m}{2}\left(\dot{\boldsymbol{\lambda}}^2 + \dot{\boldsymbol{\rho}}^2\right), \tag{3}$$

possessing an $O(6)$ symmetry that is made manifest by introducing six-dimensional coordinate hyper-vector $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$: the kinetic energy Eq. (3) can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{\mathrm{K}_{\mu\nu}^2}{2mR^2} \tag{4}$$

where $R \equiv \sqrt{\boldsymbol{\lambda}^2 + \boldsymbol{\rho}^2}$ is the hyper-radius and the "grand angular", or hyper-angular momentum tensor $\mathrm{K}_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, 6$ reads

$$\begin{aligned} \mathrm{K}_{\mu\nu} &= m\left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu\right) \\ &= \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu\right). \end{aligned} \tag{5}$$

It has 15 linearly independent components and generates an $SO(6)$ group acting in this six-dimensional space. Among these 15 generators are also the three components of the "ordinary" (total) orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}}\right)$. In addition to the $SO(6)$ group action that generates linear transformations of the 6-dimensional space with unit determinant, particle permutations also constitute a part of the symmetries of the kinetic energy, Eq. (3). The odd permutations, however, correspond to six-dimensional linear transformations with determinant equal to -1, thus extending the full symmetry group to $O(6)$.

Once the potential energy V is introduced, this large symmetry is generally broken to some extent, sometimes all the way down to the product of the three-body permutation symmetry S_3

and the rotation symmetry $SO(3)$ (or $O(3)$), and sometimes with an additional remnant dynamical symmetry. This is a motivation to split the three-particle wave-function into a hyper-radial part (which is a function solely of the hyper-radius R) and the hyper-angular part (which is a function of x^{μ}/R), where the natural basis for the latter one are 6-dimensional hyper-spherical harmonics.

Hyper-spherical harmonics (in any dimension D) transform as symmetric tensor representations of $SO(D)$ group, which is (for $D > 3$) only a subset of all tensorial representations. In turn, this means that any hyper-spherical harmonic is labeled by a single integer K that is also an irreducible representation label, matching the order of the symmetric tensor representation (K corresponds to the highest weight $(\mathrm{K}, 0, 0, 0, \dots)$ irreducible representation, following the usual definitions, or to the Young diagram with K boxes in a single row). Ordinary ($D = 3$) spherical harmonics are eigenfunctions of the square of the angular momentum operator K (this operator is, in the $D = 3$ context, usually denoted as L) with the eigenvalue $\mathrm{K}(\mathrm{K} + 1)$, whereas D-dimensional hyper-spherical harmonics are eigenfunctions of the square of the hyper-angular momentum operator K with the eigenvalue $\mathrm{K}(\mathrm{K} + D - 2)$.

In addition to the irreducible representation label K, hyper-spherical harmonics carry additional labels specifying a concrete vector within that representation, usually describing the transformation properties of the hyper-spherical harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group $SO(D)$.

In the context of three-particle wave functions, additional labels ought to be introduced in a way that respects physically important features, i.e., the remnant symmetries of the system in question. As most three-body potentials in physics are rotationally invariant, the hyper-spherical harmonics should have definite transformation properties under rotations, i.e., they should carry labels L and m (the "magnetic" quantum number) of the rotational subgroup $SO(3)$. Permutation symmetry is often a remnant symmetry, so once we construct the $SO(6)$ hyper-spherical harmonics we shall address the question of how the particle transpositions, i.e., the full $O(6)$ group, act upon them.

2.2. The $SO(6)$ group structure

The rotational group here appears as the diagonal $SO(3)$ subgroup of the six-dimensional rotations $SO(3)_{rot} = SO(3)_{diag} \subset SO(6)$, i.e., the rotations act equally on the first three coordinates ($\boldsymbol{\lambda}$) and the last three coordinates ($\boldsymbol{\rho}$) of the six-dimensional coordinate x_{μ}. As we shall shortly demonstrate, the space of 15 generators of the $SO(6)$ Lie algebra decomposes as $(3)_{rot} + (3) + (3) + (5) + (1)$ w.r.t. $SO(3)_{rot}$, so there is exactly one "additional" generator of $SO(6)$ that commutes with the rotations. This decomposition becomes manifest upon introduction of complex coordinates:

$$X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3. \tag{6}$$

One basis of the $so(6)$ algebra generating $SO(6)$ transformations of hyper-coordinates x_{μ} is given by the 15 operators $\{\mathrm{K}_{\mu\nu} \equiv i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) | \mu, \nu = 1, \dots 6\}$. Of a greater physical significance is the following basis, written in terms of the new coordinates:

$$L_{ij} \equiv -i\left(X_i^{+}\frac{\partial}{\partial X_j^{+}} + X_i^{-}\frac{\partial}{\partial X_j^{-}} - X_j^{+}\frac{\partial}{\partial X_i^{+}} - X_j^{-}\frac{\partial}{\partial X_i^{-}}\right), \tag{7}$$

$$Q_{ij} \equiv \frac{1}{2}\left(X_i^{+}\frac{\partial}{\partial X_j^{+}} - X_i^{-}\frac{\partial}{\partial X_j^{-}} + X_j^{+}\frac{\partial}{\partial X_i^{+}} - X_j^{-}\frac{\partial}{\partial X_i^{-}}\right), \tag{8}$$

$$\Delta L_{ij} \equiv -i\left(X_i^+\frac{\partial}{\partial X_j^-}+X_i^-\frac{\partial}{\partial X_j^+}-X_j^+\frac{\partial}{\partial X_i^-}-X_j^-\frac{\partial}{\partial X_i^+}\right), \tag{9}$$

$$W_{ij} \equiv \left(X_i^+\frac{\partial}{\partial X_j^-}-X_i^-\frac{\partial}{\partial X_j^+}-X_j^+\frac{\partial}{\partial X_i^-}+X_j^-\frac{\partial}{\partial X_i^+}\right). \tag{10}$$

Of these, the L_{ij} (corresponding to orbital angular momentum), the ΔL_{ij} (equal to the difference $L_{ij}^\lambda - L_{ij}^\rho$) and the W_{ij} are antisymmetric tensors, thus having three components each. The ("quadrupole") tensor Q_{ij} is symmetric, thus decomposing into an irreducible second-rank tensor (with five components) and a scalar (with one component) under rotations. The trace of Q_{ij} is the scalar:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+\frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^-\frac{\partial}{\partial X_i^-} \tag{11}$$

which (obviously) commutes with the rotation generators and its eigenvalue is therefore the natural choice for the additional label of the hyper-spherical harmonics.

Apart from its mathematical significance, the operator Q is also physically important, as it generates the so-called "democracy" transformations [2,12] that are closely related to permutations of three particles. Moreover, as mentioned in Introduction, some interactions preserve this quantum number (e.g., due to $[Q, |\boldsymbol{\lambda}\times\boldsymbol{\rho}|] = 0$, such are all three-particle potentials that are functions of the area of the subtended triangle, $|\boldsymbol{\lambda}\times\boldsymbol{\rho}|$, which potentials are of importance in QCD). Note that the polynomials in X can only have integer eigenvalues of Q: these eigenvalues correspond to the difference $d_+ - d_-$, where d_+, d_- are polynomial degrees in X^+ and X^- variables, respectively.

The centralizer of the element Q in the $so(6)$ algebra, i.e., the subalgebra of the $so(6)$ algebra consisting of elements that commute with Q, is larger than the rotational subalgebra $so(3)_{rot}$: Q commutes not only with operators L_{ij}, but also with operators Q_{ij}. The three rotation generators L_{ij} together with five linearly independent components of the traceless $\tilde{Q}_{ij} \equiv Q_{ij} - \frac{1}{3}\delta_{ij}Q$ part of the symmetric (quadrupole) tensor Q_{ij} form eight generators of an $su(3)$ subalgebra of $so(6)$.

Labeling of the $SO(6)$ hyper-spherical harmonics with labels K, Q, L and m thus corresponds to the subgroup chain $U(1)\otimes SO(3)_{rot} \subset U(3) \subset SO(6)$ Note, however, that the $SU(3)$ subgroup does not introduce any new quantum numbers into the hyper-spherical harmonics labels (K, Q, L, m). For more details on $SU(3)$ aspect of the three particle h.s. harmonics, see Section 8.3.

Yet, these four quantum numbers are generally insufficient to uniquely specify an $SO(6)$ hyper-spherical harmonic: it is well known, see Ref. [43,44,47], that $SU(3)$ representations in general have nontrivial multiplicity w.r.t. decomposition into $SO(3)$ subgroup representations, and such a multiplicity also appears here. We shall deal with this multiplicity issue in Sect. 5 in a general way, and then again in Sects. 6 and 7, in more specific ways.

3. Core polynomials

Six-dimensional hyper-spherical harmonics with hyper-angular momentum K can be expressed as harmonic homogeneous polynomials of order K in variables x_μ (when restricted to the unit hyper-sphere). Our first goal is to construct such polynomials (which we shall call "core polynomials") that have pre-determined sharp values of quantum numbers Q, L and m. Once this goal has been achieved, we shall address the problem of how to project out parts of these polynomials that have also well defined values of K.

We begin by considering polynomials:

$$\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X) = \left(\boldsymbol{X}^+ \cdot \boldsymbol{X}^+\right)^{\frac{d_+-L_+}{2}} \left(\boldsymbol{X}^- \cdot \boldsymbol{X}^-\right)^{\frac{d_--L_-}{2}} \tilde{\mathcal{Y}}^{L_+}_{3,m_+}(X^+)\tilde{\mathcal{Y}}^{L_-}_{3,m_-}(X^-), \tag{12}$$

where $d_\pm, L_\pm, m_\pm$ are integers such that $d_\pm - L_\pm$ are even and non-negative. Here $\tilde{\mathcal{Y}}^L_{3,m}(X)$ denotes an $SO(3)$ spherical harmonic function expressed as a homogeneous polynomial (of degree L) in the three coordinates X_i, cf. Ref. [58], i.e. $\tilde{\mathcal{Y}}^L_{3,m}(X)/|X|^L = \mathcal{Y}^L_{3,m}(X)$, where $\mathcal{Y}^L_{3,m}(X)$ is a standard $SO(3)$ spherical harmonic function.

These polynomials are of degree d_+ in variables X^+_i and d_- in variables X^-_i, meaning that they yield a sharp eigen-value $q = d_+ - d_-$ of the operator Q. The polynomials Eq. (12) are homogeneous functions of order $(d_+ + d_-)$ in coordinates x, but they are not harmonic, i.e., they don't have a vanishing Laplacian, which is, in this context, equivalent to stating that they are not eigenfunctions of $\mathrm{K}^2 \equiv \sum \frac{1}{2}\mathrm{K}_{\mu\nu}\mathrm{K}_{\mu\nu}$. This, in turn, implies that the polynomial $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X)$ contains components with various values of K, though none larger then $d_+ + d_-$, i.e. $\mathrm{K} \leq d_+ + d_-$.

Furthermore, the maximum value of K appearing in the decomposition of $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}$ (restricted to the unit hyper-sphere) into hyper-spherical harmonics is exactly $d_+ + d_-$. The latter statement follows from the fact that $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}$ as a polynomial is not divisible by R^2.

The $SO(3)$ rotational properties of the polynomials Eq. (12) are determined by the coupling of angular momenta L_+ and L_-; therefore $\mathcal{P}^{d_+d_-}_{L_+m_+L_-m_-}(X)$ decomposes into $SO(3)$ spherical harmonics with L ranging from $|L_+ - L_-|$ to $L_+ + L_-$. By forming linear combinations of polynomials Eq. (12) we define the following homogeneous "core polynomials", that have good quantum numbers Q, L and m and maximal 6-dimensional hyper-angular momentum equal to $\overline{\mathrm{K}}$:

$$\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) \equiv \sum_{m_+,m_-} C^{L_+L_-L}_{m_+m_-m} \mathcal{P}^{\frac{\overline{\mathrm{K}}+Q}{2}\frac{\overline{\mathrm{K}}-Q}{2}}_{L_+m_+L_-m_-}(X), \tag{13}$$

where $C^{L_+L_-L}_{m_+m_-m}$ is an "ordinary" $SO(3)$ Clebsch–Gordan coefficient.

In addition, in the definition Eq. (13), the following is required to hold (the motivation for this will be given below):

$$L_+ + L_- = L \text{ or } L_+ + L_- = L + 1. \tag{14}$$

These polynomials exist and are nonzero only when all of the exponents appearing in Eqs. (12) and (13), i.e., $\frac{\overline{\mathrm{K}}+Q}{2}$ and $\frac{\overline{\mathrm{K}}-Q}{2}$, are non-negative integers and all the Clebsch–Gordan coefficients and 3-dim spherical harmonics are nonvanishing. In particular, this implies that: $\overline{\mathrm{K}} - Q$, $\frac{\overline{\mathrm{K}}+Q}{2} - L_+$ and $\frac{\overline{\mathrm{K}}-Q}{2} - L_-$ are all even and nonnegative, $m \leq L$ and $|L_+ - L_-| \leq L \leq L_+ + L_-$ (due to (14), the last requirement is relevant only when $L_+ = 0$ or $L_- = 0$). From this, it also follows that $\overline{\mathrm{K}} \equiv L_+ + L_- \pmod 2$.

The core polynomials Eq. (13) have sharp values of quantum numbers Q, L and m irrespectively of the condition Eq. (14). The condition in Eq. (14) is only necessary to ensure that the decomposition of $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ into $SO(6)$ hyper-spherical harmonics contains a component with the hyper-angular momentum $\mathrm{K} = \overline{\mathrm{K}}$. The argument goes as follows. A 3-dim $SO(3)$ spherical harmonic polynomial $\tilde{\mathcal{Y}}^L_{3,m}(X)$ can be related to a symmetric tensor $(\tilde{\mathcal{Y}}^L_{3,m})^{i_1i_2\ldots i_L}$ of order

L, that is trace-free in every pair of indices, as $\tilde{\mathcal{Y}}^{L}_{3,m}(X) = \sum_{i_1 i_2 \ldots i_L} (\tilde{\mathcal{Y}}^{L}_{3,m})^{i_1 i_2 \ldots i_L} X_{i_1} X_{i_2} \ldots X_{i_L}$ (again, restricted to the unit hyper-sphere). Coupling of two polynomials $\tilde{\mathcal{Y}}^{L_+}_{3,m_+}(X^+)$ and $\tilde{\mathcal{Y}}^{L_-}_{3,m_-}(X^-)$ to yield a polynomial transforming as a representation with $SO(3)$ angular momentum $L < L_+ + L_-$ involves contracting indices in the product of the corresponding tensors. On the other hand, simply contracting an index from $(\tilde{\mathcal{Y}}^{L_+}_{3,m_+})^{i_1 i_2 \ldots i_{L_+}}$ with an index from $(\tilde{\mathcal{Y}}^{L_-}_{3,m_-})^{j_1 j_2 \ldots j_{L_-}}$ corresponds to a polynomial that is proportional to $\boldsymbol{X}^+ \cdot \boldsymbol{X}^- = R^2$. This, in turn, means that the entire polynomial Eq. (13) would be proportional to R^2 and thus its maximal value K in the decomposition would be less than $\overline{\mathrm{K}}$, which contradicts our original assumption.

The only allowed contraction that would not effectively lower the $\overline{\mathrm{K}}$ value is a contraction with the Levi-Civita tensor, and such a contraction can be applied only once (two successive contractions of this sort are again equivalent to a direct contraction discussed above). Such a single contraction with the Levi-Civita tensor results in a polynomial that transforms w.r.t. spatial rotations as a vector from representation of angular momentum $L = L_+ + L_- - 1$. Therefore, two distinct types of core polynomials exist: those not contracted at all, with $L = L_+ + L_-$, and those once contracted with Levi-Civita tensor, with $L = L_+ + L_- - 1$. Due to $\overline{\mathrm{K}} \equiv L_+ + L_- \pmod 2$ the two possibilities are distinguished by $\overline{\mathrm{K}} - L \equiv 0 \pmod 2$ and $\overline{\mathrm{K}} - L \equiv 1 \pmod 2$, respectively, and in general:

$$L_+ + L_- = L + (\mathrm{K} - L \pmod 2)). \tag{15}$$

The core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$, when restricted to a unit hyper-sphere, are thus equal to a linear combination of 6-dim hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X)$, with v accounting for possible multiplicity:

$$\frac{1}{R^{\overline{\mathrm{K}}}} \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) = \sum_{\mathrm{K}=0}^{\overline{\mathrm{K}}} \sum_{v} c_{\mathrm{K},v} \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X), \tag{16}$$

where at least one $c_{\overline{\mathrm{K}},v}$ is nonzero. Let V^{K} denote the space spanned by all spherical harmonics having hyper-angular momentum less or equal to K, and V^{K}_{QLm} denote a subspace of V^{K} with given values of Q,L and m. Then, the functions $\{\frac{1}{R^{\overline{\mathrm{K}}}} \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) | \overline{\mathrm{K}} = 0, 1, 2, \ldots, \mathrm{K}_{max}\}$, though not orthonormal, span the subspace $V^{\mathrm{K}_{max}}_{QLm}$. (It can be checked that the number of all core polynomials with given $\overline{\mathrm{K}}$ equals $\frac{(\overline{\mathrm{K}}+3)!(\overline{\mathrm{K}}+2)}{12\overline{\mathrm{K}}!}$, equal to the number of spherical harmonics with $\mathrm{K} = \overline{\mathrm{K}}$, Ref. [5].) Conversely, the 6-dim hyper-spherical harmonics can be obtained from the core polynomials by a procedure of orthogonalization and normalization, such as the Gram–Schmidt one.

4. Harmonic polynomials

As mentioned earlier, the core polynomials Eq. (13) are not harmonic, as they contain components with $\mathrm{K} < \overline{\mathrm{K}}$ belonging to some $V^{\overline{\mathrm{K}}-1}$. We introduce a shorthand notation $\mathcal{P}_a(X)$ for the polynomials $\mathcal{P}^{\mathrm{K}Q}_{(L_+L_-)L,m}(X)$ with fixed given values of Q,L,m, with $\mathrm{K} < \overline{\mathrm{K}}$, and L_+, L_- taking all of the allowed values. That is: $V^{\overline{\mathrm{K}}-1}_{QLm} = span\{\mathcal{P}_a(\frac{X}{R}), a = 1, 2, \ldots \dim(V^{\overline{\mathrm{K}}-1}_{QLm})\}$.

The harmonic polynomial $\mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ can be obtained from the core polynomial $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ by removing the components that belong to $V^{\overline{\mathrm{K}}-1}_{QLm}$:

$$\mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) = \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X) - \sum_a c_a R^{\overline{\mathrm{K}}-\mathrm{K}_a}\mathcal{P}_a(X), \tag{17}$$

with c_a being the coefficients to be deduced from orthogonality conditions:

$$\left\langle \mathcal{P}_a \middle| \mathcal{P}H^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m} \right\rangle = 0. \tag{18}$$

These conditions readily lead to:

$$c_a = \sum_b (M^{-1})_{ab} A_b, \quad \text{with:} \quad M_{ab} \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}_b \right\rangle,\ A_a \equiv \left\langle \mathcal{P}_a \middle| \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m} \right\rangle. \tag{19}$$

The above scalar product is naturally given by the integration over a unit 6-dimensional hypersphere:

$$\left\langle \mathcal{P}_a \middle| \mathcal{P}_b \right\rangle \equiv \int_\Omega \mathcal{P}^*_a\left(\frac{X}{R}\right)\mathcal{P}_b\left(\frac{X}{R}\right) d\Omega. \tag{20}$$

This is, in turn, can be calculated by using the following formula (cf. Eq. (A.4) in Appendix A) for integration over the sphere of monomials in x_μ:

$$\int_\Omega \frac{1}{R^6} x_1^{m_1} x_2^{m_2} \cdots x_6^{m_6} d\Omega = 2\frac{\prod_{\mu=1}^6 \frac{1+(-1)^{m_\mu}}{2}\Gamma(\frac{m_\mu+1}{2})}{\Gamma(3+\sum_\mu m_\mu)}, \tag{21}$$

where $\Gamma(n)$ is the usual gamma function.

It is now convenient to introduce a "spherical" version of the $X^\pm$ coordinates:

$$X^{(\pm)}_\pm \equiv X^{(\pm)}_1 \pm X^{(\pm)}_2, \quad X^{(\pm)}_0 \equiv X^{(\pm)}_3, \tag{22}$$

as they are particularly suitable for explicit writing of the core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ for $m = L$:

$$\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L_++L_-,L_++L_-}(X) = |X^+|^{\frac{\overline{\mathrm{K}}+Q}{2}-L_+}\,|X^-|^{\frac{\overline{\mathrm{K}}-Q}{2}-L_-}\,(X^+_+)^{L_+}(X^-_+)^{L_-}, \tag{23}$$

$$\begin{aligned}
&\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L_++L_--1,L_++L_--1}(X) \\
&\quad = \sqrt{\frac{2L_+L_-}{L_++L_-}}|X^+|^{\frac{\overline{\mathrm{K}}+Q}{2}-L_+}\,|X^-|^{\frac{\overline{\mathrm{K}}-Q}{2}-L_-}\Big((X^+_+)^{L_+}(X^-_+)^{L_--1}(X^-_0) \\
&\qquad - (X^+_+)^{L_+-1}(X^+_0)(X^-_+)^{L_-}\Big), \quad L_+, L_- \neq 0,
\end{aligned} \tag{24}$$

where $|X^\pm|^2 = \boldsymbol{X}^\pm \cdot \boldsymbol{X}^\pm = X^\pm_+ X^\pm_- + (X^\pm_0)^2$. Note that the formula Eq. (23) is only relevant when $\overline{\mathrm{K}} \equiv L \pmod 2$, and the formula Eq. (24) should be used otherwise.

Expressions for the scalar products of core polynomials with forms of Eq. (23) and Eq. (24) turn out to be relatively simple, due to the following identity (derivable from Eq. (21), or Eq. (A.4) in Appendix A):

$$\int\limits_{\Omega} |X^+|^{k^+} |X^-|^{k^-} (X_+^+)^{k_+^+} (X_-^+)^{k_-^+} (X_0^+)^{k_0^+} (X_+^-)^{k_+^-} (X_-^-)^{k_-^-} (X_0^-)^{k_0^-} \, d\Omega =$$

$$\frac{2\pi^3}{(2+\frac{\sum k}{2})!} \sum_{l=0}^{\frac{k^+}{2}} \binom{\frac{k^+}{2}}{l} \binom{\frac{k^-}{2}}{l+k_+^+ - k_-^-} 2^{2l+k_+^+ + k_-^+} (l+k_+^+)!(l+k_-^+)!(k^+ - 2l + k_0^+)! \cdot \qquad (25)$$
$$\delta(\sum k^+, \sum k^-) \cdot \delta(k_+^+ + k_+^-, k_-^+ + k_-^-),$$

where $\sum k^\pm = k^\pm + k_+^\pm + k_-^\pm + k_0^\pm$ and $\sum k = \sum k^+ + \sum k^-$, while $\delta(a,b) = \delta_{ab}$ is the Kronecker delta symbol. The formula allows us not only to directly calculate the scalar products of the core polynomials, but also any spherical integral of the product of arbitrarily many core polynomials. The result is particularly simple if all polynomials in the product have the property that $m = L$.

Furthermore, due to the $SO(3)$ rotational symmetry reasons (i.e., due to the Wigner–Eckart theorem) the scalar products of two core polynomials must be independent of the magnetic quantum number m, so that by combining Eqs. (23), (24) and (25) we may write the result in full generality:

$$\left\langle \mathcal{P}^{\overline{\mathrm{K}}'Q'}_{(L'_+L'_-)L',m'} \middle| \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m} \right\rangle = \delta_{mm'} \left\langle \mathcal{P}^{\overline{\mathrm{K}}'Q'}_{(L'_+L'_-)L',L'} \middle| \mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,L} \right\rangle$$
$$= \begin{cases} \frac{2\pi^3 \delta_{QQ'}\delta_{JJ'}\delta_{mm'}}{(2+\frac{\overline{\mathrm{K}}+\overline{\mathrm{K}}'}{2})!} \sum_{l=0}^{\frac{k^+}{2}} 2^{2l+L_+ + L'_-} \binom{\frac{k^+}{2}}{l} \times \\ \binom{\frac{k^+}{2}+L_+ - L'_+}{\frac{k^+}{2}-l} (l+L_+)!(l+L'_-)!(k^+-2l)! \\ \quad \text{if } \overline{\mathrm{K}} - L \equiv \overline{\mathrm{K}}' - L' \equiv 0 \pmod 2 \\ \frac{2\pi^3 \delta_{QQ'}\delta_{JJ'}\delta_{mm'}}{(2+\frac{\overline{\mathrm{K}}+\overline{\mathrm{K}}'}{2})!} \frac{2\sqrt{L_+L_-L'_+L'_-}}{1+L} \sum_{l=0}^{\frac{k^+}{2}} 2^{2l+L_+ + L'_-} \binom{\frac{k^+}{2}}{l} \times \\ \Big[\binom{\frac{k^+}{2}+L_+ - L'_+}{\frac{k^+}{2}-l} (l+L_+-1)!(l+L'_- -1)!(k^+-2l+1)! \frac{l+L+1}{2} \\ - \binom{\frac{k^+}{2}+L_+ - L'_+}{\frac{k^+}{2}-l-1} (l+L_+)!(l+L'_-)!(k^+-2l)! \Big] \\ \quad \text{if } \overline{\mathrm{K}} - L \equiv \overline{\mathrm{K}}' - L' \equiv 1 \pmod 2 \\ 0 \quad \text{if } \overline{\mathrm{K}} - L \not\equiv \overline{\mathrm{K}}' - L' \pmod 2, \end{cases} \qquad (26)$$

where $k^+ = \frac{\overline{\mathrm{K}}+\overline{\mathrm{K}}'}{2} + L_+ - L'_-$ and it is implied that $\binom{m}{n} \equiv 0$ whenever $n < 0$ or $n > m$. Scalar product of a polynomial of form Eq. (23) with a polynomial of form Eq. (24) always yields zero: this case corresponds to $\overline{\mathrm{K}} - L \not\equiv \overline{\mathrm{K}}' - L' \pmod 2$ which, combined with requirement that $L = L'$ leads to $\overline{\mathrm{K}} + \overline{\mathrm{K}}' \equiv 1 \pmod 2$. And, as the integration of any polynomial of odd order over the unit hyper-sphere yields zero, we conclude that the scalar product (26) when $\overline{\mathrm{K}} - L \not\equiv \overline{\mathrm{K}}' - L' \pmod 2$ is also zero.

Relations Eqs. (17), (19) and (26) combined give us expressions for $\mathcal{P}_H{}^{\mathrm{K}Q}_{(L_+L_-)L,m}(X)$ – the homogeneous harmonic polynomials of order K, that are eigenfunctions of the 6-dim hyper-angular momentum and that have well defined values of quantum numbers Q, L and m. In addition to these 4 quantum numbers that are eigenvalues of the corresponding Casimir or Cartan subalgebra operators, harmonic polynomials $\mathcal{P}_H{}^{\mathrm{K}Q}_{(L_+L_-)L,m}(X)$ are also labeled by two numbers L_+ and L_-, only one of which is independent due to the relation Eq. (15). Existence of this additional freedom demonstrates the nontrivial multiplicity of the totally symmetric tensorial representations of $SO(6)$ w.r.t. decomposition into $U(1) \otimes SO(3)_{rot}$ subrepresentations. As the L_+ is non-negative; for any given values of K, Q and L the value of L_+ can only change in steps of 2; it cannot exceed $L+1$; and it cannot take the values of 0 and $L+1$ within the same degenerate subspace, therefore the maximal multiplicity degree that can occur for a given L is $[L/2]+1$, where $[n]$ is the integer part of n. The same holds for L_-. That implies that a non-trivial multiplicity can occur only for harmonics with $L \geq 2$.

Naturally, either L_+ or L_- can be taken to label this multiplicity, though more convenient choices, both mathematically and physically, will be discussed below. A basic option for the multiplicity label is to introduce the difference

$$\Delta l \equiv L_+ - L_-, \tag{27}$$

(not to be confused with ΔL_{ij} in Eq. (9)) as the multiplicity label, which is essentially the same as choosing L_+, or L_-, for that purpose, yet Δl is more convenient, as we shall explain shortly. In this sense, the harmonic polynomials obtained in the previous section would now be labeled as $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l)L,m}(X)$.

5. Multiplicity of degenerate harmonic polynomials

In general, two harmonic polynomials $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l)L,m}(X)$ and $\mathcal{P}_{H}{}^{\mathrm{K}Q}_{(\Delta l')L,m}(X)$, that differ only in the multiplicity label, are not orthogonal. An orthonormal basis has to be introduced in the degenerate subspace of harmonic polynomials with given K, Q, L and m, and this can be done in (infinitely) many ways. For example, the Gram–Schmidt ortho-normalization procedure can be carried out, choosing as the first vector the normalized (in the sense of Eq. (20)) harmonic polynomial with the highest Δl in this subspace, and then taking the polynomial with the next-to-highest value of Δl, subtracting from it a component proportional to the first vector and normalizing it, and so on.

In this process, care should be taken to preserve the symmetry between X_+ and X_- coordinates, *viz.* of complex conjugation, that had been present thus far – we shall demonstrate in Section 7 that this symmetry is directly related to the permutational symmetry S_2. In practice this means that if we begin the orthonormalization procedure with the highest Δl value in the subspaces with $Q>0$, then we must start with the lowest Δl value in subspaces with $Q<0$ (this is due to $\Delta l \to -\Delta l$ when $X_+ \leftrightarrow X_-$). In the limiting case of $Q=0$, the optimal strategy is firstly to introduce symmetric and antisymmetric combinations of harmonic polynomials with opposite values of Δl:

$$\mathcal{P}_{H}{}^{\mathrm{K},0}_{(|\Delta l|,\pm)L,m}(X) \equiv \mathcal{P}_{H}{}^{\mathrm{K},0}_{(\Delta l)L,m}(X) \pm (-1)^{\mathrm{K}-L}\mathcal{P}_{H}{}^{\mathrm{K},0}_{(-\Delta l)L,m}(X). \tag{28}$$

Of these polynomials, those labeled with the plus sign will turn out to be symmetric w.r.t. transposition of particles 1 and 2, whereas those labeled with the minus sign will be asymmetric – and the factor of $(-1)^{\mathrm{K}-L}$ will be necessary to establish this property, see Eq. (65) in Section 7. In turn, this implies that for $Q=0$ it is sufficient to perform Gram–Schmidt procedure separately on these two subsets – since the polynomials from different subsets are mutually orthogonal, and that no ortho-normalization procedure is necessary when multiplicity degree equals (only) two.

Note that the harmonic polynomials that are nondegenerate w.r.t. numbers K, Q, L and m should also be normalized, as they are already orthogonal to all other harmonic polynomials Eq. (17).

The set of polynomials obtained by such an ortho-normalization procedure, when restricted to a unit hyper-sphere, constitutes a system of $SO(6)$ hyper-spherical harmonics that we will denote as $\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X)$, and is labeled by four quantum numbers: K, Q, L, and m, together with an additional multiplicity label Δl. Advantages of this method for multiplicity labeling are the following: i) all of the harmonics can be expressed in analytical form; ii) multiplicity lifting procedure is computationally efficient, since it relies only on Gram–Schmidt ortho-normalization; iii) the label Δl takes only integer values.

Nevertheless, from the physical viewpoint, it is often convenient to choose a basis that diagonalizes some physically significant operator in this degenerate subspace – e.g. the potential energy. Any operator $\mathcal{V}$ that has no degenerate eigenvalues when reduced to this subspace, can be used for this purpose. Moreover, there are certain operators commonly used for multiplicity lifting in the literature (in the context of $SO(3) \subset SU(3)$ multiplicity) and sticking to one of these choices is good from a compatibility aspect (some general results, such as the values of Clebsch–Gordan coefficients, can then be directly used here – cf. Section 8.3).

To address this approach in full generality, we firstly introduce an abbreviated single-letter notation for labeling harmonic polynomials spanning a given degenerate subspace $V^{\mathrm{K},Q}_{L,m}$: $\{\mathcal{P}_{H\,a} | a = 1, 2, \dots \dim V^{\mathrm{K},Q}_{L,m}\}$, and let:

$$\mathcal{V}_{ab} \equiv \langle \mathcal{P}_{H\,a} | \, \mathcal{V} \, | \mathcal{P}_{H\,b} \rangle \,, \quad M_{ab} \equiv \left\langle \mathcal{P}_{H\,a} \middle| \mathcal{P}_{H\,b} \right\rangle. \tag{29}$$

The goal is to find an orthonormal basis of hyper-spherical harmonic polynomials $\tilde{\mathcal{Y}}_a(X) = \sum_b c_{ab} \mathcal{P}_{H\,b}$ that diagonalizes $\mathcal{V}$:

$$\left\langle \tilde{\mathcal{Y}}_a \middle| \tilde{\mathcal{Y}}_b \right\rangle = \delta_{ab}, \tag{30}$$

$$\left\langle \tilde{\mathcal{Y}}_a \middle| \mathcal{V} \middle| \tilde{\mathcal{Y}}_b \right\rangle = \delta_{ab} v_a. \tag{31}$$

From Eq. (30) it follows:

$$c^\dagger M c = I, \tag{32}$$

where I is a unit matrix and $\dagger$ denotes conjugate transpose matrix. As the matrix M is hermitian, it follows that matrix $(\sqrt{M} c)$ is a unitary matrix, that we shall denote as U:

$$U \equiv \sqrt{M} c, \quad U^\dagger U = I. \tag{33}$$

From the condition Eq. (31) we know that the matrix $c^\dagger \mathcal{V} c = U^\dagger (M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}}) U$ has to be diagonal, i.e., a unitary matrix U can be found that diagonalizes the hermitian matrix $(M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}})$:

$$U^{-1} (M^{-\frac{1}{2}} \mathcal{V} M^{-\frac{1}{2}}) U = diag(v_1, v_2, \dots v_{dim}). \tag{34}$$

Therefore, resolving the multiplicity problem reduces to finding a unitary matrix U that satisfies Eq. (34); thereafter the hyper-spherical harmonic polynomials, labeled by K, Q, L, m and v_a, are calculated as:

$$\tilde{\mathcal{Y}}_a(X) = \sum_b (M^{-\frac{1}{2}} U)_{ab} \mathcal{P}_{H\,b}. \tag{35}$$

Note that the same procedure, when applied to a non-degenerate one-dimensional subspace $V^{\mathrm{K},Q}_{L,m}$ simply normalizes the corresponding harmonic polynomial.

Polynomials $\tilde{\mathcal{Y}}^{\mathrm{K}Qv}_{L,m}(X)$ obtained by this procedure, when reduced to the unit hyper-sphere – that is, when divided by R^{K}, give a set of orthonormal $SO(6)$ hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X)$, labeled by the 6-dimensional hyper-angular momentum K, the eigenvalue of the Q operator, the total (orbital) angular momentum of the system L, the projection m of the total (orbital) angular momentum and the eigenvalue of the reduced $\mathcal{V}$ operator:

$$\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X) = \tilde{\mathcal{Y}}^{\mathrm{K}Qv}_{L,m}(X)/R^{\mathrm{K}}. \tag{36}$$

In the context of the $SO(3) \subset SU(3)$ multiplicity, the operator:

$$\mathcal{V}_{JQJ} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{37}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ and Q_{ij} is given by Eq. (8)) has often been used in the literature, see Refs. [43–47], to label the multiplicity. This operator has the desirable property that it commutes both with the angular momentum L_i, and with the "democracy rotation" generator Q:

$$\left[\mathcal{V}_{JQJ}, L_i\right] = 0; \quad \left[\mathcal{V}_{JQJ}, Q\right] = 0.$$

Of course, this is not the only operator that commutes with L_i, and with Q, so there is a certain degree of freedom left in this choice that can, perhaps, be used so as to optimize the h.s. harmonics to a particular application, see e.g. Ref. [43–47]. For example, the area of the triangle "operator" $|\boldsymbol{\lambda} \times \boldsymbol{\rho}|$ commutes with L_i, and Q: $[Q, |\boldsymbol{\lambda} \times \boldsymbol{\rho}|] = 0$, $[L_i, |\boldsymbol{\lambda} \times \boldsymbol{\rho}|] = 0$, and can also be used for this purpose. We shall show below that these two operators have "opposite" transformation properties under certain permutations (transpositions) and, in some sense, represent the only two possible classes of such operators.

In Appendix C we list the hyper-spherical harmonics labeled by the operator $\mathcal{V}_{JQJ}$, up to $\mathrm{K} \le 6$, and compare them with the few explicit harmonics that already exist in the literature, Ref. [5]. There is only a handful of harmonics with non-trivial multiplicity in this range of K-values, so they can be readily calculated and examined with the alternative degeneracy-lifting ("area") operator. The result is that the two multiplicity-lifting operators are for all practical purposes equivalent. Other examples of degeneracy-lifting operators have been discussed in Refs. [45–47], irrespectively of their geometrical meaning in the three-body problem.

We note that no solution to Eq. (34) is unique and that this arbitrariness directly corresponds to the freedom of choosing multiplicative phase factors for the obtained basis functions. This arbitrariness should be fixed by adopting a definite phase convention: e.g. in the explicit calculations in the remainder of this paper, we shall adjust the overall sign of hyper-spherical harmonic in Eq. (35) so that the projection of each vector $\tilde{\mathcal{Y}}_a(X)$ on the sum $\sum_b \mathcal{P}_{Hb}$ is non-negative, i.e.,

$$\sum_b \left\langle \mathcal{P}_{Hb} \middle| \tilde{\mathcal{Y}}_a(X) \right\rangle \ge 0. \tag{38}$$

It should be clear that the process of using an operator to lift degeneracy amounts to the diagonalization of the chosen operator in a finite-dimensional space. That, in turn, boils down to solving an algebraic eigenvalue equation, that can be solved in closed form ("surds") only so long as the order of the equation is less than five (due to Galois' theory) and that solutions to higher-order degeneracy-lifting problems must necessarily be numerical.

The choice of optimal degeneracy-lifting operator(s) is a problem in SU(3) group theory that has been essentially solved in Ref. [47], where it was noted that "it is not possible to choose a complete set of operators whose eigenvalues are all integers and whose eigenfunctions can be constructed analytically. The price we have to pay for having orthonormal basis functions and a physically meaningful operator, providing the missing label in the $SU(3) \supset O(3)$ scheme, and thus providing selection rules, etc., is that many of the computations involved will be numerical by necessity." On the other hand, if we give up insisting on a "physically meaningful operator" to label the states, we can account for the multiplicity by the value of Δl, Eq. (27), and retain both the algebraic form of hyper-spherical harmonics and an integer-valued degeneracy-lifting label.

Thus, with the concept of degeneracy-lifting clarified, we see that the entire construction of three particle hyper-spherical harmonics can be automatized/programmed using (several) commercially available computer software codes for symbolic computation, with the understanding that, if using operator approach for multiplicity lifting, then for sufficiently high value(s) of harmonic labels some of the results will necessarily be numerical. More specifically, maximal $SO(3) \subset SU(3)$ multiplicity that occurs for a given K grows as $[\mathrm{K}/4] + 1$ and for a given L grows as $[L/2] + 1$. Effectively, this means that it is unavoidable to resort to numerical solutions only when $\mathrm{K} \geq 16$ and $L \geq 8$, and even then not for all harmonics (to give some impression of these numbers, we note that there are 27132 hyperspherical harmonicas with $\mathrm{K} < 16$).

Now that we have established a mathematical procedure for calculating $SO(6)$ h.s. harmonics, in the next section we will treat in detail a few examples of this procedure. In particular, we shall illustrate the application of two different multiplicity lifting operators, so as to demonstrate the concept and to clarify the limitations on computability of arbitrary h.s. harmonics, imposed by the degeneracy problem, Ref. [46].

6. Examples of harmonic construction

In order to illustrate the procedure for obtaining the hyper-spherical harmonics described in this paper, we shall explicitly calculate several h.s. harmonics with two different degeneracy-lifting operators.

For the purpose of this demonstration we look for hyper-spherical harmonics with quantum numbers $\mathrm{K} = 4$, $Q = 0$ and $L = 2$, as this is the simplest case with nontrivial multiplicity. As for the quantum number m, we will first demonstrate how to obtain the harmonic function that corresponds to maximal value $m = L$, in this particular case $m = 2$. After that we will discuss how to obtain harmonics with arbitrary values of m, $-L \leq m \leq L$.

The first step is to calculate necessary core polynomials Eq. (13). There are two core polynomials $\mathcal{P}^{\overline{\mathrm{K}}Q}_{(L_+L_-)L,m}(X)$ with quantum numbers $Q = 0$, $L = 2$ and $m = 2$, that have $\overline{\mathrm{K}} = 4$:

$$\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) = \left(X^+_+\right)^2 \left|X^-\right|^2 \qquad \text{and} \qquad \mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) = \left(X^-_+\right)^2 \left|X^+\right|^2. \tag{39}$$

It can be easily checked that these are eigenfunctions of the operators Q, $L^2 \equiv \frac{1}{2}\sum L_{ij}L_{ij}$ and $L_3 \equiv L_{12}$. These are not eigenfunctions of the square of hyper-angular momentum, however, due to the appearance of additional terms on the right-hand-side of the Eqs. (40), (41):

$$\mathrm{K}^2\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) = 4(4+6-2)\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) - 16R^2X^+_+X^-_+, \tag{40}$$

$$\mathrm{K}^2\mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) = 4(4+6-2)\mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) - 16R^2X^+_+X^-_+. \tag{41}$$

The additional terms are identical, and proportional to the core polynomial $\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)$:

$$\mathcal{P}^{\overline{4},0}_{(1,1)2,2}(X) = X^+_+X^-_+, \tag{42}$$

and this explicitly demonstrates the necessity of the procedure, described in section 4, to obtain the truly harmonic polynomials. The polynomial $\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X)$ is also the only polynomial with quantum numbers $Q = 0$, $L = 2$ and $m = 2$ that has $\overline{\mathrm{K}} < 4$. In the notation of section 4 this means that in this case the space $V^{\overline{\mathrm{K}}-1}_{QLm}$ is one dimensional, and therefore that the calculation (highly) simplifies as the index a takes only one value.

In order to find the harmonic polynomial $\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)$ from the core polynomial $\mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X)$, we follow the projection procedure, Eq. (17) and use Eq. (26), to readily find:

$$M_{11} = \frac{\pi^3}{3}, \quad (M^{-1})_{11} = \frac{3}{\pi^3}, \quad A_1 = \frac{4\pi^3}{15}, \quad c_1 = \frac{4}{5}, \tag{43}$$

leading to

$$\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X) = \mathcal{P}^{\overline{4},0}_{(2,0)2,2}(X) - \frac{4}{5}R^2\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X) = \left(X^+_+\right)^2\left|X^-\right|^2 - \frac{4}{5}R^2 X^+_+ X^-_+. \tag{44}$$

In an identical manner one obtains:

$$\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X) = \mathcal{P}^{\overline{4},0}_{(0,2)2,2}(X) - \frac{4}{5}R^2\mathcal{P}^{\overline{2},0}_{(1,1)2,2}(X) = \left(X^-_+\right)^2\left|X^+\right|^2 - \frac{4}{5}R^2 X^+_+ X^-_+. \tag{45}$$

Now it can be verified that these polynomials are indeed harmonic, in the sense that they satisfy the Laplace equation:

$$\nabla^2\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X) = \nabla^2\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X) = 0 \tag{46}$$

and that they are eigen-functions of the operator K^2:

$$\begin{aligned} \mathrm{K}^2\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X) &= 4(4+6-2)\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X), \\ \mathrm{K}^2\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X) &= 4(4+6-2)\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X). \end{aligned} \tag{47}$$

Being harmonic and having good quantum numbers K, Q, L and m, these polynomials indeed represent the sought-after hyper-spherical harmonics, i.e. functions $\mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X)/R^4$ and $\mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X)/R^4$, reduced to the $R = 1$ unit sphere. The fact that there are two different polynomials with the same set of numbers K, Q, L, m means that there is nontrivial multiplicity present.

These functions have certain shortcomings, however: first, these two functions are not mutually orthogonal:

$$\left\langle \mathcal{P}_H{}^{\overline{4},0}_{(2,0)2,2}(X) \middle| \mathcal{P}_H{}^{\overline{4},0}_{(0,2)2,2}(X) \right\rangle = -\frac{8\pi^3}{225}. \tag{48}$$

Secondly, these states are not normalized, as yet.

In order to obtain an ortho-normal basis of harmonic functions and to have the multiplicity labeled in some more precise way, we can follow one of the procedures laid out in section 5.

In the following we shall demonstrate three ways to label the multiplicity: i) by the difference Δl in Sect. 6.1; ii) by the transposition-odd operator $\mathcal{V}_{JQJ}$, Eq. (37) in Sect. 6.2; and iii) by using the transposition-even area operator $|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$ in Sect. 6.3. In Section 7 we discuss the particle permutation properties of the harmonics and show that the symmetric (even) and antisymmetric (odd) multiplicity-lifting operators are the only two relevant classes.

6.1. Δl as the multiplicity label

As this is a $Q = 0$ subspace, we will define symmetric and antisymmetric combinations Eq. (28) of the polynomials $\mathcal{P}_{H\,(2,0)2,2}^{\overline{4},0}(X)$ and $\mathcal{P}_{H\,(0,2)2,2}^{\overline{4},0}(X)$, that, after normalization, take form:

$$\mathcal{P}_{H\,(|\Delta l|=2,+)2,2}^{4,0}(X) = \frac{3\left(-8R^2 X_+^+ X_+^- + 5\left(X_+^-\right)^2 \left|X^+\right|^2 + 5\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\sqrt{7}\pi^{3/2} R^4} \tag{49}$$

$$\mathcal{P}_{H\,(|\Delta l|=2,-)2,2}^{4,0}(X) = \frac{\sqrt{15}\left(\left(X_+^-\right)^2 \left|X^+\right|^2 - \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\pi^{3/2} R^4} \tag{50}$$

By virtue of different transformation properties w.r.t. transposition of first two particles, these combinations are now already mutually orthogonal, even without any Gram–Schmidt procedure (however, had the multiplicity degree been larger than 2 such a procedure would have been necessary). Once we have found and normalized these combinations, we can return to the labeling $\mathcal{Y}_{L,m}^{KQ\Delta l}(X)$, where Δl takes both positive and negative values:

$$\begin{aligned}
&\mathcal{Y}_{2,2}^{4,0,\Delta l=2}(X) \\
&\quad\equiv \frac{1}{\sqrt{2}}\left(\mathcal{P}_{H\,(|\Delta l|=2,+)2,2}^{4,0}(X) + \mathcal{P}_{H\,(|\Delta l|=2,-)2,2}^{4,0}(X)\right) \\
&\quad= \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2 \left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2 \left|X^-\right|^2}{14\pi^{3/2} R^4},
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X) \\
&\quad\equiv \frac{1}{\sqrt{2}}\left(\mathcal{P}_{H\,(|\Delta l|=2,+)2,2}^{4,0}(X) - \mathcal{P}_{H\,(|\Delta l|=2,-)2,2}^{4,0}(X)\right) \\
&\quad= \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2 \left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2 \left|X^-\right|^2}{14\pi^{3/2} R^4}.
\end{aligned} \tag{52}$$

By using relation Eq. (25) we can also explicitly verify orthonormality of the obtained hyper-spherical harmonics:

$$\begin{aligned}
&\int_\Omega \mathcal{Y}_{2,2}^{*\,4,0,\Delta l=2}(X)\,\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)\,d\Omega = 0, \\
&\int_\Omega |\mathcal{Y}_{2,2}^{4,0,\Delta l=2}(X)|^2 d\Omega = \int_\Omega |\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)|^2 d\Omega = 1.
\end{aligned} \tag{53}$$

6.2. Harmonics with antisymmetric degeneracy lifting operator

Next we demonstrate the use the operator $\mathcal{V}_{JQJ}$ in Eq. (37) to label the multiplicity. Combining Eqs. (7)–(8) and Eq. (25) we obtain the following values for matrices $\mathcal{V}$ and M, defined by Eqs. (29):

$$\mathcal{V} = \frac{14\pi^3}{15}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \frac{2\pi^3}{225}\begin{pmatrix} 11 & -4 \\ -4 & 11 \end{pmatrix} \tag{54}$$

leading to

$$\begin{aligned} M^{-\frac{1}{2}} &= \frac{1}{2\pi^{3/2}}\sqrt{\frac{15}{7}}\begin{pmatrix} \sqrt{\left(11+\sqrt{105}\right)} & \sqrt{\left(11-\sqrt{105}\right)} \\ \sqrt{\left(11-\sqrt{105}\right)} & \sqrt{\left(11+\sqrt{105}\right)} \end{pmatrix}, \\ M^{-\frac{1}{2}}\mathcal{V}M^{-\frac{1}{2}} &= \sqrt{105}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{55}$$

As the matrix $M^{-\frac{1}{2}}\mathcal{V}M^{-\frac{1}{2}}$ is already diagonal (generally this is not so), the U matrix is trivial and from Eqs. (35), (36) we finally obtain the hyper-spherical harmonics:

$$\begin{aligned} &\mathcal{Y}_{2,2}^{4,0,\sqrt{105}}(X) \\ &= \frac{\sqrt{\frac{15}{7}\left(11+\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H_{(2,0)2,2}^{\bar{4},0}(X)/R^4 + \frac{\sqrt{\frac{15}{7}\left(11-\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H_{(0,2)2,2}^{\bar{4},0}(X)/R^4 \\ &= \frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2|X^+|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2|X^-|^2}{14\pi^{3/2}R^4} \end{aligned} \tag{56}$$

$$\begin{aligned} &\mathcal{Y}_{2,2}^{4,0,-\sqrt{105}}(X) \\ &= \frac{\sqrt{\frac{15}{7}\left(11+\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H_{(0,2)2,2}^{\bar{4},0}(X)/R^4 + \frac{\sqrt{\frac{15}{7}\left(11-\sqrt{105}\right)}}{2\pi^{3/2}}\mathcal{P}H_{(2,0)2,2}^{\bar{4},0}(X)/R^4 \\ &= \frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2|X^+|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2|X^-|^2}{14\pi^{3/2}R^4} \end{aligned} \tag{57}$$

We observe that two obtained hyper-spherical harmonics are identical to $\mathcal{Y}_{2,2}^{4,0,\Delta l=2}(X)$ and $\mathcal{Y}_{2,2}^{4,0,\Delta l=-2}(X)$.

6.3. Harmonics with symmetric degeneracy lifting operator

By an identical procedure, only taking this time operator $\mathcal{V}$ to be square of area of the subtended triangle, we obtain the only two K = 4 h.s. harmonics with degeneracy:

$$\mathcal{Y}_{2,2}^{4,0,\frac{1}{8}}(X) = \frac{\sqrt{15}\left(\left(X_+^-\right)^2|X^+|^2 - \left(X_+^+\right)^2|X^-|^2\right)}{2\pi^{3/2}R^4} \tag{58}$$

which turns out to be antisymmetric w.r.t. particle transpositions, and

$$\mathcal{Y}_{2,2}^{4,0,\frac{47}{280}}(X) = \frac{3\left(-8R^2X_+^+X_+^- + 5\left(X_+^-\right)^2|X^+|^2 + 5\left(X_+^+\right)^2|X^-|^2\right)}{2\sqrt{7}\pi^{3/2}R^4} \tag{59}$$

which is symmetric under permutations.

Of course, these two h.s. harmonics have to be equal to the symmetric and antisymmetric combinations $\mathcal{P}H^{4,0}_{(|\Delta l|=2,+)2,2}(\lambda,\rho)$ and $\mathcal{P}H^{4,0}_{(|\Delta l|=2,-)2,2}(\lambda,\rho)$, and thus also equal to linear combinations of harmonics $\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}$ and $\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}$ (previously obtained by using $\mathcal{V}_{JQJ}$ as the degeneracy-lifting operator). This is indeed the case:

$$\mathcal{Y}^{4,0,\frac{1}{8}}_{2,2}(\lambda,\rho)=\mathcal{P}H^{4,0}_{(|\Delta l|=2,-)2,2}(\lambda,\rho)=\frac{1}{\sqrt{2}}\left(\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(\lambda,\rho)-\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\right)$$

and

$$\mathcal{Y}^{4,0,\frac{47}{280}}_{2,2}(\lambda,\rho)=\mathcal{P}H^{4,0}_{(|\Delta l|=2,+)2,2}(\lambda,\rho)=\frac{1}{\sqrt{2}}\left(\mathcal{Y}^{4,0,\sqrt{105}}_{2,2}(\lambda,\rho)+\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\right).$$

6.4. Harmonics with $m < L$

Finally, as we have thus far presented an algorithm for the construction only of hyper-spherical harmonics with $m = L$, we should clarify how one can obtain the hyper-spherical harmonics with $m < L$. This can be done in (at least) two ways. One is to repeat the procedure above, this time starting from the core polynomials with some given value of m, such that $m < L$. However, in this case all of the intermediate expressions will be significantly more complicated. More optimal way is to find the corresponding hyper-spherical harmonic with $m = L$ first and than to use lowering operators $L_- \equiv L_1 - iL_2$ to obtain harmonics with lower values of m, following the well known recurrence formula:

$$L_-\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(X)=\sqrt{L(L+1)-m(m-1)}\,\mathcal{Y}^{\mathrm{K}Qv}_{L,m-1}(X). \tag{60}$$

For example:

$$\begin{aligned}\mathcal{Y}^{4,0,-\sqrt{105}}_{2,1}(X)&=L_-\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(X)/2\\&=\frac{1}{35\pi^{3/2}R^4}\left(5\sqrt{\frac{7}{2}}X_0^-\left(\left(\sqrt{105}-15\right)\,X_+^-\left|X^+\right|^2+12R^2X_+^+\right)\right.\\&\qquad\left.-5X_0^+\,\left(\sqrt{105\left(11+\sqrt{105}\right)}X_+^+\left|X^-\right|^2-6\sqrt{14}R^2X_+^-\right)\right).\end{aligned} \tag{61}$$

Clearly the spherical harmonics can also be expressed in terms of the initial variables, the Jacobi vectors, e.g.:

$$\begin{aligned}&\mathcal{Y}^{4,0,-\sqrt{105}}_{2,2}(\lambda,\rho)\\&=\frac{1}{14\pi^{3/2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)^2}\\&\times\left(\sqrt{105\left(11+\sqrt{105}\right)}\left((\lambda_1-i\rho_1)^2+(\lambda_2-i\rho_2)^2+(\lambda_3-i\rho_3)^2\right)\right.\\&\times(\lambda_1+i(\lambda_2+\rho_1+i\rho_2))^2-12\sqrt{14}(\lambda_1+i\lambda_2-i\rho_1+\rho_2)\\&\times\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)(\lambda_1+i(\lambda_2+\rho_1+i\rho_2))+\sqrt{105\left(11-\sqrt{105}\right)}\end{aligned}$$

$$\times\,(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2)^2 \Big((\lambda_1 + i\rho_1)^2 + (\lambda_2 + i\rho_2)^2 + (\lambda_3 + i\rho_3)^2\Big)\Big), \tag{62}$$

by inverting definitions Eq. (6) and Eq. (22). By comparing the forms of Eq. (56) and Eq. (62) it becomes clear that the expressions are much more compact when written in terms of spherical complex coordinates $X^{\pm}_{\pm}, X^{\pm}_0$, Eq. (22).

Next, we turn to consider the permutation properties of these h.s. harmonics, which, in turn, fixes (some of) the phase ambiguities, and corroborates our basic claim stated in the title of this paper.

7. Permutation properties

From the viewpoint of applications of the three particle hyperspherical harmonics in physics, it is of some importance that the wave functions have simple and manifest transformation properties with respect to both the spatial rotations and permutations of the three particles. Of course, the rotational properties of the functions $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ are manifestly given by the values of corresponding $SO(3)_{rot}$ labels L and m, so it is the permutation properties that must be established here.

Properties of functions $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ under particle permutations are (readily) inferred from the (simple) transformation properties of the coordinates $X^{\pm}_i$. Namely, under the transpositions (two-body permutations) $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs $(1, 2)$, $(2, 3)$ and $(3, 1)$, the Jacobi coordinates transform as:

$$\begin{aligned}
&\mathcal{T}_{12} : \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho},\\
&\mathcal{T}_{23} : \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda},\\
&\mathcal{T}_{31} : \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned} \tag{63}$$

That induces the following transformations of complex coordinates $X^{\pm}_i$:

$$\mathcal{T}_{12} : X^{\pm}_i \to X^{\mp}_i, \quad \mathcal{T}_{23} : X^{\pm}_i \to e^{\pm\frac{2i\pi}{3}} X^{\mp}_i, \quad \mathcal{T}_{31} : X^{\pm}_i \to e^{\mp\frac{2i\pi}{3}} X^{\mp}_i. \tag{64}$$

Note that Eqs. (63) imply that the transpositions $\mathcal{T}_{ij}$ correspond to $O(6)$ transformations of x_μ with $\det\mathcal{T}_{ij} = -1$, i.e. they form a set of (parity-like in odd-D spaces; though in D=6 the usual parity (i.e., the reflection of all 6 coordinates) transformation's determinant equals $+1$) "reflection transformation" in the 6-D space and as such do not belong to $SO(6)$ group of proper hyper-rotations. Such reflections generally lead to an appearance of phases, see below.

It follows from Eqs. (7)–(10) and Eq. (64) that none of the quantum numbers K, L and m change under permutations of particles, whereas the value of the "democracy label" Q is inverted under all transpositions: $Q \to -Q$. The fact that K is not changed by particle transpositions implies that the set of $SO(6)$ hyper-spherical harmonics with given K also carry an irreducible representation of the entire $O(6)$ group (and, in this sense, these functions are equally $O(6)$ hyper-spherical harmonics). Group-theoretically, change in the label Q is a consequence of the fact that the discrete group of permutations S_3 is not a subgroup of the $U(1)$ group generated by operator Q, but of the group $O(2) = U(1) \rtimes S_2$ instead. The behavior of the multiplicity label v under transpositions manifestly depends on the choice of the multiplicity-lifting operator $\mathcal{V}$, but this choice is effectively reduced to the choice of the sign change of v under transpositions.

To explain this, we first note that the set of even permutations of the three particles constitutes a discrete subgroup of the $SO(6)$ group of hyper-rotations generated by $\mathrm{K}_{\mu\nu}$: each transposition corresponds to an orthogonal matrix with determinant -1, so that the combination of even number of transpositions has determinant $+1$. More specifically, by comparing the actions on the 6-dimensional coordinates, it turns out that $\mathcal{T}_{12}\mathcal{T}_{23} = e^{\frac{2i\pi Q}{3}}$ and $\mathcal{T}_{23}\mathcal{T}_{12} = e^{-\frac{2i\pi Q}{3}}$, as (11) and (64) yield $\mathcal{T}_{12}\mathcal{T}_{23}X_i^{\pm}\mathcal{T}_{23}\mathcal{T}_{12} = e^{\frac{2i\pi Q}{3}}X_i^{\pm}e^{-\frac{2i\pi Q}{3}}$. However, due to the requirement that multiplicity lifting operator $\mathcal{V}$ must commute with generator Q, we conclude that even permutations of particles must leave the operator $\mathcal{V}$ invariant: $\mathcal{T}_{12}\mathcal{T}_{23}\mathcal{V}\mathcal{T}_{23}\mathcal{T}_{12} = e^{\frac{2i\pi Q}{3}}\mathcal{V}e^{-\frac{2i\pi Q}{3}} = \mathcal{V}$ and $\mathcal{T}_{23}\mathcal{T}_{12}\mathcal{V}\mathcal{T}_{12}\mathcal{T}_{23} = e^{-\frac{2i\pi Q}{3}}\mathcal{V}e^{\frac{2i\pi Q}{3}} = \mathcal{V}$. On the other hand, only one-dimensional irreducible representations of permutation group S_3 (i.e. symmetric and antisymmetric representation) have this property that the even permutations are mapped onto the unit operator, whereas the remaining two-dimensional (mixed) representation does not have this property.

In other words, this means that multiplicity lifting operator $\mathcal{V}$ itself can transform according to the one-dimensional antisymmetric representation of S_3, or transform as the one-dimensional symmetric representation, or be a nontrivial linear combination of antisymmetric and symmetric components. We dismiss the third option (linear combinations) both on physical grounds, as there can hardly be physical motivation for the introduction of such an operator, and on practical grounds, as that choice would lead to unnecessarily complicated transformation properties. Put together, these two reasons render such a choice of operator inappropriate for multiplicity lifting. Therefore we consider only two choices for the multiplicity-lifting operator: a) operators that are antisymmetric under permutations, i.e., $\mathcal{T}_{ij}\mathcal{V}\mathcal{T}_{ij} = -\mathcal{V}$; and b) the symmetric ones under permutations, i.e., $\mathcal{T}_{ij}\mathcal{V}\mathcal{T}_{ij} = \mathcal{V}$. For example, the $\mathcal{V}_{JQJ}$ operator is of the antisymmetric type, whereas the triangle area operator is a representative of the symmetric type.

In conclusion, the action of a single transposition $\mathcal{T}_{ij}$ on the label v of a permutation symmetric h.s. harmonic can lead at most to a (minus) sign for the multiplicity label: $v \to \pm v$. In the next subsection, we shall also show that only the antisymmetric degeneracy-lifting operators lead to a completely unambiguous set of permutation properties of h.s. harmonics.

As far as the permutation properties are concerned, the choice of Δl, Eq. (27), for the multiplicity label v is no different than using eigenvalues of any antisymmetric multiplicity lifting operator, because the value of $\Delta l = L_+ - L_-$ (obviously) changes the sign upon the interchange of X_+ and X_-.[2] Therefore, the case of using Δl to label multiplicity need not be treated separately, as it is already included in the case of a general antisymmetric multiplicity lifting operator.

Apart from the changes in labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyper-spherical harmonic. For values of K, Q, L and m with no multiplicity, the transformation properties of h.s. harmonics under (two-particle) particle transpositions coincide with the corresponding properties of the core polynomials, so that Eq. (13) and Eq. (64) readily lead to:

$$\begin{aligned}
\mathcal{T}_{12} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \\
\mathcal{T}_{23} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}e^{\frac{2Qi\pi}{3}}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \\
\mathcal{T}_{31} &: \mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L}e^{-\frac{2Qi\pi}{3}}\mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}),
\end{aligned} \tag{65}$$

[2] The fact that an operator with eigenvalues that exactly match the values Δl cannot be (easily) written down, does not change anything in principle, because such an operator $\mathcal{V}_{\Delta l}$ can always be formally defined by its action on the hyperspherical harmonics: $\mathcal{V}_{\Delta l}\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X) = \Delta l\mathcal{Y}^{\mathrm{K}Q\Delta l}_{L,m}(X)$.

where v' is the transposition-transformed value of v: $v' = \mathcal{T}_{ij}(v)$. We note that the phase factor $(-1)^{\mathrm{K}-L}$ in Eqs. (65) comes about from the transformation properties of the Clebsch–Gordan coefficient $C^{L_+ L_- L}_{m_+ m_- m}$ in Eq. (13) under the replacement $L_+, m_+ \leftrightarrow L_-, m_-$, induced by the transformation $X_i^+ \leftrightarrow X_i^-$: $C^{L_- L_+ L}_{m_- m_+ m} = (-1)^{L_+ + L_- - L} C^{L_+ L_- L}_{m_+ m_- m}$ and $(-1)^{L_+ + L_- - L} = (-1)^{\mathrm{K}-L}$ due to Eq. (15).

7.1. Irreducible representations of the permutation group

There are three distinct irreducible representations of the S_3 permutation group – two one-dimensional (the symmetric S and the antisymmetric A ones) and a two-dimensional (the mixed M one). In order to determine to which representation of the permutation group any particular h.s. harmonic $\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs, we start by considering multiplicity free cases.

7.1.1. Multiplicity-free case

When $Q = 0$, we can see from Eq. (65) that the action of transpositions reduces to

$$\mathcal{T}_{ij} : \mathcal{Y}^{\mathrm{K},0,v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K},0,v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \tag{66}$$

We obtained this relation by replacement $v' = v$, which necessarily follows from the "multiplicity-free" assumption, i.e. the assumption than numbers K, Q, L and m uniquely specify this h.s. harmonic (in particular, $v = v'$ is always true for permutation-symmetric $\mathcal{V}$, whereas here it holds only as $v = v' = 0$ for permutation-antisymmetric $\mathcal{V}$). Thus, multiplicity-free h.s. harmonics $\mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belong either to the symmetric (S) representation of S_3, for even values of $\mathrm{K} - L$, or to the antisymmetric (A) representation, for odd values of $\mathrm{K} - L$.

When $Q \neq 0$, the action of permutations on h.s. harmonics is reduced to two-dimensional subspaces spanned by pairs of harmonics $\left\{\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})\right\}$, as can be seen from Eq. (65). In this basis, the three transposition operators of Eq. (65) have the following matrix representations:

$$\mathcal{T}_{12} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathcal{T}_{23} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & e^{\frac{2i\pi Q}{3}} \\ e^{-\frac{2}{3}i\pi Q} & 0 \end{pmatrix},$$
$$\mathcal{T}_{31} \to (-1)^{\mathrm{K}-L} \begin{pmatrix} 0 & e^{-\frac{2}{3}i\pi Q} \\ e^{\frac{2i\pi Q}{3}} & 0 \end{pmatrix}. \tag{67}$$

For $Q \not\equiv 0 \pmod 3$, this representation of the permutation group S_3 is irreducible, therefore such h.s. harmonics belong to two dimensional mixed representation.

For $Q \equiv 0 \pmod 3$, this 2×2 matrix representation reduces to two one-dimensional representations, one of which is symmetric and the other antisymmetric; the representations are spanned by the following pair of linear combinations of the harmonics:

$$\begin{aligned} \mathcal{T}_{ij} : & \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) + \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right) \\ & \to (-1)^{\mathrm{K}-L} \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) + \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right), \end{aligned} \tag{68}$$

$$\mathcal{T}_{ij} : \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) - \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right)$$
$$\to (-1)^{\mathrm{K}-L+1} \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}Qv}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) - \mathcal{Y}^{\mathrm{K},-Q,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right). \tag{69}$$

7.1.2. Cases with multiplicity

When there is a (nontrivial) multiplicity of h.s. harmonics with a given set of quantum numbers K, Q, L, m, then the transformation properties under permutations clearly depend also on the choice of multiplicity lifting operator $\mathcal{V}$.

As explained above, there are two types of admissible choices for the multiplicity-lifting operator: a) the antisymmetric ones; and b) the symmetric ones under permutations. For both of these types, the relations Eqs. (65) still hold with $v' = -v$ and $v' = v$, respectively, albeit possibly with additional phase factors on the right-hand sides of Eqs. (65). These additional phases can be, in principle, introduced by multiplicity lifting procedure (Section 5), and hence the transformation properties in cases with multiplicity can no longer be inferred from the corresponding properties of the core polynomials.

However, these possible additional phase factors can be absorbed into the definition of h.s. harmonics, i.e. into the phase convention, in all cases, except one: when both the operator $\mathcal{V}$ is symmetric and $Q = 0$. Only in this one case of nontrivial multiplicity with $v' = v$ does the same h.s. harmonic appear on both sides of Eqs. (65), i.e., Eqs. (65) then lead to Eq. (66) with the aforementioned phase factor $e^{i\phi}$ on the righthand side:

$$\mathcal{T}_{ij} : \mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to e^{i\phi} (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}), \tag{70}$$

and it is clear that no redefinition of $\mathcal{Y}^{\mathrm{K}0v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ can remove this factor. Due to idempotency of transpositions, this phase factor $e^{i\phi}$ can be either $+1$ or -1. In the former case the h.s. harmonic obtains under transpositions a factor of $(-1)^{\mathrm{K}-L}$, and of $(-1)^{\mathrm{K}-L+1}$ in the latter, but which one of the two cannot be established without providing further details about the chosen operator $\mathcal{V}$.

Exactly such an example was illustrated in the Section 6.3: *a priori* – i.e. based solely on the values of the labels – it is not possible to determine which one of the h.s. harmonics $\mathcal{Y}^{4,0,\frac{1}{8}}_{2,2}$ and $\mathcal{Y}^{4,0,\frac{47}{280}}_{2,2}$ in Eqs. (58), (59) belongs to the symmetric and which one to the antisymmetric representations of S_3.

In all other cases (i.e. apart from the case of symmetric degeneracy-lifting operator $\mathcal{V}$ at $Q = 0$) the same reasoning as in Eqs. (67)–(69) holds and we again conclude that for $Q \not\equiv 0$ (mod 3), the h.s. harmonics belong to the mixed representation of S_3, whereas for $Q \equiv 0$ (mod 3) the two linear combinations Eq. (68) and Eq. (69) belong to the one-dimensional representations, acquiring, respectively, factor of $(-1)^{\mathrm{K}-L}$ and $(-1)^{\mathrm{K}-L+1}$ under transpositions.

7.1.3. Summary of the permutation properties

In order to summarize the above results it is convenient to introduce the following linear combinations of the h.s. harmonics, which are no longer eigenfunctions of Q operator, but are instead eigenfunctions of transposition $\mathcal{T}_{12}$:

$$\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \pm (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K},-|Q|,v'}_{L,m}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \right). \tag{71}$$

The normalization factor $\frac{1}{\sqrt{2}}$ ought to be changed to $\frac{1}{2}$ in the cases when the both terms in the bracket are equal (instead of orthogonal) and do not cancel each out.

Apart from the specific case when $\mathcal{V}$ is symmetric under permutations, the multiplicity is nontrivial, and $Q = 0$ (and when the conclusions further depend on the details of the operator $\mathcal{V}$), the following holds:

1. the transposition $\mathcal{T}_{12}$ is a pure sign: $\mathcal{T}_{12} : \mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho}) \to \pm\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho})$,
2. for $Q \not\equiv 0 \pmod 3$, the harmonics $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,\pm}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belong to the mixed representation M,
3. for $Q \equiv 0 \pmod 3$, the harmonic $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,+}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs to the symmetric representation S and $\mathcal{Y}^{\mathrm{K}|Q|v}_{L,m,-}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ belongs to the antisymmetric representation A.

Note that the above statements also implicitly contain our previous conclusions about the behavior of $Q = 0$ multiplicity-free harmonics Eq. (66), and thus summarize all of the previous results specifying the representation of S_3 to which any given harmonic belongs.

Above, we have tacitly assumed that the phase convention, i.e., the choice of how to fix the otherwise arbitrary phases of harmonics, obeys Eq. (65). This assumption is satisfied in the (specific) case of the multiplicity-resolving operator Eq. (37) together with the phase convention, Eq. (38): $\mathcal{V}_{JQJ}$ is antisymmetric (which is readily derived from Eq. (7) and Eq. (8)), thus any transposition simply flips the sign of v: $v' = -v$, and the relations Eqs. (65) hold in full generality. *q.e.d.*

8. Matrix elements of $SO(6)$ harmonics

In applications to the quantum mechanical three-body problem, Ref. [50], one often needs to know the $SO(6)$ hyper-angular matrix elements of the form

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha, \phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5) \rangle \tag{72}$$

This kind of integral can be readily evaluated using formulas from Appendix A so long as the HS harmonics $\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)$ are explicitly known as polynomials of the integration variables, with the result expressed in terms of Γ function, see Eq. (A.4), see Tables 1 and 2. By the procedure laid out in the previous sections, it is possible to find the required polynomial expressions and thus to evaluate matrix elements of the type shown in Eq. (72) yielding algebraic numbers whenever the multiplicity of the hyperspherical harmonics with equal K, Q and L is less than five[3] or the integer Δl is used as a multiplicity label, as already explained in Sect. 5.

In a great number of practical applications, however, there is no need to calculate explicitly the h.s. harmonics, apart from evaluating matrix elements of the form shown in Eq. (72). As the calculation of h.s. harmonic functions, as well as the application of the formulas from Appendix A, can be considerably involved for higher values of K, it is very useful to have some more direct method for evaluation of such matrix elements involving three h.s. harmonics. In this section we shall therefore discuss matrix elements Eq. (72) *per se*, both in the $SO(6)$ and $SU(3)$ context, finally evaluating them as a closed form expression, apart from a single $SU(3)$ Clebsch–Gordan coefficient.

[3] There are alleviating circumstances here that sometimes allow algebraic solutions in cases with multiplicities higher than five, e.g. due to discrete S_3 numbers in $Q = 0$ cases, or in cases discussed in Sects. 3. and 4. of Ref. [46].

Table 1
The values of the three-body potential hyper-angular diagonal matrix elements $\langle \mathcal{Y}^{4,|0|,0}_{0,0,+} \rangle_{\rm ang}$, $\langle \mathcal{Y}^{6,|6|,0}_{0,0,+} \rangle_{\rm ang}$ and $\langle \mathcal{Y}^{8,|0|,0}_{0,0,+} \rangle_{\rm ang}$, for K $\leq$ 4 states (for all allowed orbital waves L). The correspondence between the S_3 permutation group irreps. and SU(6)$_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$. The table values are independent of the angular moment projection m.

K	(K, $\|Q\|$, L, ν, $\pm$)	$[SU(6), L^P]$	$\pi\sqrt{\pi}\langle \mathcal{Y}^{4,\|0\|,0}_{0,0,+} \rangle_{\rm ang}$	$\pi\sqrt{2\pi}\langle \mathcal{Y}^{6,\|6\|,0}_{0,0,+} \rangle_{\rm ang}$	$\pi\sqrt{\pi}\langle \mathcal{Y}^{8,\|0\|,0}_{0,0,+} \rangle_{\rm ang}$
2	$(2,\|2\|,0,0,\pm)$	$[70, 0^+]$	$\frac{1}{\sqrt{3}}$	0	0
2	$(2,\|0\|,2,0,+)$	$[56, 2^+]$	$\frac{\sqrt{3}}{5}$	0	0
2	$(2,\|2\|,2,-3,\pm)$	$[70, 2^+]$	$-\frac{1}{5\sqrt{3}}$	0	0
2	$(2,\|0\|,1,0,-)$	$[20, 1^+]$	$-\frac{1}{\sqrt{3}}$	0	0
3	$(3,\|3\|,1,-1,-)$	$[20, 1^-]$	$\frac{1}{\sqrt{3}}$	-1	0
3	$(3,\|3\|,1,-1,+)$	$[56, 1^-]$	$\frac{1}{\sqrt{3}}$	1	0
3	$(3,\|1\|,1,3,\pm)$	$[70, 1^-]$	0	0	0
3	$(3,\|1\|,2,-5,\pm)$	$[70, 2^-]$	$-\frac{1}{\sqrt{3}}$	0	0
3	$(3,\|1\|,3,-2,\pm)$	$[70, 3^-]$	$\frac{5}{7\sqrt{3}}$	0	0
3	$(3,\|3\|,3,-6,+)$	$[56, 3^-]$	$-\frac{\sqrt{3}}{7}$	$\frac{2}{7}$	0
3	$(3,\|3\|,3,-6,-)$	$[20, 3^-]$	$-\frac{\sqrt{3}}{7}$	$-\frac{2}{7}$	0
4	$(4,\|4\|,0,0,\pm)$	$[70, 0^+]$	$\frac{\sqrt{3}}{2}$	0	$\frac{1}{2\sqrt{5}}$
4	$(4,\|0\|,0,0,+)$	$[56, 0^+]$	0	0	$\frac{2}{\sqrt{5}}$
4	$(4,\|2\|,1,2,\pm)$	$[70, 1^+]$	0	0	$-\frac{1}{\sqrt{5}}$
4	$\left(4,\|0\|,2,\sqrt{105},+\right)$	$[56, 2^+]$	$-\frac{12\sqrt{3}}{35}$	0	$\frac{\sqrt{5}}{7}$
4	$\left(4,\|0\|,2,\sqrt{105},-\right)$	$[20, 2^+]$	0	0	$-\frac{1}{\sqrt{5}}$
4	$(4,\|2\|,2,2,\pm)$	$[70, 2^+]$	$\frac{4\sqrt{3}}{35}$	0	$\frac{\sqrt{5}}{7}$
4	$(4,\|4\|,2,-3,\pm)$	$[70', 2^+]$	$\frac{2\sqrt{3}}{7}$	0	$-\frac{1}{7\sqrt{5}}$
4	$(4,\|2\|,3,-13,\pm)$	$[70, 3^+]$	$-\frac{5\sqrt{3}}{14}$	0	$\frac{1}{14\sqrt{5}}$
4	$(4,\|0\|,3,0,-)$	$[20, 3^+]$	$-\frac{3\sqrt{3}}{14}$	0	$-\frac{\sqrt{5}}{14}$
4	$(4,\|0\|,4,0,+)$	$[56, 4^+]$	$\frac{5\sqrt{3}}{14}$	0	$\frac{3}{14\sqrt{5}}$
4	$(4,\|2\|,4,-5,\pm)$	$[70, 4^+]$	$\frac{3\sqrt{3}}{14}$	0	$-\frac{\sqrt{5}}{42}$
4	$(4,\|4\|,4,-10,\pm)$	$[70', 4^+]$	$-\frac{3\sqrt{3}}{14}$	0	$\frac{1}{42\sqrt{5}}$

8.1. Some matrix elements and their properties

Decomposition into hyperspherical harmonics with manifest permutation properties highly simplifies solving of Schrödinger's equation. The benefits are most notable when the three-body potential is permutation symmetric. The decomposition of any such potential into h.s. harmonics has a low number of nonzero components due to the permutation symmetry constraints: e.g. up to K $\leq$ 11 the only h.s. harmonics that can appear in such decomposition are $\mathcal{Y}^{0,|0|,0}_{0,0,+}$, $\mathcal{Y}^{4,|0|,0}_{0,0,+}$, $\mathcal{Y}^{8,|0|,0}_{0,0,+}$ and $\mathcal{Y}^{6,|6|,0}_{0,0,+}$. In Tables 1 and 2, we show the nonzero matrix elements between states of

Table 2
The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{2\pi}\langle[SU(6)_f, L_f^P]|\mathcal{Y}_{0,0,+}^{6,|6|,0}|[SU(6)_i, L_i^P]\rangle_{\rm ang}$, for various K = 4 states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{2\pi}\langle\mathcal{Y}_{0,0,+}^{6,\|6\|,0}\rangle_{\rm ang}$
4	$[70, 2^+]$	$[70', 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70', 2^+]$	$[70, 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70, 4^+]$	$[70', 4^+]$	$\frac{8}{21}$
4	$[70', 4^+]$	$[70, 4^+]$	$\frac{8}{21}$
4	$[20, L^+]$	$[20, L^+]$	0
4	$[56, L^+]$	$[56, L^+]$	0
4	$[20, L^+]$	$[56, L^+]$	0

same K, K ≤ 4. These matrix elements are sufficient to evaluate the matrix elements of permutation symmetric sums of arbitrary one-, two- and three-body operators, such as the three-body potential, and thus to solve Schrödinger's equation in the first order of perturbation theory (in K ≤ 4 subspace). These harmonics have been applied to three homogeneous confining potentials in Ref. [49].

We observe that generally, the $SO(6)$ matrix elements obey the following selection rules that reduce the number of non-zero values: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $m' = m''$. Moreover, Q is an Abelian (i.e. additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$.

The aforementioned selection rules naturally follow since the hyper-angular matrix element Eq. (72) can be reduced to a product of two $SO(6)$ group Clebsch–Gordan coefficients.

8.2. Matrix elements as functions of $SO(6)$ Clebsch–Gordan coefficients

In the case of $SO(3)$ harmonics holds the Gaunt formula [60]

$$\int Y_{LM}^*(\theta,\phi) Y_{l_1 m_1}(\theta,\phi) Y_{l_2 m_2}(\theta,\phi) \sin\theta d\theta d\phi$$
$$= \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}\right]^{1/2} C_{m_1 m_2 M}^{l_1\, l_2\, L} C_{0\ 0\ 0}^{l_1\, l_2\, L}, \tag{73}$$

where $C_{m_1 m_2 M}^{l_1\, l_2\, L}$ is the $SO(3)$ Clebsch–Gordan coefficient. In the context of the Wigner–Eckart theorem, the Clebsch–Gordan coefficient $C_{0\ 0\ 0}^{l_1\, l_2\, L}$ in Eq. (73) is proportional to/defines the "reduced matrix element" $\langle L||T_{l_1}||l_2\rangle$, in this case of the $SO(3)$ spherical harmonic $T_{l_1} = \mathbf{Y}_{l_1}$: $\langle LM|T_{l_1 m_1}|l_2 m_2\rangle = \langle LM|l_2 l_1 m_2 m_1\rangle\langle L||T_{l_1}||l_2\rangle$. Of course, the precise definition of the reduced matrix element depends on the conventions used, see e.g. Refs. [58,59], but the right-hand side of Eq. (73) is independent of convention.

The equivalent formula holds also for $SO(n)$ groups with higher-values of n (see Appendix B for derivation), and the integral of three $SO(6)$ harmonics is, similarly, proportional to products of two $SO(6)$ group Clebsch–Gordan coefficients:

$$\int_{\mathcal{M}} \mathcal{Y}^{*K}_{[m]}(\Omega_5)\mathcal{Y}^{K_1}_{[m_1]}(\Omega_5)\mathcal{Y}^{K_2}_{[m_2]}(\Omega_5)d\Omega_5$$
$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(K_1)\dim(K_2)}{\dim(K)}}\, C^{K_1\ K_2\ K}_{[m_1]\,[m_2]\,[m]} C^{K_1\ \ K_2\ \ K}_{[0_H]\ [0_H]\ [0_H]}, \tag{74}$$

where $V_{\mathcal{M}} = \pi^3$ is the "volume"of the coset space $\mathcal{M} = SO(6)/SO(5)$, $\dim_{SO(6)}(\mathrm{K}) = \frac{(\mathrm{K}+3)!(\mathrm{K}+2)}{12\mathrm{K}!} = \frac{(\mathrm{K}+3)(\mathrm{K}+2)^2(\mathrm{K}+1)}{12}$ and $[0_H]$ are labels of the vector that is invariant w.r.t. $SO(5)$ subgroup (because a sphere in 6 dimensions is isomorphic with $SO(6)/SO(5)$ coset space, see Appendix B).

Now, this does not necessarily simplify the problem at hand, because the SO(6) Clebsch–Gordan coefficients are not well known in general. The prospect of evaluating these Clebsch–Gordan coefficients or finding their values in literature is further complicated by the fact that the physical context dictates the choice of the basis, $SO(6) \supset U(3) \supset SO(3) \times U(1)$, which is not the simplest one from the mathematical viewpoint, instead of the simpler one $SO(6) \supset U(3) \supset U(2) \supset U(1)$. It is a (physical) necessity to have manifest transformation properties w.r.t. the (physical) angular momentum $SO(3)$ that spoils such attempts – if it were not for this, the multiplicity problem would not arise and both the construction of $SO(6)$ H.H. and calculation of $SO(6)$ Clebsch–Gordan coefficients would be much simpler.

Nevertheless, certain general properties of the matrix elements Eq. (72) can be inferred based on elemental $SO(6)$ group-theoretical arguments. For example, the following dimensional C.G. series (the reduction of tensor products) immediately give information, for some lower dimensional cases, when the value of the matrix element in Eq. (72) is allowed to be nonzero:

$$\begin{aligned}
\mathbf{6} \otimes \mathbf{1} &= \mathbf{6} \\
\mathbf{6} \otimes \mathbf{6} &= \mathbf{1} \oplus \mathbf{20} \\
\mathbf{6} \otimes \mathbf{20} &= \mathbf{6} \oplus \mathbf{50} \\
\mathbf{6} \otimes \mathbf{50} &= \mathbf{20} \oplus \mathbf{105} \\
&\vdots
\end{aligned} \tag{75}$$

where boldface numbers denote dimensions of $SO(6)$ irreps. In general:

$$[1] \otimes [\mathrm{K}'] = [\mathrm{K}' - 1] \oplus [\mathrm{K}' + 1].$$

This result, and the property that the left-hand and right-hand sides of these equations do not agree arithmetically, e.g., $6 \times 6 = 36 \neq 1 + 20 = 21$, are direct consequences of the fact that h.s. harmonics transform as totally symmetric tensors of $SO(6)$ and that the rest of irreps (e.g. antisymmetric ones) are absent from the right hand side.

Unlike the $SO(6)$ Clebsch–Gordan coefficients, the $SU(3)$ Clebsch–Gordan coefficients are quite well known in various bases, Refs. [51–57,63], including the multiplicity problem of $SU(3)$ to $SO(3)$ reduction, Ref. [43,44,46,47]. We shall exploit this fact in the following subsection.

8.3. Matrix elements as functions of SU(3) Clebsch–Gordan coefficients

We have already shown that the $U(3)$ subgroup appears as an intermediary step in the reduction $SO(6) \supset U(3) \supset U(1) \otimes SO(3)$ that dictates our choice of basis. On the other hand the $SU(3)$ subgroup does not introduce any new quantum numbers into the hyper-spherical harmonics labels (K, Q, L, m) – the reason being that $SU(3)$ irreducible representations contained within the $SO(6)$ harmonics are already fully determined by the integers K and Q. Namely, coordinates X_i^+, Eq. (6) transform as the fundamental $SU(3)$ unitary irreducible representation (UIR) of the $U(3)$ subgroup, i.e., one box Young diagram, whereas the coordinates X_i^- transform as the conjugate representation of $U(3)$, i.e., a two-box column Young diagram. Therefore, an $SU(3)$ representation with given K and Q corresponds to a Young diagram with K boxes in the first row, and $(\mathrm{K}-Q)/2$ boxes in the second one.[4]

Moreover, it easy to see, Ref. [14,15], that three-particle hyper-spherical harmonics can be also viewed as functions on the $SU(3)/SU(2)$ coset space. In decomposition of the Hilbert space of square integrable functions over $SU(3)/SU(2)$ coset space into $SU(3)$ irreducible components, each $SU(3)$ UIR appears exactly once. In other words, there is exactly one three particle harmonic transforming as each of the $SU(3)$ UIR's (and the state vectors within), i.e. there is one set of harmonics for each allowed combination of K and Q (where by "allowed", we mean $|Q| \leq \mathrm{K}$ and $Q \equiv \mathrm{K} \pmod 2$). That much ought to be clear already from our construction of hyper-spherical harmonics, as the polynomials with given degrees K and Q cannot constitute more than one copy of the same $SU(3)$ UIR, and yet there are polynomials for each combination of (K, Q) (one of the ways to verify the first part of this statement is to note that there is only one polynomial with the highest weight for that representation).[5] In accordance with this, an $SO(6)$ symmetric tensor representation of order K decomposes to $SU(3)$ UIR's (K, Q), $Q = -\mathrm{K}, -\mathrm{K}+2, \ldots \mathrm{K}$, each UIR appearing only once in the decomposition – as the sum of dimensions of $SU(3)$ irreducible representations building up the order-K harmonics, confirms:

$$
\begin{aligned}
n_{\mathrm{K}} &= \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots\mathrm{K}} \dim(\mathrm{K}, Q) = \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots\mathrm{K}} \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2) \\
&= \frac{1}{8}(\mathrm{K}+2) \sum_{Q=-\mathrm{K},-\mathrm{K}+2,\ldots\mathrm{K}} (\mathrm{K}+2)^2 - (Q)^2 = \frac{1}{12}(\mathrm{K}+1)(\mathrm{K}+2)^2(\mathrm{K}+3),
\end{aligned}
$$

where $\dim(\mathrm{K}, Q) = \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$ and indeed $\dim_{O(6)}(\mathrm{K}) = n_{\mathrm{K}} = \frac{(\mathrm{K}+3)!(\mathrm{K}+2)}{12\mathrm{K}!} = \frac{1}{12}(\mathrm{K}+3)(\mathrm{K}+2)^2(\mathrm{K}+1)$.

The embeddings of $S_3 \otimes SO(3)$ and $SU(3)$ multiplets in $SO(6)$ multiplets is illustrated in Table 3. Note that each (complete) $SU(3)$ irreducible representations appears once and only once among all the H.H.s – there is no repetition. Moreover, one must be careful not to double-count the self-conjugate irreps, such as the $(1,1) = 8$ one.

The same multiplicity issue that we have dealt with in Sect. 5 has been studied in the $SU(3)$ context. It is well known, see Refs. [43,46,47], that $SU(3)$ representations in general have non-trivial multiplicity w.r.t. decomposition into $SO(3)$ subgroup representations, see e.g. the two $\mathrm{K}=4$, $L=2$ states $\in$ **27**-plet in Table 3.[6] Different multiplicity lifting operators were considered in the literature, and the corresponding bases constructed Refs. [43,44,46,47].

[4] Notice that giving the pair K, Q differs from the usual $SU(3)$ irreducible representation labeling described by two integers (p, q) that correspond to a Young diagram with $p+q$ boxes in the first row, and q boxes in the second one, see e.g. Ref. [62].)

[5] Another way to prove this property is by invoking the Frobenius reciprocity theorem, as in Ref. [14,15].

[6] Note that non-trivial multiplicities do not exist for $L = 0, 1$ states, thus also explaining why these two series of states have been explicitly constructed in Refs. [5,8,19].

Table 3
The labels of distinct K ≤ 4 h.s. harmonics $\mathcal{Y}^{\mathrm{K},Q,\nu}_{L,m}$ (three-body states, with allowed orbital angular momentum value L; only $L = m$ labels are shown). The correspondence between the S_3 permutation group irreps. and SU(6)$_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$. The number of states in an SO(3) irrep is $n_L = 2L+1$, and the number of states in an O(6) multiplet/H.H. is $n_{\mathrm{K}} = \sum \mathrm{dim}_{S_3} \times n_L = \sum \mathrm{dim}_{SU(3)}$, where the sum goes over all the O(3) multiplets, or over all SU(3) multiplets contained in the O(6) H.H. The sign $\pm$ in front of Q values (second column) denotes mixing of mutually conjugate $SU(3)$ representations, which occurs as a consequence of $S_3 \not\subset U(3)$.

K	$(\mathrm{K}, Q, L, m, \nu)$	$[SU(6), L^P]$	S_3 irrep.	$SU(3)$ irrep.	(λ_1, λ_2)	$\mathrm{dim}_{S_3} \times n_L$	N_{K}
0	$(0,0,0,0,0)$	$[56, 0^+]$	S	$\mathbf{1}$	(0,0)	1×1	1
1	$(1,\pm1,1,1,\mp1)$	$[70, 1^-]$	M	$\mathbf{3}, \bar{\mathbf{3}}$	(1,0), (0,1)	2×3	6
2	$(2,\pm2,0,0,0)$	$[70, 0^+]$	M	$\mathbf{6}, \bar{\mathbf{6}}$	(2,0), (0,2)	2×1	20
2	$(2,\mp2,2,2,\pm3)$	$[70, 2^+]$	M	$\mathbf{6}, \bar{\mathbf{6}}$	(2,0), (0,2)	2×5	20
2	$(2,0,2,2,0)$	$[56, 2^+]$	S	$\mathbf{8}$	(1,1)	1×5	20
2	$(2,0,1,1,0)$	$[20, 1^+]$	A	$\mathbf{8}$	(1,1)	1×3	20
3	$(3,\mp3,1,1,\pm1)$	$[20, 1^-]$	A	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×3	50
3	$(3,\mp3,1,1,\pm1)$	$[56, 1^-]$	S	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×3	50
3	$(3,\pm3,3,3,\mp6)$	$[56, 3^-]$	S	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×7	50
3	$(3,\pm3,3,3,\mp6)$	$[20, 3^-]$	A	$\mathbf{10}, \overline{\mathbf{10}}$	(3,0), (0,3)	1×7	50
3	$(3,\pm1,1,1,\pm3)$	$[70, 1^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×3	50
3	$(3,\mp1,2,2,\pm5)$	$[70, 2^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×5	50
3	$(3,\mp1,3,3,\pm2)$	$[70, 3^-]$	M	$\mathbf{15}, \overline{\mathbf{15}}$	(2,1), (1,2)	2×7	50
4	$(4,\pm4,0,0,0)$	$[70, 0^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0), (0,4)	2×1	105
4	$(4,\pm4,2,2,\mp3)$	$[70', 2^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0),(0,4)	2×5	105
4	$(4,\mp4,4,4,\pm10)$	$[70', 4^+]$	M	$\mathbf{15}', \overline{\mathbf{15}}'$	(4,0), (0,4)	2×9	105
4	$(4,\pm2,1,1,\pm2)$	$[70, 1^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1), (1,3)	2×3	105
4	$(4,\pm2,2,2,\pm2)$	$[70, 2^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1),(1,3)	2×5	105
4	$(4,\mp2,3,3,\pm13)$	$[70, 3^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1),(1,3)	2×7	105
4	$(4,\mp2,4,4,\pm5)$	$[70, 4^+]$	M	$\mathbf{24}, \overline{\mathbf{24}}$	(3,1), (1,3)	2×9	105
4	$(4,0,0,0,0)$	$[56, 0^+]$	S	$\mathbf{27}$	(2,2)	1×1	105
4	$\left(4,0,2,2,\mp\sqrt{105}\right)$	$[56, 2^+]$	S	$\mathbf{27}$	(2,2)	1×5	105
4	$\left(4,0,2,2,\mp\sqrt{105}\right)$	$[20, 2^+]$	A	$\mathbf{27}$	(2,2)	1×5	105
4	$(4,0,3,3,0)$	$[20, 3^+]$	A	$\mathbf{27}$	(2,2)	1×7	105
4	$(4,0,4,4,0)$	$[56, 4^+]$	S	$\mathbf{27}$	(2,2)	1×9	105

Of special interest is the fact that the $SU(3)$ Clebsch–Gordan coefficients are known for certain choices of multiplicity lifting operator, [51–57,63]. This means that an $SU(3)$ analogon of formula, Eq. (74), has substantial practical utility (Appendix B.3):

$$\int_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}Q\nu}_{L,m}(X)\mathcal{Y}^{\mathrm{K}_1 Q_1 \nu_1}_{L_1,m_1}(X)\mathcal{Y}^{\mathrm{K}_2 Q_2 \nu_2}_{L_2,m_2}(X)dX^3 =$$

$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\mathrm{dim}(\mathrm{K}_1,Q_1)\,\mathrm{dim}(\mathrm{K}_2,Q_2)}{\mathrm{dim}(\mathrm{K},Q)}}\, C^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \ \{\mathrm{K},Q\}}_{\{L_1,m_1,\nu_1\}\,\{L_2,m_2,\nu_2\}\,\{L,m,\nu\}}\, C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ \ 0_H\ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ 0_H}, \quad (76)$$

where X_i are the complex coordinates defined in Eq. (6), the integration is over $SU(3)/SU(2)$ coset space which is parameterized by X subjected to constraint $|X| = 1$, 0_H is the unique vector from the given $SU(3)$ UIR that is invariant w.r.t. $SU(2)$ subgroup, $V_{\mathcal{M}} = \pi^3$, and dim(K, Q) $= \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$. Since $SU(3)$ Clebsch–Gordan coefficients appearing in the above formula are known entities, whose numerical evaluation has been carefully studied in the literature, Refs. [54,55,57], in various $SU(3)$ bases, (and no longer the obscure CG coefficients of the $SO(6)$ group), it means that Eq. (76) can be used to evaluate matrix elements, Eq. (72), in practice.

Furthermore, another nice thing is that this formula can be made even more explicit, that is, less dependent on knowledge of (tables of) $SU(3)$ CG coefficients:

1) The first Clebsch–Gordan coefficient in Eq. (76) factors into an $SO(3)_{\rm rot}$ part and the reduced Clebsch–Gordan coefficient, see pp. 360 in Ref. [63],

$$C^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \ \{\mathrm{K},Q\}}_{\{L_1,m_1,v_1\}\,\{L_2,m_2,v_2\}\,\{L,m,v\}} = C^{L_1L_2L}_{m_1m_2m}\, C_r{}^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\{L_1,v_1\}\ \ \{L_2,v_2\}\ \ \{L,v\}}. \tag{77}$$

When applied to Eq. (72), the $SO(3)$ coefficient becomes simply a product of two Kronecker delta functions: $C^{L'0L''}_{m'0m''} = \delta_{L',L''}\delta_{m',m''}$. The values of the reduced Clebsch–Gordan coefficient can be found in literature, Refs. [54,55,57], at least in their numerical form.

2) The second of the two SU(3) Clebsch–Gordan coefficients (the “reduced matrix element”) in Eq. (76) does not depend on the L, m, v labels and can be explicitly evaluated in closed form, as follows,

$$\begin{aligned} C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{\ \ 0_H\ \ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ \ 0_H} &= \Bigg(A_0^{\mathrm{K}_1,Q} A_0^{\mathrm{K}_2,Q} A_0^{\mathrm{K},Q} \sqrt{\frac{\pi^3 \dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}} \\ &\times \sum_{\mathrm{K}_1'=|Q_1|,|Q_1|+2,\ldots}^{\mathrm{K}_1} \ \sum_{\mathrm{K}_2'=|Q_2|,|Q_2|+2,\ldots}^{\mathrm{K}_2} \ \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K}_1,Q_1}_{\mathrm{K}_1'} \Pi^{\mathrm{K}_2,Q_2}_{\mathrm{K}_2'} \Pi^{\mathrm{K},-Q}_{\mathrm{K}'} \\ &\times \frac{2\pi^3}{(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+1)(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+2)} \delta_{Q_1+Q_2,Q}\Bigg)^{\frac{1}{2}} \end{aligned} \tag{78}$$

where

$$A_0^{\mathrm{K},Q} = (-1)^{\frac{\mathrm{K}-|Q|}{2}} \left(\sum_{\mathrm{K}_1,\mathrm{K}_2=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi^{\mathrm{K},Q}_{\mathrm{K}_1} \Pi^{\mathrm{K},Q}_{\mathrm{K}_2} \frac{2\pi^3}{(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+1)(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+2)} \right)^{-\frac{1}{2}} \tag{79}$$

and

$$\Pi^{\mathrm{K},Q}_{\mathrm{K}'} = \prod_{\mathrm{K}''=|Q|,|Q|+2,\ldots}^{\mathrm{K}'-2} \left(1 - \frac{(\mathrm{K}+2)^2 - Q^2}{(\mathrm{K}''+2)^2 - Q^2}\right). \tag{80}$$

Combining the simplifications shown above, we finally obtain:

$$\begin{aligned} \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Qv}_{00}(\alpha,\phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle &= \frac{1}{\sqrt{\pi^3}} \sqrt{\frac{\dim(\mathrm{K},Q)\dim(\mathrm{K}',Q')}{\dim(\mathrm{K}'',Q'')}} \\ &\times \delta_{L',L''}\delta_{m',m''} C_r{}^{\{\mathrm{K},Q\}\ \ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}} C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ \ 0_H\ \ \ \ \ \ \ 0_H\ \ \ \ \ \ \ \ 0_H}, \end{aligned} \tag{81}$$

The remaining $SU(3)/SO(3)$ reduced Clebsch–Gordan coefficient cannot be further simplified/analytically evaluated in general, for two reasons: i) its value depends on the choice of multiplicity lifting operator; ii) due to conclusions of Moshinsky et al. [47], irrespectively of the choice of multiplicity lifting operator some of the values inevitably have to be numerical. Therefore, Eq. (81), together with Eqs. (78), (79), (80), represents our final result.

These results can be difficult to interpret without specifying the phase- and other conventions, both for the state vectors (both $SO(6)$ and $SU(3)$) and for the Clebsch–Gordan coefficients.

8.4. Conventions for $SO(6)$ and $SU(3)$ states and Clebsch–Gordan coefficients

It is well known [64] that the Clebsch–Gordan coefficients of $SO(3)$, and/or $SU(2)$ can be chosen to be real under a specific convention on the phase of the state vectors. Our construction of the $SO(6)$ hyperspherical harmonics in Sects. 3, 4 provides a specific set of conventions that lead to the reality of matrix elements in Eq. (72). That, in turn, does not guarantee the reality of the $SO(6)$ Clebsch–Gordan coefficients, but the converse statement [7] is assured. We shall henceforth assume that (all of) the relevant $SO(6)$ Clebsch–Gordan coefficients are real with our choice of phases for the $SO(6)$ hyperspherical harmonics.

Even after such a convention is imposed, however, there is one "remnant" sign ambiguity left over, in the form of the overall sign of the Clebsch–Gordan matrix, which is conventionally fixed, say by the Condon–Shortley definition. Such a remnant sign ambiguity does not affect the Gaunt formula either in the $SO(3)$, or in the $SO(6)$ case, because the right-hand sides of Eqs. (73), (74) are bi-linear in their respective Clebsch–Gordan coefficients. Similarly, we shall assume[8] that the $SU(3)$ Clebsch–Gordan coefficients appearing in Eq. (81) are real, as well, which is a common/standard convention, see Refs. [51,53–55,57].

The above relation between the $SO(6)$ and $SU(3)$ Clebsch–Gordan coefficients calls for yet another comment about the conventions adopted here. When dealing with Clebsch–Gordan coefficients of an $SU(n)$ group with $n > 2$, there is also a certain freedom related to the so called "outer multiplicity", Ref. [62]. This freedom amounts to the fact that not only some of the phase factors depend on the adopted phase conventions (as in the $SU(2)$ case), but that there are other more general phase ambiguities. Namely, the Kronecker product of two irreducible $SU(n)$ representations contains more often than not, a multiplicity, i.e., the reduction of the Kronecker product (the "Clebsch–Gordan series") contains more than one copy of one and the same irreducible representation, Ref. [62]. Consequently, the $SU(3)$ Clebsch–Gordan tables must generally have a number of (different) coefficients for each triplet $(\{\mathrm{K}, Q\}\,\{\mathrm{K}', Q'\}\,\{\mathrm{K}'', Q''\})$ of $SU(3)$ state labels.[9] Therefore, in cases when the Clebsch–Gordan series contains outer multiplicity, an additional label, or some other method of identification must be specified to distinguish between otherwise identical copies of irreducible representations. This ambiguity "spills over" into the evaluation of Eq. (81), as follows.

In the case when existing programs (e.g. Refs. [55,57]) for the evaluation ("tables") of $SU(3)$ Clebsch–Gordan coefficients feature more than one coefficient for the given triplet $(\{\mathrm{K}, Q\}\,\{\mathrm{K}', Q'\}\,\{\mathrm{K}'', Q''\})$ of $SU(3)$ labels, the question arises how to tell which one corresponds to the decomposition of hyperspherical harmonics and should be plugged into Eq. (81), that is, how does one tell which one (of sometimes many) copies of the same UIR appearing after the reduction of the Kronecker product is relevant to application here?

The answer to this question, and the behavior of the decomposition of the three-particle hyperspherical harmonics product into $SU(3)$ UIR's, are governed by the very value of the coefficient $C^{\{\mathrm{K},Q\}\,\{\mathrm{K}',Q'\}\,\{\mathrm{K}'',Q''\}}_{\;0_H\;\;\;\;0_H\;\;\;\;0_H}$: The value of this coefficient is nonzero in only one, of many, copies of

[7] The reality of the $SO(6)$ Clebsch–Gordan coefficients guarantees the reality of matrix elements in Eq. (72).

[8] As explained earlier, checking this convention would be equivalent to an explicit calculation of all $SO(6)$ coefficients, which is beyond our scope here.

[9] Up till now, we had worked under the assumption that only one well-defined Clebsch–Gordan coefficient exists for each triplet of $SU(3)$ state labels. The basis for this assumption was the fact that such an outer multiplicity does not appear in the context of products of $SO(6)$ hyperspherical harmonics, as we already noted that in the decomposition of the Hilbert space of functions over $SU(3)/SU(2)$ cosets, each $SU(3)$ UIR appears exactly once.

the same $SU(3)$ UIR, and that fact effectively implies that there is no outer multiplicity in the Clebsch–Gordan decomposition of Kronecker products of three-particle hyperspherical harmonics.

In practice, this means the following. Let there be n copies of UIR (K, Q) appearing in the product of (K$_1$, Q_1) and (K$_2$, Q_2), distinguished by the value of an additional label $\alpha = 1, 2, \dots n$ (obviously, there is freedom in choosing orthonormal bases within the sum of these irreducible spaces). A look-up in a Clebsch–Gordon table in a such case generally reveals the $n \geq 2$ values for the coefficients $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\quad\ \ 0_H\quad\ \ 0_H}$: $C_0^1, C_0^2, \dots C_0^n$. Nevertheless, the "overall magnitude" of these values $\sqrt{(C_0^1)^2 + (C_0^2)^2 + \cdots + (C_0^n)^2}$ is independent of any conventions and must coincide with the value determined by Eq. (78). Let us, for convenience call $C_r^1, C_r^2, \dots C_r^n$ the multiple values of the reduced CG coefficient $C_r{}^{\{\mathrm{K},Q\}\ \ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}}$ needed in Eq. (81). Then, the proper value of the CG coefficient to be plugged in Eq. (81) is the one "projected" on the relevant UIR, where $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\quad\ \ 0_H\quad\ \ 0_H}$ coefficient is nonzero:

$$C_r{}^{\{\mathrm{K},Q\}\ \ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\{0,0\}\ \ \{L',v'\}\ \ \{L'',v''\}} = \frac{(C_r^1 C_0^1 + C_r^2 C_0^2 + \cdots + C_r^n C_0^n)}{\sqrt{(C_0^1)^2 + (C_0^2)^2 + \cdots + (C_0^n)^2}}.$$

This formula further assumes that the value of $C^{\{\mathrm{K},Q\}\ \{\mathrm{K}',Q'\}\ \{\mathrm{K}'',Q''\}}_{\ 0_H\quad\ \ 0_H\quad\ \ 0_H}$ coefficient is taken to be positive, which is in agreement with our convention, that amounts to fixing the positive sign in Eq. (78), when compared with Eq. (B.21).

Explicit applications of the described method by using programs for the evaluation of $SU(3)$ Clebsch–Gordan coefficients, Refs. [54,55,57], confirm the above analysis (taking into account additional sign conventions used by the authors of these tables) and yield results identical with the values obtained by integration of explicit expressions for HSH (Appendix A).

9. Summary, discussion and conclusions

In summary, we have constructed the three-body permutation symmetric $SO(6)$ hyperspherical harmonics in three spatial dimensions. We used a method of constructing homogeneous harmonic polynomials that are labeled by SO(6) group's indices. In this way we arrived at the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset U(3) \rtimes S_2 \subset SO(6)$ (where $SO_{rot}(3)$ is the group of spatial rotations, $O(2)$ is the group of so-called "democracy" transformations where the permutation group S_3 is a (discrete) subgroup of the so-called "kinematic rotations", Ref. [2], or equivalently the "democracy" transformation (continuous) group $O(2)$, Ref. [12]).

The constructed symmetrized hyperspherical harmonics can be used to reformulate the three-body Schrödinger equation in three spatial dimensions, [48,50]. Then we calculated a certain type of integrals that appear in the three-body Schrödinger equation. We reduced these integrals at first to a product of two $SO(6)$ Clebsch–Gordan coefficients, and then to the product of two $SU(3)$ Clebsch–Gordan coefficients, that are readily available in the literature, Refs. [51–57,63], at least in their numerical form.

Next we give a brief discussion of some previous attempts at constructing symmetrized three-body hyperspherical harmonics and their relation to ours.

The first attempts to systematically construct all hyperspherical wave functions with well defined permutational symmetry go back to Aquilanti et al., Ref. [18] and subsequent papers.

They used something they called "tree pruning" technique (that appears to be related to the "tree" method of Vilenkin, Kuznetsov, and Smorodinskii, Ref. [61], see below) to obtain certain partial results in both 2D and 3D. Ultimately, this approach has not yielded a definitive answer – for a recent review of this approach see Ref. [28].

Second, Barnea and Novoselsky constructed hyperspherical wave functions with orthogonal and permutational symmetry in Ref. [20], where they used "a recursive algorithm for the (efficient) construction of N-body wave functions that belong to a given irreducible representation (irrep) of the orthogonal group and are at the same time characterized by a well-defined permutational symmetry." Whereas, in the final instance, Barnea and Novoselsky's work ought to be related to ours, we note the following basic differences: a) their work is based on a different subgroup chain, with a missing link in comparison with ours: $O(3) \otimes S_3 \subset O(6)$ vs. our $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$; b) theirs is an essentially recursive-numerical method relying on the knowledge of tables of the symmetric group S_3 Clebsch–Gordan coefficients, whereas ours is a group-theoretical approach; c) their S_3 hyperspherical states are expressed in terms of S_2 hyperspherical states, that are coupled, via the "tree" method of Vilenkin, Kuznetsov, and Smorodinskii, Ref. [61]; whereas ours makes no reference to any two-body substate; d) they evaluated only matrix elements of two-body operators, whereas we can treat all kinds of three-body operators.

Third, it ought to be said that Wang and Kuppermann Ref. [23] used symbolic algebra programs to calculate certain three- and four-body hyperspherical harmonics that were used in atomic and molecular physics. Their method does not seem to be based on a clearly defined algorithm, or group structure, however.

Last, but not least, we re-iterate that Dragt, Ref. [6] had used the $SU(3) \subset SO(6)$ chain of algebras to label three-particle scattering states as early as 1965, with follow-up work in Refs. [8,14,15], albeit with an emphasis on the applications to three-body decays, as opposed to our emphasis on applications to the three-body bound-state problem. Of course, these results, particularly those in the all but forgotten/unnoticed Refs. [14,15], must be closely related to ours, but this relation is not straightforward to see, due to their use of different kinematic variables (Dalitz–Fabri coordinates vs. hyperspherical angles) and to different construction methods. Ref. [15] in particular shows tables of some ($L \leq 6$, K ≤ 12) harmonics and their matrix elements. Ref. [14] on the other hand, gives "classification of three particle states according to an orthonormal $SU(3) \supset SO(3)$ basis" and then some. These authors simply could not evaluate the triple-harmonic matrix elements in terms of $SU(3)$ Clebsch–Gordan coefficients without the benefit of more recent developments, such as those in Refs. [54,55].

We conclude that we have provided (all of) the previously missing pieces that are sufficient for a complete reduction and efficient solution of the three-body Schrödinger equation, as in Refs. [48,50], and thus we opened the doors to simplified algebraic and faster numerical solutions to many specific physical three-body problems.

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Appendix A. Integrals over the $SO(6)$ hyper-sphere

Let $f(x_1, x_2, \ldots x_n)$ be a homogeneous function of degree K of n coordinates $x_i, i = 1, 2, \ldots n$:

$$f_{\mathrm{K}}(ax_1, ax_2, \ldots ax_n) = a^{\mathrm{K}} f_{\mathrm{K}}(x_1, x_2, \ldots x_n), \quad a \neq 0. \tag{A.1}$$

In particular, it holds $f_{\mathrm{K}}(x_1, x_2, \ldots x_n) = R^{\mathrm{K}} f(\frac{x_1}{R}, \frac{x_2}{R}, \ldots \frac{x_n}{R})$, where R is the radius in the n-dimensional space: $R = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The weighted integral of the function $f_{\mathrm{K}}(x_1, x_2, \ldots x_n)$ by the factor e^{-aR^2} over the entire volume of the hyper-space V can be evaluated via an unit-radius hyper-sphere Ω surface integral as follows:

$$\int_V f_{\mathrm{K}}(x_1, x_2, \ldots x_n) e^{-aR^2} dV = \int_0^\infty R^{\mathrm{K}} e^{-aR^2} R^{n-1} dR \int_\Omega f(\frac{x_1}{R}, \frac{x_2}{R}, \ldots \frac{x_n}{R}) d\Omega. \tag{A.2}$$

This leads to the following connection of hyper-sphere surface and volume integrals:

$$\int_\Omega f(\frac{x_1}{R}, \frac{x_2}{R}, \ldots \frac{x_n}{R}) d\Omega = \frac{\int_V f_{\mathrm{K}}(x_1, x_2, \ldots x_n) e^{-aR^2} dV}{\frac{1}{2} a^{-\frac{\mathrm{K}+n}{2}} \Gamma(\frac{\mathrm{K}+n}{2})}. \tag{A.3}$$

In the particular case when the function is a homogeneous polynomial $f_{\mathrm{K}}(x_1, x_2, \ldots x_n) = x_1^{\mathrm{K}_1} x_2^{\mathrm{K}_2} \cdots x_n^{\mathrm{K}_n}$ with $\sum_i \mathrm{K}_i = \mathrm{K}$ the right-hand-side can be explicitly evaluated:

$$\int_\Omega \frac{1}{R^{\mathrm{K}}} x_1^{\mathrm{K}_1} x_2^{\mathrm{K}_2} \cdots x_n^{\mathrm{K}_n} d\Omega = \frac{\prod_{i=1}^n \frac{1+(-1)^{\mathrm{K}_i}}{2} a^{-\frac{\mathrm{K}_i+1}{2}} \Gamma(\frac{\mathrm{K}_i+1}{2})}{\frac{1}{2} a^{-\frac{\mathrm{K}+n}{2}} \Gamma(\frac{\mathrm{K}+n}{2})} = 2 \frac{\prod_{i=1}^n \frac{1+(-1)^{\mathrm{K}_i}}{2} \Gamma(\frac{\mathrm{K}_i+1}{2})}{\Gamma(\frac{\mathrm{K}+n}{2})}. \tag{A.4}$$

If the hyper-spherical integrand on the left side is evaluated on the unit radius sphere, the $\frac{1}{R^{\mathrm{K}}}$ factor can be obviously left out.

Appendix B. Three-body $SO(6)$ hyperspherical harmonics as Wigner D-functions on the $SU(3)/SU(2)$ coset space

B.1. Spherical harmonics as Wigner D-functions

Spherical (hyperspherical) harmonics in a generalized sense are functions on a given manifold $\mathcal{M}$ of interest that have certain given properties w.r.t. action of some group G that acts transitively on $\mathcal{M}$ (here we will constrain to the cases when G is a compact Lie group). More precisely, (hyper)spherical harmonics are usually required to transform as basis vectors of unitary irreducible representations of G, and are given labels accordingly. E.g. harmonic function $\mathcal{Y}_m^L(\Omega), \Omega \in \mathcal{M}$ is required to transform under action of G as basis vector m of the irreducible representation L of G:

$$g : \mathcal{Y}_m^L(\Omega) \to \sum_{m'} D_{m'm}^L(g) \mathcal{Y}_{m'}^L(\Omega), \qquad g \in G, \tag{B.1}$$

where $D_{m'm}^L(g)$ is matrix representing element g in irreducible representation given by label(s) L:

$$D^L_{m'm}(g) = \langle{}^L_{m'}\big|D(g)\big|{}^L_m\rangle. \tag{B.2}$$

Seen as functions of g, matrix elements $D^L_{m'm}(g)$ are known as Wigner D-functions.

We proceed by considering square integrable functions on the G group manifold. Hilbert space of such functions we denote as $\mathcal{L}^2(G)$ and its vectors are of the form:

$$|\phi\rangle = \int\limits_G \phi(g)\,|g\rangle\,dg, \qquad g \in G, \tag{B.3}$$

where ϕ is a square integrable function on the manifold of group G parameters, $|g\rangle$ are (generalized) basis vectors and dg is the normalized Haar measure.

The usual (left) action of the group G elements is given by:

$$g'\,|\phi\rangle = g'\int \phi(g)\,|g\rangle\,dg = \int \phi(g)\,\big|g'g\big\rangle dg, \qquad g', g \in G. \tag{B.4}$$

Note that this action induces the following transformation of the function ϕ:

$$g' : \phi(g) \to \phi'(g) = \phi({g'}^{-1}g). \tag{B.5}$$

The functions belonging to $\mathcal{L}^2(G)$ that transform according to (B.1) are (complex conjugate) Wigner D-functions $D^{*L}_{mk}(g)$, as can be easily verified:

$$g' : D^{*L}_{mk}(g) \to D^{*L}_{mk}({g'}^{-1}g) = \sum_{m'} {D^{*L}_{mm'}}^{-1}(g')D^{*L}_{m'k}(g) = \sum_{m'} D^L_{m'm}(g')D^{*L}_{m'k}(g). \tag{B.6}$$

Notice that index k above, that corresponds to a vector of UIR L, is arbitrary.

If the manifold $\mathcal{M}$ were to coincide with the group G manifold, then it is these functions $D^{*L}_{mk}(g)$ that would play the role of the (hyper) spherical harmonics. However, in most cases of practical interest that is not the case. Instead, each point Ω of the manifold $\mathcal{M}$ has nontrivial stabilizer (isotropy) subgroup $H_\Omega \subset G$ such that $H_\Omega \cdot \Omega = \Omega$. As we already assumed that action of G is transitive on $\mathcal{M}$, all stabilizer subgroups H_Ω are mutually conjugate, isomorphic to some group H, and the manifold $\mathcal{M}$ is homogeneous space G/H.

To obtain hyperspherical harmonics on $\mathcal{M}$ we need functions of Ω that transform according to (B.1). Let $g(\Omega)$ be a mapping from $\mathcal{M}$ to a set of coset representatives in G. Then arbitrary group element g can be written as $g(\Omega)h$ for some $\Omega \in \mathcal{M}$ and $h \in H$. To obtain (hyper)spherical functions on $\mathcal{M}$ in the sense of definition (B.1), we can use the arbitrariness of choice of vector k in Eq. (B.6), by choosing it to be invariant w.r.t. action of subgroup H, i.e. $k = 0_H$ where:

$$D(h)\big|{}^L_{0_H}\big\rangle = \big|{}^L_{0_H}\big\rangle, \forall h \in H. \tag{B.7}$$

Now the Wigner D-function $D^{*L}_{mk}(g)$ reduces to a function on the coset space $\mathcal{M}$ by:

$$D^{*L}_{m0_H}(g) = D^{*L}_{m0_H}(g(\Omega)h) = \langle{}^L_m\big|D(g(\Omega))D(h)\big|{}^L_{0_H}\big\rangle = D^{*L}_{m0_H}(g(\Omega)) \equiv D^{*L}_{m0_H}(\Omega). \tag{B.8}$$

Therefore, functions $D^{*L}_{m0_H}(\Omega)$, where $\big|{}^L_{0_H}\big\rangle$ is invariant w.r.t. H action, are functions on manifold $\mathcal{M} = G/H$ that properly transform, in the sense of (B.1), under the action of group G, i.e. transform as vector $\big|{}^L_m\big\rangle$ of the UIR labeled by L. Thus we can establish proportionality between generalized (hyper)spherical harmonics and Wigner D-functions:

$$\mathcal{Y}^L_m(\Omega) = N(L)D^{*L}_{m0_H}(\Omega), \tag{B.9}$$

with $N(L)$ being a proportionality constant possibly dependent on L. In addition to transformation properties (B.1), it is usual to require that a (hyper)spherical harmonic should be normalized to unity. From:

$$\begin{aligned}\int\limits_{\mathcal{M}} \mathcal{Y}^{*L}_{m}(\Omega)\mathcal{Y}^{L'}_{m'}(\Omega)d\Omega &= \int\limits_{\mathcal{M}} N(L)N(L')D^{L}_{m0_H}(\Omega)D^{*L}_{m'0_H}(\Omega)d\Omega \\ &= \frac{N(L)N(L')}{V_H}\int\limits_{\mathcal{M}}\int\limits_{H} D^{L}_{m0_H}(g(\Omega)h)D^{*L}_{m'0_H}(g(\Omega)h)d\Omega dh \\ &= \frac{N(L)N(L')}{V_H}\int\limits_{G} D^{L}_{m0_H}(g)D^{*L}_{m'0_H}(g)dg \\ &= \frac{N(L)N(L')}{V_H}\frac{V_G}{dim(L)}\delta_{JJ'}\delta_{mm'},\end{aligned} \tag{B.10}$$

where $d\Omega$ is a measure on $\mathcal{M}$, V_H and V_G are volumes of H and G group manifolds (corresponding to measures dh and dg) and $dim(L)$ is the dimension of the UIR L, it follows $N(L) = \sqrt{\frac{dim(L)}{V_{\mathcal{M}}}}$. (Compactness of the subgroup H and of the manifold M follows from the presumed compactness of G.) Finally we conclude:

$$\mathcal{Y}^{L}_{m}(\Omega) = \sqrt{\frac{dim(L)}{V_{\mathcal{M}}}}D^{*L}_{m0_H}(\Omega), \tag{B.11}$$

while, of course, arbitrariness of choice of overall complex phase in the definition necessarily remains.

B.2. Integral of three (hyper)spherical harmonics

Expressing (hyper)spherical harmonics via Wigner D-functions allows immediate evaluation of integral of three (or more) of h.s. harmonics in terms of group G Clebsch–Gordan coefficients. Namely:

$$\begin{aligned}&\int\limits_{\mathcal{M}} \mathcal{Y}^{*L}_{m}(\Omega)\mathcal{Y}^{L_1}_{m_1}(\Omega)\mathcal{Y}^{L_2}_{m_2}(\Omega)d\Omega \\ &\quad= \sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{\mathcal{M}} D^{L}_{m0_H}(\Omega)D^{*L_1}_{m_10_H}(\Omega)D^{*L_2}_{m_20_H}(\Omega)d\Omega \\ &\quad= \frac{1}{V_H}\sqrt{\frac{\dim(L)\dim(L_1)\dim(L_2)}{V^3_{\mathcal{M}}}}\int\limits_{G} D^{L}_{m0_H}(g)D^{*L_1}_{m_10_H}(g)D^{*L_2}_{m_20_H}(g)dg \\ &\quad= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(L_1)\dim(L_2)}{\dim(L)}}C^{L_1\,L_2\,L}_{m_1\,m_2\,m}C^{L_1\,L_2\,L}_{0_H\,0_H\,0_H},\end{aligned} \tag{B.12}$$

where $C^{L_1\,L_2\,L}_{m_1\,m_2\,m}$ denotes a G group Clebsch–Gordan coefficient and the well known formulas for integral of three Wigner D-functions were used.

B.3. Application to three particle systems

It is known, Ref. [14,15], that three-particle hyper-spherical harmonics can be viewed as functions on the $SU(3)/SU(2)$ coset space. Thus all of the previous considerations directly apply to this case. Of particular importance for the evaluation of interaction matrix elements is the formula for integral of three three-particle h.s. harmonics, that now takes the form:

$$\int_{\mathcal{M}} \mathcal{Y}^{*\mathrm{K}Q\nu}_{L,m}(X)\mathcal{Y}^{\mathrm{K}_1Q_1\nu_1}_{L_1,m_1}(X)\mathcal{Y}^{\mathrm{K}_2Q_2\nu_2}_{L_2,m_2}(X)dX^3$$
$$= \frac{1}{\sqrt{V_{\mathcal{M}}}}\sqrt{\frac{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}{\dim(\mathrm{K},Q)}}\, C^{\{\mathrm{K}_1,Q_1\}\ \ \{\mathrm{K}_2,Q_2\}\ \ \{\mathrm{K},Q\}}_{\{L_1,m_1,\nu_1\}\{L_2,m_2,\nu_2\}\{L,m,\nu\}} C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{0_H\qquad 0_H\qquad 0_H}, \tag{B.13}$$

where X_i are the complex coordinates (6), $\dim(\mathrm{K},Q) = \frac{1}{8}(\mathrm{K}+2)(\mathrm{K}-Q+2)(\mathrm{K}+Q+2)$ and $V_{\mathcal{M}} = \pi^3$. The $SU(3)$ Clebsch–Gordan coefficients appearing in the above formula are well known entities, whose (numerical) evaluation is well studied in the literature [57], in various $SU(3)$ bases.

The second of the two Clebsch–Gordan coefficients can be evaluated in the following way. From (B.13) it follows that

$$|C^{\{\mathrm{K}_1,Q_1\}\ \{\mathrm{K}_2,Q_2\}\ \{\mathrm{K},Q\}}_{0_H\qquad 0_H\qquad 0_H}|$$
$$= \left(\sqrt{V_{\mathcal{M}}}\sqrt{\frac{\dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}}\int_{\mathcal{M}} \mathcal{Y}^{*\{\mathrm{K},Q\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K}_1,Q_1\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K}_2,Q_2\}}_{0_H}(X)dX^3\right)^{\frac{1}{2}}. \tag{B.14}$$

Now, $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ is an $SU(3)$ harmonic on $SU(3)/SU(2)$ that is invariant w.r.t. $SU(2)$ subgroup, and for the sake of concreteness, let it be the subgroup that nontrivially acts on indices 1 and 2. That means that $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ is of the form:

$$\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X) = \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} A^{\mathrm{K},Q}_{\mathrm{K}'}\cdot (X_3^+)^{\frac{\mathrm{K}'+Q}{2}}(X_3^-)^{\frac{\mathrm{K}'-Q}{2}}, \tag{B.15}$$

where sum over K' goes only over odd or over even integers and $A^{\mathrm{K},Q}_{\mathrm{K}'}$ are algebraic coefficients that can be determined from two requirements: i) the functions $\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)$ must be $SO(6)$ harmonics, i.e., $\Delta\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X) = 0$ and ii) they have to be normalized to unity, i.e.,

$$\int_{\mathcal{M}} \mathcal{Y}^{*\{\mathrm{K},Q\}}_{0_H}(X)\mathcal{Y}^{\{\mathrm{K},Q\}}_{0_H}(X)dX^3 = 1. \tag{B.16}$$

From the first requirement one obtains:

$$A^{\mathrm{K},Q}_{\mathrm{K}'} = A^{\mathrm{K},Q}_{0}\Pi^{\mathrm{K},Q}_{\mathrm{K}'} \tag{B.17}$$

where $A^{\mathrm{K},Q}_{0}$ is a remaining constant, yet to be determined, and

$$\Pi^{\mathrm{K},Q}_{\mathrm{K}'} = \prod_{\mathrm{K}''=|Q|,|Q|+2,\ldots}^{\mathrm{K}'-2}\left(1-\frac{(\mathrm{K}+2)^2-Q^2}{(\mathrm{K}''+2)^2-Q^2}\right), \tag{B.18}$$

where the product over K″, yet again takes only every other integer value, even or odd, as the case may be.

By plugging Eq. (B.15) into Eq. (B.16) and by using integration formulas from Appendix A one determines the absolute value of the remaining constant as

$$|A_0^{\mathrm{K},Q}| = \left(\sum_{\mathrm{K}_1,\mathrm{K}_2=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi_{\mathrm{K}_1}^{\mathrm{K},Q} \Pi_{\mathrm{K}_2}^{\mathrm{K},Q} \frac{2\pi^3}{(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+1)(\frac{\mathrm{K}_1+\mathrm{K}_2}{2}+2)} \right)^{-\frac{1}{2}}. \tag{B.19}$$

Although the phase factor will turn out to be irrelevant for our purposes, for completeness' sake we note that the phase of the constant $A_0^{\mathrm{K},Q}$ and thus the overall phase of the harmonic $\mathcal{Y}_{0_H}^{\{\mathrm{K},Q\}}(X)$ can be recovered from the consistency requirement obtained from (B.11) by taking Ω to be the coset of the unit group element:

$$\int_{\mathcal{M}} \sqrt{\frac{\dim(\mathrm{K},Q)}{\pi^3}} \mathcal{Y}_{0_H}^{*\{\mathrm{K},Q\}}(X) \mathcal{Y}_{L,m}^{\mathrm{K}'Q'\nu}(X) dX^3 = \mathcal{Y}_{L,m}^{\mathrm{K}'Q'\nu}(X)\Big|_{X_1=0,X_2=0,X_3=1}. \tag{B.20}$$

Thus we conclude that $A_0^{\mathrm{K},Q} = (-1)^{\frac{\mathrm{K}-|Q|}{2}} |A_0^{\mathrm{K},Q}|$.

Finally, we plug the resulting expressions for $\mathcal{Y}_{0_H}^{\{\mathrm{K},Q\}}(X)$ into Eq. (B.14) and obtain:

$$\begin{aligned} |C_{0_H\ \ 0_H\ \ 0_H}^{\{\mathrm{K}_1,Q_1\}\,\{\mathrm{K}_2,Q_2\}\,\{\mathrm{K},Q\}}| &= \Bigg(A_0^{\mathrm{K}_1,Q} A_0^{\mathrm{K}_2,Q} A_0^{\mathrm{K},Q} \sqrt{\frac{\pi^3 \dim(\mathrm{K},Q)}{\dim(\mathrm{K}_1,Q_1)\dim(\mathrm{K}_2,Q_2)}} \cdot \\ & \sum_{\mathrm{K}_1'=|Q_1|,|Q_1|+2,\ldots}^{\mathrm{K}_1} \sum_{\mathrm{K}_2'=|Q_2|,|Q_2|+2,\ldots}^{\mathrm{K}_2} \sum_{\mathrm{K}'=|Q|,|Q|+2,\ldots}^{\mathrm{K}} \Pi_{\mathrm{K}_1'}^{\mathrm{K}_1,Q_1} \Pi_{\mathrm{K}_2'}^{\mathrm{K}_2,Q_2} \Pi_{\mathrm{K}'}^{\mathrm{K},-Q} \\ & \times \frac{2\pi^3}{(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+1)(\frac{\mathrm{K}_1'+\mathrm{K}_2'+\mathrm{K}'}{2}+2)} \delta_{Q_1+Q_2,Q} \Bigg)^{\frac{1}{2}}. \end{aligned} \tag{B.21}$$

Appendix C. Tables of hyper-spherical harmonics

Below we explicitly list all hyper-spherical harmonics up to K = 6, where multiplicity is resolved by using the operator Eq. (37). We list only the harmonics with $m = L$ and $Q \geq 0$, as the rest can be easily obtained by acting on them with standard lowering operators Eq. (60) and by using the permutation symmetry Eqs. (63), (65): $\mathcal{Y}_{L,m}^{\mathrm{K}Q\nu}(\lambda, \rho) = (-1)^{\mathrm{K}-L} \mathcal{Y}_{L,m}^{\mathrm{K}-Q-\nu}(\lambda, -\rho)$. We write the $\mathrm{K} \leq 3$ harmonics in both complex spherical and Jacobi coordinates.

Of the harmonics listed below, expressions for $\mathcal{Y}_{0,0}^{4,0,0}(X)$ and $\mathcal{Y}_{0,0}^{6,6,0}(X)$ can be compared with the corresponding expressions in [5], where a few particular examples for $L = 0$ are explicitly shown. After taking into account the differences in notation it is easily verified that the expressions coincide.

$$\mathcal{Y}_{0,0}^{0,0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}_{1,1}^{1,1,-1}(X) = \frac{\sqrt{\frac{3}{2}} X_+^+}{\pi^{3/2} R} = \frac{\sqrt{\frac{3}{2}} (\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))}{\pi^{3/2} \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}}$$

$$\mathcal{Y}_{1,1}^{2,0,0}(X) = \frac{\sqrt{3}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2} R^2} = \frac{2\sqrt{3}(\lambda_3(\rho_2 - i\rho_1) + i(\lambda_1 + i\lambda_2)\rho_3)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,0,0}_{2,2}(X) = \frac{\sqrt{3}X^+_+X^-_+}{\pi^{3/2}R^2} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,0}_{0,0}(X) = \frac{\sqrt{2}\left|X^+\right|^2}{\pi^{3/2}R^2}$$
$$= \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{2,2,-3}_{2,2}(X) = \frac{\sqrt{\frac{3}{2}}\left(X^+_+\right)^2}{\pi^{3/2}R^2} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}^{3,1,3}_{1,1}(X) = \frac{\sqrt{6}\left(X^-_+\left|X^+\right|^2 - \frac{1}{2}R^2X^+_+\right)}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{6}}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$
$$\times \Bigg(\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)\left(\left(\lambda_1 + i\rho_1\right)^2 + \left(\lambda_2 + i\rho_2\right)^2 + \left(\lambda_3 + i\rho_3\right)^2\right)$$
$$- \frac{1}{2}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)\Bigg)$$

$$\mathcal{Y}^{3,1,-5}_{2,2}(X) = \frac{\sqrt{5}X^+_+\left(X^-_+X^+_0 - X^+_+X^-_0\right)}{\pi^{3/2}R^3}$$
$$= \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,1,-2}_{3,3}(X) = \frac{\sqrt{15}\left(X^+_+\right)^2X^-_+}{2\pi^{3/2}R^3}$$
$$= \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-1}_{1,1}(X) = \frac{\sqrt{3}X^+_+\left|X^+\right|^2}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{3,3,-6}_{3,3}(X) = \frac{\sqrt{5}\left(X^+_+\right)^3}{2\pi^{3/2}R^3} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}^{4,0,0}_{0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^-\right|^2\left|X^+\right|^2\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,-\sqrt{105}}(X) = \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2 \left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2 \left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,\sqrt{105}}(X) = \frac{-12\sqrt{14}R^2 X_+^+ X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2 \left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2 \left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,0,0}(X) = \frac{3\sqrt{5}X_+^+ X_+^- \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,0,0}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_+^+\right)^2 \left(X_+^-\right)^2}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{1,1}^{4,2,2}(X) = \frac{3\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,2,2}(X) = \frac{\sqrt{\frac{3}{7}}X_+^+ \left(5X_+^- \left|X^+\right|^2 - 2R^2 X_+^+\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,2,-13}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_+^+\right)^2 \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,2,-5}(X) = \frac{\sqrt{15}\left(X_+^+\right)^3 X_+^-}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{0,0}^{4,4,0}(X) = \frac{\sqrt{3}\left|X^+\right|^4}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,4,-3}(X) = \frac{3\sqrt{\frac{5}{14}}\left(X_+^+\right)^2 \left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,4,-10}(X) = \frac{\sqrt{15}\left(X_+^+\right)^4}{4\pi^{3/2}R^4}$$

$$\mathcal{Y}_{1,1}^{5,1,-3}(X) = -\frac{\sqrt{3}\left(X_+^+\left(R^4 - 3\left|X^-\right|^2\left|X^+\right|^2\right) + R^2 X_+^- \left|X^+\right|^2\right)}{\pi^{3/2}R^5}$$

$$\mathcal{Y}_{2,2}^{5,1,13}(X) = \frac{\sqrt{\frac{5}{2}}\left(X_+^+ X_0^- - X_+^- X_0^+\right)\left(R^2 X_+^+ - 3X_+^- \left|X^+\right|^2\right)}{\pi^{3/2}R^5}$$

$$\mathcal{Y}_{3,3}^{5,1,-7-\sqrt{241}}(X) = \frac{\sqrt{\frac{35}{723}}X_+^+\left(\sqrt{482-26\sqrt{241}}X_+^-\left(3X_+^-\left|X^+\right|^2 - 2R^2X_+^+\right) + \sqrt{2651+163\sqrt{241}}X_+^+\left(X_+^+\left|X^-\right|^2 - R^2X_+^-\right)\right)}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{3,3}^{5,1,-7+\sqrt{241}}(X) = \frac{\sqrt{\frac{35}{723}}X_+^+\left(\sqrt{482+26\sqrt{241}}X_+^-\left(3X_+^-\left|X^+\right|^2 - 2R^2X_+^+\right) + \sqrt{2651-163\sqrt{241}}X_+^+\left(X_+^+\left|X^-\right|^2 - R^2X_+^-\right)\right)}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{4,4}^{5,1,-8}(X) = \frac{3\sqrt{7}\left(X_+^+\right)^2 X_+^- \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^5}$$

$$\mathcal{Y}_{5,5}^{5,1,-3}(X) = \frac{\sqrt{105}\left(X_+^+\right)^3 \left(X_+^-\right)^2}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{1,1}^{5,3,5}(X) = \frac{\sqrt{\frac{3}{2}}\left|X^+\right|^2 \left(3X_+^- \left|X^+\right|^2 - 2R^2 X_+^+\right)}{\pi^{3/2}R^5}$$

$$\mathcal{Y}_{2,2}^{5,3,-3}(X) = \frac{\sqrt{15}X_+^+ \left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^5}$$

$$\mathcal{Y}_{3,3}^{5,3,0}(X) = -\frac{\sqrt{\frac{35}{6}}\left(X_+^+\right)^2 \left(R^2 X_+^+ - 3X_+^- \left|X^+\right|^2\right)}{2\pi^{3/2}R^5}$$

$$\mathcal{Y}_{4,4}^{5,3,-24}(X) = \frac{\sqrt{21}\left(X_+^+\right)^3 \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^5}$$

$$\mathcal{Y}_{5,5}^{5,3,-9}(X) = \frac{\sqrt{\frac{105}{2}}\left(X_+^+\right)^4 X_+^-}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{1,1}^{5,5,-1}(X) = \frac{3X_+^+ \left|X^+\right|^4}{\sqrt{2}\pi^{3/2}R^5}$$

$$\mathcal{Y}_{3,3}^{5,5,-6}(X) = \frac{\sqrt{\frac{35}{3}}\left(X_+^+\right)^3 \left|X^+\right|^2}{2\pi^{3/2}R^5}$$

$$\mathcal{Y}_{5,5}^{5,5,-15}(X) = \frac{\sqrt{\frac{21}{2}}\left(X_+^+\right)^5}{4\pi^{3/2}R^5}$$

$$\mathcal{Y}_{1,1}^{6,0,0}(X) = -\frac{\sqrt{6}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(R^4 - 3\left|X^-\right|^2\left|X^+\right|^2\right)}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{2,2}^{6,0,0}(X)$$
$$= -\frac{\sqrt{\frac{10}{7}}\left(X_+^+ X_+^- \left(R^4 - 7\left|X^-\right|^2\left|X^+\right|^2\right) + 2R^2\left(X_+^-\right)^2\left|X^+\right|^2 + 2R^2\left(X_+^+\right)^2\left|X^-\right|^2\right)}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{3,3}^{6,0,-3\sqrt{105}}(X)$$
$$= \frac{\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(-8\sqrt{5}R^2 X_+^+ X_+^- + \left(7\sqrt{5} - 3\sqrt{21}\right)\left(X_+^-\right)^2\left|X^+\right|^2 + \left(7\sqrt{5} + 3\sqrt{21}\right)\left(X_+^+\right)^2\left|X^-\right|^2\right)}{2\sqrt{6}\pi^{3/2}R^6}$$

$$\mathcal{Y}_{3,3}^{6,0,3\sqrt{105}}(X)$$
$$= \frac{\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left(-8\sqrt{5}R^2 X_+^+ X_+^- + \left(7\sqrt{5} + 3\sqrt{21}\right)\left(X_+^-\right)^2\left|X^+\right|^2 + \left(7\sqrt{5} - 3\sqrt{21}\right)\left(X_+^+\right)^2\left|X^-\right|^2\right)}{2\sqrt{6}\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,0,-\sqrt{385}}(X)$$
$$= \frac{\sqrt{\frac{3}{154}}X_+^+ X_+^- \left(-6\sqrt{70}R^2 X_+^+ X_+^- + 7\sqrt{23 - \sqrt{385}}\left(X_+^-\right)^2\left|X^+\right|^2 + 7\sqrt{23 + \sqrt{385}}\left(X_+^+\right)^2\left|X^-\right|^2\right)}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,0,\sqrt{385}}(X) = \frac{\sqrt{\frac{3}{154}} X_+^+ X_+^- \left(-6\sqrt{70} R^2 X_+^+ X_+^- + 7\sqrt{23+\sqrt{385}} \left(X_+^-\right)^2 \left|X^+\right|^2 + 7\sqrt{23-\sqrt{385}} \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{5,5}^{6,0,0}(X) = \frac{\sqrt{105} \left(X_+^+\right)^2 \left(X_+^-\right)^2 \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{6,6}^{6,0,0}(X) = \frac{\sqrt{35} \left(X_+^+\right)^3 \left(X_+^-\right)^3}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{0,0}^{6,2,0}(X) = \frac{6\left|X^-\right|^2 \left|X^+\right|^4 - 4R^4 \left|X^+\right|^2}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{2,2}^{6,2,4-\sqrt{193}}(X) = -\frac{\sqrt{\frac{5}{1158}}}{21\pi^{3/2} R^6} \left(\sqrt{5597+355\sqrt{193}} X_+^+ \left(X_+^+ \left(5R^4 - 21\left|X^-\right|^2 \left|X^+\right|^2\right) + 12R^2 X_+^- \left|X^+\right|^2\right) + \sqrt{4439-305\sqrt{193}} \left(24R^2 X_+^+ X_+^- \left|X^+\right|^2 - 21\left(X_+^-\right)^2 \left|X^+\right|^4 - 4R^4 \left(X_+^+\right)^2\right)\right)$$

$$\mathcal{Y}_{2,2}^{6,2,4+\sqrt{193}}(X) = \frac{\sqrt{\frac{5}{1158}}}{21\pi^{3/2} R^6} \left(\sqrt{4439+305\sqrt{193}} \left(-24R^2 X_+^+ X_+^- \left|X^+\right|^2 + 21\left(X_+^-\right)^2 \left|X^+\right|^4 + 4R^4 \left(X_+^+\right)^2\right) - \sqrt{5597-355\sqrt{193}} X_+^+ \left(X_+^+ \left(5R^4 - 21\left|X^-\right|^2 \left|X^+\right|^2\right) + 12R^2 X_+^- \left|X^+\right|^2\right)\right)$$

$$\mathcal{Y}_{3,3}^{6,2,8}(X) = \frac{X_+^+ \left(X_+^+ X_0^- - X_+^- X_0^+\right) \left(2R^2 X_+^+ - 7X_+^- \left|X^+\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,2,-40}(X) = \frac{3\sqrt{\frac{5}{506}} \left(X_+^+\right)^2 \left(-20R^2 X_+^+ X_+^- + 7\left(X_+^-\right)^2 \left|X^+\right|^2 + 14\left(X_+^+\right)^2 \left|X^-\right|^2\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{4,4}^{6,2,6}(X) = \frac{\sqrt{\frac{21}{23}} \left(X_+^+\right)^2 \left(-8R^2 X_+^+ X_+^- + 12\left(X_+^-\right)^2 \left|X^+\right|^2 + \left(X_+^+\right)^2 \left|X^-\right|^2\right)}{2\pi^{3/2} R^6}$$

$$\mathcal{Y}_{5,5}^{6,2,-19}(X) = \frac{\sqrt{\frac{35}{2}} \left(X_+^+\right)^3 X_+^- \left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{6,6}^{6,2,-7}(X) = \frac{\sqrt{105} \left(X_+^+\right)^4 \left(X_+^-\right)^2}{4\pi^{3/2} R^6}$$

$$\mathcal{Y}_{1,1}^{6,4,4}(X) = \frac{3\sqrt{2} \left(X_+^- X_0^+ - X_+^+ X_0^-\right) \left|X^+\right|^4}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{2,2}^{6,4,4}(X) = \frac{\sqrt{\frac{10}{21}} X_+^+ \left|X^+\right|^2 \left(7X_+^- \left|X^+\right|^2 - 4R^2 X_+^+\right)}{\pi^{3/2} R^6}$$

$$\mathcal{Y}_{3,3}^{6,4,-11}(X) = \frac{\sqrt{\frac{35}{2}}\left(X_+^+\right)^2\left(X_+^- X_0^+ - X_+^+ X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,4,-3}(X) = \frac{\sqrt{\frac{3}{11}}\left(X_+^+\right)^3\left(7X_+^-\left|X^+\right|^2 - 2R^2X_+^+\right)}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{5,5}^{6,4,-38}(X) = \frac{\sqrt{\frac{35}{2}}\left(X_+^+\right)^4\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{6,6}^{6,4,-14}(X) = \frac{\sqrt{\frac{21}{2}}\left(X_+^+\right)^5 X_+^-}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{0,0}^{6,6,0}(X) = \frac{2\left|X^+\right|^6}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{2,2}^{6,6,-3}(X) = \frac{\sqrt{5}\left(X_+^+\right)^2\left|X^+\right|^4}{\pi^{3/2}R^6}$$

$$\mathcal{Y}_{4,4}^{6,6,-10}(X) = \frac{\sqrt{\frac{105}{11}}\left(X_+^+\right)^4\left|X^+\right|^2}{2\pi^{3/2}R^6}$$

$$\mathcal{Y}_{6,6}^{6,6,-21}(X) = \frac{\sqrt{7}\left(X_+^+\right)^6}{4\pi^{3/2}R^6}$$

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Algebraic Bethe ansatz for the XXZ Heisenberg spin chain with triangular boundaries and the corresponding Gaudin model

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Abstract

The implementation of the algebraic Bethe ansatz for the XXZ Heisenberg spin chain in the case, when both reflection matrices have the upper-triangular form is analyzed. The general form of the Bethe vectors is studied. In the particular form, Bethe vectors admit the recurrent procedure, with an appropriate modification, used previously in the case of the XXX Heisenberg chain. As expected, these Bethe vectors yield the strikingly simple expression for the off-shell action of the transfer matrix of the chain as well as the spectrum of the transfer matrix and the corresponding Bethe equations. As in the XXX case, the so-called quasi-classical limit gives the off-shell action of the generating function of the corresponding trigonometric Gaudin Hamiltonians with boundary terms.

1. Introduction

The quantum inverse scattering method (QISM) is an approach to construct and solve quantum integrable systems [1–3]. In the framework of the QISM the algebraic Bethe ansatz is a powerful algebraic approach, which yields the spectrum and corresponding eigenstates for the systems for which highest weight type representations are relevant, like for example quantum

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spin systems, Gaudin models, etc. In particular, the Heisenberg spin chain [4], with periodic boundary conditions, has been studied by the algebraic Bethe ansatz [1,3], including the question of completeness and simplicity of the spectrum [5].

A way to introduce non-periodic boundary conditions compatible with the integrability of one-dimensional solvable quantum systems was developed in [6]. The boundary conditions are expressed in the form of the left and right reflection matrices. The compatibility conditions between the bulk and the boundary of the system take the form of the so-called reflection equation, at the left site, and the dual reflection equation, at the right site of the system. The matrix form of the exchange relations between the entries of the Sklyanin monodromy matrix is analogous to the reflection equation. Together with the dual reflection equation they yield the commutativity of the open transfer matrix [6–8].

There is a renewed interest in applying the algebraic Bethe ansatz to the open XXX and XXZ chains with non-periodic boundary conditions compatible with the integrability of the systems [9–17]. Other approaches include the Bethe ansatz based on the functional relation between the eigenvalues of the transfer matrix and the quantum determinant and the associated T-Q relation [18–20], functional relations for the eigenvalues of the transfer matrix based on fusion hierarchy [21] and the Vertex-IRF correspondence [22,23]. For a review of the coordinate Bethe ansatz for non-diagonal boundaries see [24]. For the latest results, as well as an excellent review, on the application of the separation of variables method on the 6-vertex model and the associate XXZ quantum chains see [25]. However, we will focus on applying the algebraic Bethe ansatz to the XXZ Heisenberg spin chain in the case when system admits the so-called pseudo-vacuum, or the reference state. In his seminal work on boundary conditions in quantum integrable models Sklyanin has studied the XXZ spin chain with diagonal boundaries [6]. As opposed to the case of the open XXX Heisenberg chain were both reflection matrices can be simultaneously brought to a triangular form by a single similarity transformation which leaves the R-matrix invariant and it is independent of the spectral parameter [10–12], here the triangularity of the K-matrices has to be imposed by hand. The algebraic Bethe ansatz was applied to the XXZ spin-$\frac{1}{2}$ chain with upper triangular reflection matrices [13,14]. The spectrum and the corresponding Bethe equations were obtained [13] and the Bethe vectors were defined using a family of creations operators [14].

This work is centered on the study of the Bethe vectors which are fundamental in the implementation of the algebraic Bethe ansatz for the XXZ Heisenberg spin chain when the corresponding reflection matrices have the upper-triangular form. Seeking the Bethe vectors $\widetilde{\Psi}_M(\mu_1, \mu_2, \ldots, \mu_M)$ which would in the scaling limit coincide with the ones of the XXX Heisenberg chain [12], we have also found certain identities yielding the general form of the Bethe vectors for a fixed M. The general form of Bethe vectors is given as a sum of a particular vector and the linear combination of lower order Bethe vectors. Due to certain identities this linear combination of lower order Bethe vectors corresponds the same eigenvalue as the particular vector. Although we have obtained explicitly the Bethe vectors $\widetilde{\Psi}_M(\mu_1, \mu_2, \ldots, \mu_M)$ for $M = 1, 2, 3, 4$, unfortunately they do not admit a compact closed form for an arbitrary M. However, a detailed analysis yields a particular form of the Bethe vectors $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ which admits the recurrence formulas for the coefficient functions analogous to the once used in the study of the XXX Heisenberg chain [12]. These Bethe vectors are defined explicitly, for an arbitrary natural number M, as some polynomial functions of the creation operators. Also, the off-shell action of the transfer matrix on these Bethe vectors is strikingly simple since it almost coincides with the corresponding action in the case when the two boundary matrices are diagonal. As expected, the off-shell action yields the spectrum of the transfer matrix and the corresponding Bethe equations. To explore further these results we use the so-called quasi-classical limit and obtain the off-shell action of the generating function of the trigonometric Gaudin Hamiltonians with boundary terms, on the corresponding Bethe vectors.

Originally in his approach, Gaudin defined these models as a quasi-classical limit of the integrable quantum chains [26,27]. The Gaudin models were extended to any simple Lie algebra, with arbitrary irreducible representation at each site of the chain [27]. Sklyanin studied the rational $s\ell(2)$ model in the framework of the quantum inverse scattering method using the $s\ell(2)$ invariant classical r-matrix [28]. A generalization of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang-Baxter equation [29] was relatively straightforward [30,31]. Therefore, considerable attention has been devoted to Gaudin models corresponding to the classical r-matrices of simple Lie algebras [32–34] and Lie superalgebras [35–39].

Hikami showed how the quasi-classical expansion of the XXZ transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the case when both reflection matrices are diagonal [40]. Then the algebraic Bethe ansatz was applied to open Gaudin model in the context of the Vertex-IRF correspondence [41–43]. Also, results were obtained for the open Gaudin models based on Lie superalgebras [44]. An approach to study the open Gaudin models based on the classical reflection equation [45] and the non-unitary r-matrices [46–48] was developed, see [49–53] and the references therein. For a review of the open Gaudin model see [54].

In [55] we have derived the generating function of the trigonometric Gaudin Hamiltonians with boundary terms following Sklyanin's approach for the periodic boundary conditions [28, 56]. Analogously to the rational case [52,12], our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXZ Heisenberg chain and the central element, the so-called Sklyanin determinant. Here we use this result with the objective to derive the off-shell action of the generating function. As we will show below, the quasi-classical expansion of the Bethe vectors we have defined for the XXZ Heisenberg spin chain yields the Bethe vectors of the corresponding Gaudin model. The importance of these Bethe vectors stems from the striking simplicity of the off-shell action of the generating function of the trigonometric Gaudin Hamiltonians with boundary terms.

This paper is organized as follows. In Section 2 we review the suitable R-matrix as well as the Lax operator and the corresponding monodromy as the fundamental tools of the quantum inverse scattering method in the study of the inhomogeneous XXZ Heisenberg spin chain. The general solutions of the relevant reflection equation and the corresponding dual reflection equation are surveyed in Section 3. In Section 4 we briefly expose the Sklyanin approach to the inhomogeneous XXZ Heisenberg spin chain with non-periodic boundary conditions, in particular the derivation of the relevant commutation relations. The implementation of the algebraic Bethe ansatz and most notably the study of the Bethe vectors, as one of the main results of the paper, are presented in Section 5. The corresponding Gaudin model is studied through the quasi-classical limit in Section 6. Our conclusions are presented in the Section 7. In Appendix A are given some basic definitions for the convenience of the reader. The commutation relations relevant for the implementation of the algebraic Bethe ansatz for the XXZ Heisenberg chain are given in the Appendix B. Finally, detailed presentation of the illustrative example of the Bethe vector $\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3)$, including its general form and some important identities, are given in Appendix C.

2. Inhomogeneous XXZ Heisenberg spin chain

The starting point in our study of the XXZ Heisenberg spin chain is the R-matrix [1,2,57,58]

$$R(\lambda,\eta)=\begin{pmatrix}\sinh(\lambda+\eta) & 0 & 0 & 0\\ 0 & \sinh(\lambda) & \sinh(\eta) & 0\\ 0 & \sinh(\eta) & \sinh(\lambda) & 0\\ 0 & 0 & 0 & \sinh(\lambda+\eta)\end{pmatrix}. \tag{2.1}$$

This R-matrix satisfies the Yang-Baxter equation [59,57,58,1,2] in the space $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu), \tag{2.2}$$

and it also has other relevant properties such as

$U(1)$ symmetry	$\left[\sigma_1^3+\sigma_2^3, R_{12}(\lambda)\right]=0;$
unitarity	$R_{12}(\lambda)R_{21}(-\lambda)=\sinh(\eta-\lambda)\sinh(\eta+\lambda)\mathbb{1};$
parity invariance	$R_{21}(\lambda)=R_{12}(\lambda);$
temporal invariance	$R_{12}^t(\lambda)=R_{12}(\lambda);$
crossing symmetry	$R(\lambda)=\mathcal{J}_1 R^{t_2}(-\lambda-\eta)\mathcal{J}_1,$

where t_2 denotes the transpose in the second space and the two-by-two matrix $\mathcal{J}$ is proportional to the Pauli matrix σ^2, i.e. $\mathcal{J}=\imath\sigma^2$.

Here we study the inhomogeneous XXZ spin chain with N sites, characterized by the local space $V_m=\mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. The Hilbert space of the system is

$$\mathcal{H}=\bigotimes_{m=1}^{N} V_m=(\mathbb{C}^{2s+1})^{\otimes N}. \tag{2.3}$$

We introduce the Lax operator [60–66] as the following two-by-two matrix in the auxiliary space $V_0=\mathbb{C}^2$,

$$\mathbb{L}_{0m}(\lambda)=\frac{1}{\sinh(\lambda)}\begin{pmatrix}\sinh\left(\lambda\mathbb{1}_m+\eta S_m^3\right) & \sinh(\eta)S_m^-\\ \sinh(\eta)S_m^+ & \sinh\left(\lambda\mathbb{1}_m-\eta S_m^3\right)\end{pmatrix}, \tag{2.4}$$

the operators S_m^α, with $\alpha=+,-,3$ and $m=1,2,\dots,N$, are defined in the Appendix A. It obeys

$$\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0m}(\eta-\lambda)=\frac{\sinh(s_m\eta+\lambda)\sinh((s_m+1)\eta-\lambda)}{\sinh(\lambda)\sinh(\eta-\lambda)}\mathbb{1}_0, \tag{2.5}$$

where s_m is the value of spin in the space V_m.

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the R-matrix,

$$\mathbb{L}_{0m}(\lambda)=\frac{1}{\sinh(\lambda)}R_{0m}(\lambda-\eta/2).$$

Taking into account the commutation relations (A.2), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m)=\mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu). \tag{2.6}$$

The so-called monodromy matrix

$$T(\lambda)=\mathbb{L}_{0N}(\lambda-\alpha_N)\cdots\mathbb{L}_{01}(\lambda-\alpha_1) \tag{2.7}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter η and the inhomogeneous parameters $\{\alpha_j, j=1,\dots,N\}$. Notice that $T(\lambda)$

is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2.8}$$

From RLL-relations (2.6) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu). \tag{2.9}$$

To construct integrable spin chains with non-periodic boundary condition, we will follow Sklyanin's approach [6]. Accordingly, before defining the essential operators and corresponding algebraic structure, in the next section we will introduce the relevant boundary K-matrices.

3. Reflection equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk model, was developed in [6]. Boundary conditions on the left and right sites of the chain are encoded in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $\mathbb{C}^2$ at the first site $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu). \tag{3.1}$$

Due to the properties of the R-matrix (2.1) the dual reflection equation can be presented in the following form

$$R_{12}(\mu-\lambda)K_1^+(\lambda)R_{21}(-\lambda-\mu-2\eta)K_2^+(\mu) = K_2^+(\mu)R_{12}(-\lambda-\mu-2\eta)K_1^+(\lambda)R_{21}(\mu-\lambda). \tag{3.2}$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda-\eta) \tag{3.3}$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of (3.3) into the dual reflection equation (3.2) one gets the reflection equation (3.1) with shifted arguments.

The general, spectral parameter dependent, solutions of the reflection equation (3.1) and the dual reflection equation (3.2) can be written as follows [67–69]

$$K^-(\lambda) = \begin{pmatrix} \kappa^- \sinh(\xi^- + \lambda) & \psi^- \sinh(2\lambda) \\ \phi^- \sinh(2\lambda) & \kappa^- \sinh(\xi^- - \lambda) \end{pmatrix}, \tag{3.4}$$

$$K^+(\lambda) = \begin{pmatrix} \kappa^+ \sinh(\xi^+ - \lambda - \eta) & -\psi^+ \sinh(2(\lambda+\eta)) \\ -\phi^+ \sinh(2(\lambda+\eta)) & \kappa^+ \sinh(\xi^+ + \lambda + \eta) \end{pmatrix}. \tag{3.5}$$

Due to the fact that the reflection matrices $K^{\mp}(\lambda)$ are defined up to multiplicative constants the values of parameters $\kappa^{\mp}$ are not essential, as long as they are different from zero. Therefore they could be set to be one without any loss of generality. In particular, this will be evident throughout the Sections 5 and 6. However, for completeness, we will keep them in our presentation.

Although the R-matrix (2.1) has the $U(1)$ symmetry the reflection matrices $K^{\mp}(\lambda)$ (3.4) and (3.5) cannot be brought to the upper triangular form by the symmetry transformations like in the case of the XXX Heisenberg spin chain [10,12]. Therefore, as we will see in the Section 5, triangularity of the reflections matrices has to be imposed as extra conditions on the parameters of the reflection matrices.

4. Inhomogeneous XXZ Heisenberg spin chain with boundary terms

In order to develop the formalism necessary to describe an integrable spin chain with non-periodic boundary condition, we use the Sklyanin approach [6]. The main tool in this framework is the corresponding monodromy matrix

$$\mathcal{T}_0(\lambda) = T_0(\lambda) K_0^-(\lambda) \widetilde{T}_0(\lambda), \tag{4.1}$$

it consists of the matrix $T(\lambda)$ (2.7), a reflection matrix $K^-(\lambda)$ (3.4) and the matrix

$$\widetilde{T}_0(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} = \mathbb{L}_{01}(\lambda + \alpha_1 + \eta) \cdots \mathbb{L}_{0N}(\lambda + \alpha_N + \eta). \tag{4.2}$$

It is important to notice that the identity (2.5) can be rewritten in the form

$$\mathbb{L}_{0m}(\lambda - \alpha_m)\mathbb{L}_{0m}(\eta - \lambda + \alpha_m) = \left(\frac{\sinh(\lambda - \alpha_m + s_m\eta)\sinh(-\lambda + \alpha_m + (s_m+1)\eta)}{\sinh(\lambda - \alpha_m)\sinh(-\lambda + \alpha_m + \eta)}\right)\mathbb{1}_0 . \tag{4.3}$$

It follows from the equation above and the RLL-relations (2.6) that the RTT-relations (2.9) can be recast as follows

$$\widetilde{T}_{0'}(\mu) R_{00'}(\lambda + \mu) T_0(\lambda) = T_0(\lambda) R_{00'}(\lambda + \mu) \widetilde{T}_{0'}(\mu), \tag{4.4}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu) R_{00'}(\mu - \lambda) = R_{00'}(\mu - \lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda). \tag{4.5}$$

Using the RTT-relations (2.9), (4.4), (4.5) and the reflection equation (3.1) it is straightforward to show that the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ in $V_0 \otimes V_{0'}$ are

$$R_{00'}(\lambda - \mu)\mathcal{T}_0(\lambda) R_{0'0}(\lambda + \mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu) R_{00'}(\lambda + \mu)\mathcal{T}_0(\lambda) R_{0'0}(\lambda - \mu), \tag{4.6}$$

using the notation of [6]. From the above equation we can read off the commutation relations of the entries of the monodromy matrix

$$\mathcal{T}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}. \tag{4.7}$$

Following Sklyanin [6], as in the case of the XXX Heisenberg spin chain [10,12], we introduce the operator

$$\widehat{\mathcal{D}}(\lambda) = \mathcal{D}(\lambda) - \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\mathcal{A}(\lambda). \tag{4.8}$$

For convenience, the commutation relations relevant for the implementation of the algebraic Bethe ansatz for the XXZ Heisenberg chain are given in the Appendix B.

The exchange relations (4.6) admit a central element, the so-called Sklyanin determinant,

$$\Delta[\mathcal{T}(\lambda)] = \mathrm{tr}_{00'} P^-_{00'} \mathcal{T}_0(\lambda - \eta/2) R_{00'}(2\lambda) \mathcal{T}_{0'}(\lambda + \eta/2). \tag{4.9}$$

Analogously to the XXX Heisenberg spin chain [12], the element $\Delta[\mathcal{T}(\lambda)]$ can be expressed in form

$$\Delta\left[\mathcal{T}(\lambda)\right] = \sinh(2\lambda)\widehat{\mathcal{D}}(\lambda - \eta/2)\mathcal{A}(\lambda + \eta/2) - \sinh(2\lambda + \eta)\mathcal{B}(\lambda - \eta/2)\mathcal{C}(\lambda + \eta/2). \quad (4.10)$$

The open chain transfer matrix is given by the trace of the monodromy $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^+(\lambda)$ [6],

$$t(\lambda) = \mathrm{tr}_0\left(K_0^+(\lambda)\mathcal{T}_0(\lambda)\right). \quad (4.11)$$

The reflection matrix $K^+(\lambda)$ (3.5) is the corresponding solution of the dual reflection equation (3.2). The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda), t(\mu)] = 0, \quad (4.12)$$

is guaranteed by the dual reflection equation (3.2) and the exchange relations (4.6) of the monodromy matrix $\mathcal{T}(\lambda)$ [6].

5. Algebraic Bethe Ansatz

In this section, we study the implementation of the algebraic Bethe ansatz for the XXZ Heisenberg spin chain when both reflection matrices $K^{\mp}(\lambda)$ are upper triangular. As opposed to the case of the XXX Heisenberg spin chain where the general reflection matrices could be put into the upper triangular form without any loss of generality [10,12], here the triangularity of the reflection matrices has to be imposed as extra conditions on the parameters of the reflection matrices $K^{\mp}(\lambda)$ (3.4) and (3.5). Our aim is to obtain the Bethe vectors whose scaling limit corresponds to the ones of the XXX Heisenberg chain [12].

As our starting point in the implementation of the algebraic Bethe ansatz, we observe that in every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \quad (5.1)$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \quad (5.2)$$

From the definitions (2.4), (2.7) and (5.1) it is straightforward to obtain the action of the entries of the monodromy matrix $T(\lambda)$ (2.7) on the vector Ω_+

$$A(\lambda)\Omega_+ = a(\lambda)\Omega_+, \quad \text{with} \quad a(\lambda) = \prod_{m=1}^{N} \frac{\sinh(\lambda - \alpha_m + \eta s_m)}{\sinh(\lambda - \alpha_m)}, \quad (5.3)$$

$$D(\lambda)\Omega_+ = d(\lambda)\Omega_+, \quad \text{with} \quad d(\lambda) = \prod_{m=1}^{N} \frac{\sinh(\lambda - \alpha_m - \eta s_m)}{\sinh(\lambda - \alpha_m)}, \quad (5.4)$$

$$C(\lambda)\Omega_+ = 0. \quad (5.5)$$

Analogously, from the definitions (2.4), (4.2) and (5.1) it is straightforward to obtain the action of the entries of the monodromy matrix $\widetilde{T}(\lambda)$ (4.2) on the vector Ω_+

$$\widetilde{A}(\lambda)\Omega_+ = \widetilde{a}(\lambda)\Omega_+, \quad \text{with} \quad \widetilde{a}(\lambda) = \prod_{m=1}^{N} \frac{\sinh(\lambda + \alpha_m + \eta(1 + s_m))}{\sinh(\lambda + \alpha_m + \eta)}, \quad (5.6)$$

$$\widetilde{D}(\lambda)\Omega_+ = \widetilde{d}(\lambda)\Omega_+, \quad \text{with} \quad \widetilde{d}(\lambda) = \prod_{m=1}^{N} \frac{\sinh(\lambda + \alpha_m + \eta(1 - s_m))}{\sinh(\lambda + \alpha_m + \eta)}, \quad (5.7)$$

$$\widetilde{C}(\lambda)\Omega_+ = 0. \quad (5.8)$$

Since the left reflection matrix cannot be brought to the upper triangular form by the $U(1)$ symmetry transformations we have to impose an extra condition on the parameters of $K^-(\lambda)$. By setting

$$\phi^- = 0$$

in (3.4) the reflection matrix $K^-(\lambda)$ becomes upper triangular and according to definition of the Sklyanin monodromy matrix (4.1) we have

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \kappa^- \sinh(\xi^- + \lambda) & \psi^- \sinh(2\lambda) \\ 0 & \kappa^- \sinh(\xi^- - \lambda) \end{pmatrix} \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix}. \tag{5.9}$$

From the above equation, using the relations which follow from (4.4) we obtain

$$\begin{aligned} \mathcal{A}(\lambda) &= \kappa^- \sinh(\xi^- + \lambda)\, A(\lambda)\widetilde{A}(\lambda) \\ &\quad + \left(\psi^- \sinh(2\lambda)\, A(\lambda) + \kappa^- \sinh(\xi^- - \lambda)\, B(\lambda)\right) \widetilde{C}(\lambda) \end{aligned} \tag{5.10}$$

$$\begin{aligned} \mathcal{D}(\lambda) &= \kappa^- \sinh(\xi^- + \lambda) \left(\widetilde{B}(\lambda) C(\lambda) - \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)} \left(D(\lambda)\widetilde{D}(\lambda) - \widetilde{A}(\lambda) A(\lambda)\right)\right) \\ &\quad + \left(\psi^- \sinh(2\lambda)\, C(\lambda) + \kappa^- \sinh(\xi^- - \lambda)\, D(\lambda)\right) \widetilde{D}(\lambda) \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathcal{B}(\lambda) &= \kappa^- \sinh(\xi^- + \lambda) \left(\frac{\sinh(2\lambda)}{\sinh(2\lambda + \eta)}\, \widetilde{B}(\lambda) A(\lambda) - \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\, B(\lambda)\widetilde{D}(\lambda)\right) \\ &\quad + \left(\psi^- \sinh(2\lambda)\, A(\lambda) + \kappa^- \sinh(\xi^- - \lambda)\, B(\lambda)\right) \widetilde{D}(\lambda) \end{aligned} \tag{5.12}$$

$$\mathcal{C}(\lambda) = \kappa^- \sinh(\xi^- + \lambda)\, C(\lambda)\widetilde{A}(\lambda) + \left(\psi^- \sinh(2\lambda)\, C(\lambda) + \kappa^- \sinh(\xi^- - \lambda)\, D(\lambda)\right) \widetilde{C}(\lambda). \tag{5.13}$$

The action of the entries of the Sklyanin monodromy matrix on the vector Ω_+ follows from the above relations (5.10)–(5.13) and the formulae (5.3)–(5.5) and (5.6)–(5.8)

$$\mathcal{C}(\lambda)\Omega_+ = 0, \tag{5.14}$$

$$\mathcal{A}(\lambda)\Omega_+ = \alpha(\lambda)\Omega_+, \quad \text{with} \quad \alpha(\lambda) = \kappa^- \sinh(\xi^- + \lambda)\, a(\lambda)\widetilde{a}(\lambda), \tag{5.15}$$

$$\mathcal{D}(\lambda)\Omega_+ = \delta(\lambda)\Omega_+, \quad \text{with} \tag{5.16}$$

$$\begin{aligned} \delta(\lambda) &= \kappa^- \left(\sinh(\xi^- - \lambda) - \sinh(\xi^- + \lambda) \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\right) d(\lambda)\widetilde{d}(\lambda) \\ &\quad + \kappa^- \sinh(\xi^- + \lambda) \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\, a(\lambda)\widetilde{a}(\lambda). \end{aligned}$$

In what follows we will also use the fact that Ω_+ is an eigenvector of the operator $\widehat{\mathcal{D}}(\lambda)$ (4.8)

$$\widehat{\mathcal{D}}(\lambda)\Omega_+ = \widehat{\delta}(\lambda)\Omega_+, \quad \text{with} \quad \widehat{\delta}(\lambda) = \delta(\lambda) - \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\, \alpha(\lambda), \tag{5.17}$$

or explicitly

$$\widehat{\delta}(\lambda) = \kappa^- \left(\sinh(\xi^- - \lambda) - \sinh(\xi^- + \lambda) \frac{\sinh(\eta)}{\sinh(2\lambda + \eta)}\right) d(\lambda)\widetilde{d}(\lambda). \tag{5.18}$$

The transfer matrix of the inhomogeneous XXZ chain

$$t_0(\lambda) = \mathrm{tr}_0 \left(K^+(\lambda) \mathcal{T}(\lambda) \right), \tag{5.19}$$

with the triangular K-matrix

$$K^+(\lambda) = \begin{pmatrix} \kappa^+ \sinh(\xi^+ - \lambda - \eta) & -\psi^+ \sinh(2(\lambda + \eta)) \\ 0 & \kappa^+ \sinh(\xi^+ + \lambda + \eta) \end{pmatrix}, \tag{5.20}$$

i.e. the matrix $K^+(\lambda) = K^-(-\lambda - \eta)$ were we have set

$$\phi^+ = 0,$$

can be expressed using Sklyanin's $\widehat{\mathcal{D}}(\lambda)$ operator (4.8)

$$t(\lambda) = \kappa_1(\lambda)\, \mathcal{A}(\lambda) + \kappa_2(\lambda)\, \widehat{\mathcal{D}}(\lambda) + \kappa_{12}(\lambda)\, \mathcal{C}(\lambda), \tag{5.21}$$

with

$$\kappa_1(\lambda) = \kappa^+ \sinh(\xi^+ - \lambda)\, \frac{\sinh(2(\lambda + \eta))}{\sinh(2\lambda + \eta)}, \quad \kappa_2(\lambda) = \kappa^+ \sinh(\xi^+ + \lambda + \eta),$$

$$\kappa_{12}(\lambda) = -\psi^+ \sinh(2(\lambda + \eta)). \tag{5.22}$$

Evidently, due to (5.14)–(5.18), the vector Ω_+ (5.2) is an eigenvector of the transfer matrix

$$t(\lambda)\Omega_+ = \left(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda) \right) \Omega_+ = \Lambda_0(\lambda)\Omega_+. \tag{5.23}$$

For simplicity we have suppressed the dependence of the eigenvalue $\Lambda_0(\lambda)$ on the boundary parameters κ^+, ξ^+ and ψ^+ as well as the quasi-classical parameter η.

Let us consider

$$\begin{aligned} \widetilde{\Psi}_1(\mu) = \mathcal{B}(\mu)\Omega_+ - \frac{\psi^+}{\kappa^+} \Bigg(& \frac{\sinh(2\mu)}{\sinh(2\mu + \eta)} \cosh(\xi^+ - \mu)\, \alpha(\mu) \\ & - \cosh(\xi^+ + \mu + \eta)\, \widehat{\delta}(\mu) \Bigg) \Omega_+. \end{aligned} \tag{5.24}$$

A straightforward calculation, using the relations (B.2), (B.3) and (B.4), shows that the off-shell action of the transfer matrix (5.21) on $\widetilde{\Psi}_1(\mu)$ is given by

$$\begin{aligned} t(\lambda)\widetilde{\Psi}_1(\mu) = \Lambda_1(\lambda, \mu)\widetilde{\Psi}_1(\mu) + \kappa^+ \sinh(\xi^+ - \mu) & \frac{\sinh(\eta)\sinh(2(\lambda + \eta))}{\sinh(\lambda - \mu)\sinh(\lambda + \mu + \eta)} \times \\ \times F_1(\mu)\widetilde{\Psi}_1(\lambda), & \end{aligned} \tag{5.25}$$

where the eigenvalue $\Lambda_1(\lambda, \mu)$ is given by

$$\begin{aligned} \Lambda_1(\lambda, \mu) = \kappa_1(\lambda) & \frac{\sinh(\lambda + \mu)\sinh(\lambda - \mu - \eta)}{\sinh(\lambda - \mu)\sinh(\lambda + \mu + \eta)} \alpha(\lambda) \\ & + \kappa_2(\lambda) \frac{\sinh(\lambda - \mu + \eta)\sinh(\lambda + \mu + 2\eta)}{\sinh(\lambda - \mu)\sinh(\lambda + \mu + \eta)} \widehat{\delta}(\lambda). \end{aligned} \tag{5.26}$$

Evidently $\Lambda_1(\lambda, \mu)$ depends also on boundary parameters κ^+, ξ^+ and the quasi-classical parameter η, but these parameters are omitted in order to simplify the formulae. The unwanted term on the right hand side (5.25) is annihilated by the Bethe equation

$$F_1(\mu) = \frac{\sinh(2\mu)}{\sinh(2\mu + \eta)}\, \alpha(\mu) - \frac{\sinh(\xi^+ + \mu + \eta)}{\sinh(\xi^+ - \mu)}\, \widehat{\delta}(\mu) = 0, \tag{5.27}$$

or equivalently,

$$\frac{\alpha(\mu)}{\widehat{\delta}(\mu)} = \frac{\sinh(2(\mu+\eta))\,\kappa_2(\mu)}{\sinh(2\mu)\,\kappa_1(\mu)} = \frac{\sinh(2\mu+\eta)\sinh(\xi^+ + \mu + \eta)}{\sinh(2\mu)\ \sinh(\xi^+ - \mu)}. \tag{5.28}$$

Thus we have shown that $\widetilde{\Psi}_1(\mu)$ (5.32) is a Bethe vector of the transfer matrix (5.21). Moreover, the vector $\widetilde{\Psi}_1(\mu)$ in the scaling limit yields the corresponding Bethe vector of the XXX Heisenberg spin chain [12] and it was this connection that led us to this particular form of the Bethe vector. However, it is important to note that this is not the only possible form of the Bethe vector. Namely, we notice the following important identity

$$\Lambda_0(\lambda) - \Lambda_1(\lambda,\mu) = \kappa^+ \sinh(\xi^+ - \lambda)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)} F_1(\lambda). \tag{5.29}$$

It follows that $\widetilde{\Psi}_1(\mu)$ (5.24) can be generalized by adding a term proportional to $F_1(\mu)$

$$\widetilde{\Psi}_1(\mu, C_1) = \widetilde{\Psi}_1(\mu) + C_1\,\frac{\psi^+}{\kappa^+}\ \sinh(\xi^+ - \mu) F_1(\mu)\Omega_+, \tag{5.30}$$

where C_1 is independent of μ. A direct consequence of the above identity is the off-shell action of the transfer matrix on $\widetilde{\Psi}_1(\mu, C_1)$,

$$\begin{aligned} t(\lambda)\widetilde{\Psi}_1(\mu, C_1) = \Lambda_1(\lambda,\mu)\widetilde{\Psi}_1(\mu, C_1) + \kappa^+ \sinh(\xi^+ - \mu)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)} \times \\ \times F_1(\mu)\widetilde{\Psi}_1(\lambda, C_1). \end{aligned} \tag{5.31}$$

Therefore $\widetilde{\Psi}_1(\mu, C_1)$ (5.30) can be considered as the general form of the Bethe vector of the transfer matrix (5.21) corresponding to the eigenvalue $\Lambda_1(\lambda,\mu)$ (5.26).

By setting $C_1 = 1$ in (5.30) we obtain another particular solution for the Bethe vector, that will turn out to be more suitable for the recurrence procedure

$$\Psi_1(\mu) = \widetilde{\Psi}_1(\mu, C_1 = 1) = \mathcal{B}(\mu)\Omega_+ + b_1(\mu)\Omega_+, \tag{5.32}$$

where $b_1(\mu)$ is given by

$$b_1(\mu) = \left(-\frac{\psi^+}{\kappa^+}\right)\left(\frac{\sinh(2\mu)}{\sinh(2\mu+\eta)} e^{-(\xi^+ - \mu)}\,\alpha(\mu) - e^{-(\xi^+ + \mu + \eta)}\,\widehat{\delta}(\mu)\right). \tag{5.33}$$

We seek the Bethe vector $\widetilde{\Psi}_2(\mu_1,\mu_2)$ in the form

$$\begin{aligned} \widetilde{\Psi}_2(\mu_1,\mu_2) = \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ + \widetilde{b}_2^{(1)}(\mu_2;\mu_1)\mathcal{B}(\mu_1)\Omega_+ + \widetilde{b}_2^{(1)}(\mu_1;\mu_2)\mathcal{B}(\mu_2)\Omega_+ \\ + \widetilde{b}_2^{(2)}(\mu_1,\mu_2)\Omega_+. \end{aligned} \tag{5.34}$$

One possible choice of the coefficient functions $\widetilde{b}_2^{(1)}(\mu_1;\mu_2)$ and $\widetilde{b}_2^{(2)}(\mu_1,\mu_2)$ is given by

$$\begin{aligned} \widetilde{b}_2^{(1)}(\mu_1;\mu_2) = \left(-\frac{\psi^+}{\kappa^+}\right)\left(\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\frac{\sinh(\mu_1+\mu_2)\sinh(\mu_1-\mu_2-\eta)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2+\eta)} \times \right. \\ \times \cosh(\xi^+ - \mu_1)\,\alpha(\mu_1) \\ \left. - \frac{\sinh(\mu_1-\mu_2+\eta)\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2+\eta)}\ \cosh(\xi^+ + \mu_1 + \eta)\,\widehat{\delta}(\mu_1)\right), \end{aligned} \tag{5.35}$$

and

$$\widetilde{b}_2^{(2)}(\mu_1,\mu_2) = \left(\frac{\psi^+}{\kappa^+}\right)^2 \left(\frac{\sinh(\mu_1+\mu_2)}{\sinh(\mu_1+\mu_2+\eta)}\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)}\times\right.$$
$$\times\cosh(2\xi^+ - \mu_1 - \mu_2 + \eta)\alpha(\mu_1)\alpha(\mu_2)$$
$$-\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\frac{\sinh(\mu_1-\mu_2-\eta)}{\sinh(\mu_1-\mu_2)}\cosh(2\xi^+ - \mu_1 + \mu_2 + 2\eta)\,\alpha(\mu_1)\widehat{\delta}(\mu_2)$$
$$-\frac{\sinh(\mu_2-\mu_1-\eta)}{\sinh(\mu_2-\mu_1)}\frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)}\cosh(2\xi^+ + \mu_1 - \mu_2 + 2\eta)\,\widehat{\delta}(\mu_1)\alpha(\mu_2)$$
$$\left.+\frac{\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1+\mu_2+\eta)}\cosh(2\xi^+ + \mu_1 + \mu_2 + 3\eta)\,\widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)\right). \tag{5.36}$$

Due to the fact that the operators $\mathcal{B}(\mu_1)$ and $\mathcal{B}(\mu_2)$ commute (B.1) and that $\widetilde{b}_2^{(2)}(\mu_1,\mu_2) = \widetilde{b}_2^{(2)}(\mu_2,\mu_1)$ it follows that $\widetilde{\Psi}_2(\mu_1,\mu_2)$ is symmetric with respect to the interchange of the variables μ_1 and μ_2.

Starting from the definitions (5.21) and (5.34), using the relations (B.8), (B.9) and (B.10), from the Appendix B, to push the operators $\mathcal{A}(\lambda)$, $\widehat{\mathcal{D}}(\lambda)$ and $\mathcal{C}(\lambda)$ to the right and after rearranging some terms, we obtain the off-shell action of transfer matrix $t(\lambda)$ on $\widetilde{\Psi}_2(\mu_1,\mu_2)$

$$t(\lambda)\widetilde{\Psi}_2(\mu_1,\mu_2) = \Lambda_2(\lambda,\{\mu_i\})\widetilde{\Psi}_2(\mu_1,\mu_2) + \sum_{i=1}^{2}\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\times$$
$$\times\,\kappa^+\sinh(\xi^+-\mu_i)\;F_2(\mu_i;\mu_{3-i})\widetilde{\Psi}_2(\lambda,\mu_{3-i}), \tag{5.37}$$

where the eigenvalue is given by

$$\Lambda_2(\lambda,\{\mu_i\}) = \kappa_1(\lambda)\,\alpha(\lambda)\prod_{i=1}^{2}\frac{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i-\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}$$
$$+\,\kappa_2(\lambda)\,\widehat{\delta}(\lambda)\prod_{i=1}^{2}\frac{\sinh(\lambda-\mu_i+\eta)\sinh(\lambda+\mu_i+2\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \tag{5.38}$$

and the two unwanted terms in (5.37) are canceled by the Bethe equations which follow from

$$F_2(\mu_i;\mu_{3-i}) = \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)}\frac{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)}\alpha(\mu_i)$$
$$-\frac{\sinh(\xi^+ + \mu_i + \eta)}{\sinh(\xi^+ - \mu_i)}\times$$
$$\times\frac{\sinh(\mu_i-\mu_{3-i}+\eta)\sinh(\mu_i+\mu_{3-i}+2\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)}\widehat{\delta}(\mu_i) = 0, \tag{5.39}$$

with $i = \{1,2\}$. Therefore the Bethe equations are

$$\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{\sinh(2(\mu_i+\eta))\,\kappa_2(\mu_i)}{\sinh(2\mu_i)\,\kappa_1(\mu_i)}\frac{\sinh(\mu_i-\mu_{3-i}+\eta)\sinh(\mu_i+\mu_{3-i}+2\eta)}{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}, \tag{5.40}$$

where $i = \{1,2\}$. This shows that $\widetilde{\Psi}_2(\mu_1,\mu_2)$ (5.34) is a Bethe vector of the transfer matrix (5.21) and, again, it is the one which in the scaling limit corresponds to the Bethe vector of the XXX chain [12].

Furthermore, due to the following identities

$$\Lambda_1(\lambda,\mu_2)-\Lambda_2(\lambda,\mu_1,\mu_2)=\kappa^+\sinh(\xi^+-\lambda)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_1)\sinh(\lambda+\mu_1+\eta)}F_2(\lambda;\mu_2), \tag{5.41}$$

$$\Lambda_1(\lambda,\mu_1)-\Lambda_2(\lambda,\mu_1,\mu_2)=\kappa^+\sinh(\xi^+-\lambda)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_2)\sinh(\lambda+\mu_2+\eta)}F_2(\lambda;\mu_1), \tag{5.42}$$

$$\begin{aligned}&\frac{F_2(\mu_2;\mu_1)\,F_1(\mu_1)-F_2(\mu_1;\mu_2)\,F_2(\mu_2;\lambda)}{\sinh(\lambda-\mu_1)\sinh(\lambda+\mu_1+\eta)}\\&\quad+\frac{F_2(\mu_1;\mu_2)\,F_1(\mu_2)-F_2(\mu_2;\mu_1)\,F_2(\mu_1;\lambda)}{\sinh(\lambda-\mu_2)\sinh(\lambda+\mu_2+\eta)}=0,\end{aligned} \tag{5.43}$$

the Bethe vector $\widetilde{\Psi}_2(\mu_1,\mu_2)$ (5.34) can be generalized

$$\begin{aligned}\widetilde{\Psi}_2(\mu_1,\mu_2,C_1,C_2)&=\widetilde{\Psi}_2(\mu_1,\mu_2)+C_2\frac{\psi^+}{\kappa^+}\big(\sinh(\xi^+-\mu_1)F_2(\mu_1;\mu_2)\widetilde{\Psi}_1(\mu_2,C_1)\\&\quad+\sinh(\xi^+-\mu_2)F_2(\mu_2;\mu_1)\widetilde{\Psi}_1(\mu_1,C_1)\big),\end{aligned} \tag{5.44}$$

where C_2 is independent of μ_1 and μ_2 and $\widetilde{\Psi}_1(\mu_i,C_1)$ is the Bethe vector given in (5.30), so that the off-shell action of transfer matrix $t(\lambda)$ on $\widetilde{\Psi}_2(\mu_1,\mu_2,C_1,C_2)$ reads

$$\begin{aligned}t(\lambda)\widetilde{\Psi}_2(\mu_1,\mu_2,C_1,C_2)&=\Lambda_2(\lambda,\{\mu_i\})\widetilde{\Psi}_2(\mu_1,\mu_2,C_1,C_2)\\&\quad+\sum_{i=1}^{2}\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\kappa^+\sinh(\xi^+-\mu_i)\,\times\\&\quad\times F_2(\mu_i;\mu_{3-i})\widetilde{\Psi}_2(\lambda,\mu_{3-i},C_1,C_2).\end{aligned} \tag{5.45}$$

Once more in (5.44) we find that the general form of Bethe vectors can be expressed as a sum of a particular vector and a linear combination of lower order Bethe vectors. Due to identities (5.41)–(5.43) this linear combination of lower order Bethe vectors corresponds the same eigenvalue as the particular vector (5.45). This is indeed the case with Bethe vectors of any order, for details see Appendix C. To our knowledge, the existence of this freedom in the choice of the Bethe vector has hitherto remained unnoticed in the literature. In certain cases, it seems that omission to note this freedom can be traced to imposing, by some authors [13], too strong requirements on the vanishing of the off-shell terms. Namely, all the terms (including vacuum ones) should be required to vanish only once the Bethe equations are imposed, and not necessarily to be identically zero.

However, in order to have the recurrence procedure for defining the higher order Bethe vectors it is instructive to set $C_1=-\tanh(\eta)$, $C_2=1$ in (5.44) and to consider a particular Bethe vector

$$\begin{aligned}\Psi_2(\mu_1,\mu_2)&=\widetilde{\Psi}_2(\mu_1,\mu_2,C_1=-\tanh(\eta),C_2=1)\\&=\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_++b_2^{(1)}(\mu_2;\mu_1)\mathcal{B}(\mu_1)\Omega_++b_2^{(1)}(\mu_1;\mu_2)\mathcal{B}(\mu_2)\Omega_+\\&\quad+b_2^{(2)}(\mu_1,\mu_2)\Omega_+,\end{aligned} \tag{5.46}$$

where the functions $b_2^{(1)}(\mu_1;\mu_2)$ and $b_2^{(2)}(\mu_1,\mu_2)$ are given by

$$
\begin{aligned}
b_2^{(1)}(\mu_1;\mu_2) = &\left(-\frac{\psi^+}{\kappa^+}\right)\left(\frac{\sinh(\mu_1+\mu_2)\sinh(\mu_1-\mu_2-\eta)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2+\eta)}\times\right.\\
&\times\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}e^{-(\xi^+-\mu_1)}\,\alpha(\mu_1)\\
&\left.-\frac{\sinh(\mu_1-\mu_2+\eta)\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1-\mu_2)\sinh(\mu_1+\mu_2+\eta)}\,e^{-(\xi^++\mu_1+\eta)}\,\widehat{\delta}(\mu_1)\right),
\end{aligned}
\tag{5.47}
$$

and

$$
\begin{aligned}
b_2^{(2)}(\mu_1,\mu_2) = &\left(\frac{\psi^+}{\kappa^+}\right)^2\left(\frac{\sinh(\mu_1+\mu_2)}{\sinh(\mu_1+\mu_2+\eta)}\times\right.\\
&\times\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\,\frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)}\,e^{-(2\xi^+-\mu_1-\mu_2+\eta)}\alpha(\mu_1)\alpha(\mu_2)\\
&-\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\,\frac{\sinh(\mu_1-\mu_2-\eta)}{\sinh(\mu_1-\mu_2)}\,e^{-(2\xi^+-\mu_1+\mu_2+2\eta)}\,\alpha(\mu_1)\widehat{\delta}(\mu_2)\\
&-\frac{\sinh(\mu_2-\mu_1-\eta)}{\sinh(\mu_2-\mu_1)}\,\frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)}\,e^{-(2\xi^++\mu_1-\mu_2+2\eta)}\,\widehat{\delta}(\mu_1)\alpha(\mu_2)\\
&\left.+\frac{\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1+\mu_2+\eta)}\,e^{-(2\xi^++\mu_1+\mu_2+3\eta)}\,\widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)\right).
\end{aligned}
\tag{5.48}
$$

A key observation here is that the above function $b_2^{(2)}(\mu_1,\mu_2)$ can be expressed in terms of the coefficient functions $b_2^{(1)}(\mu_1;\mu_2)$ (5.47) and $b_1(\mu_i)$ (5.33) as follows

$$
b_2^{(2)}(\mu_1,\mu_2) = \frac{1}{1+e^{2\eta}}\left(b_2^{(1)}(\mu_1;\mu_2)\,b_1(\mu_2) + b_2^{(1)}(\mu_2;\mu_1)\,b_1(\mu_1)\right).
\tag{5.49}
$$

This relation is essential in the recurrence procedure for obtaining general form of the Bethe vectors. It coincides, up to the multiplicative factor, with the recurrence relation defining the function $b_2^{(2)}(\mu_1,\mu_2)$ in the case of the corresponding Bethe vector of the XXX Heisenberg spin chain, the equation (V.25) in [12].

Although, as we have seen, the Bethe vectors $\Psi_1(\mu)$ (5.32) and $\Psi_2(\mu_1,\mu_2)$ (5.46) correspond to the particular choice of parameters C_i in (5.30) and (5.44), respectively, it turns out that these vectors admit the recurrence procedure analogous to the one applied in the case of the XXX Heisenberg spin chain [12]. Before addressing the general case of the Bethe vector $\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)$, for an arbitrary positive integer M, we will present below the $M=3$ case as an insightful example. The Bethe vector $\Psi_3(\mu_1,\mu_2,\mu_3)$ we propose is a symmetric function of its arguments and it is given as the following sum of eight terms

$$
\begin{aligned}
\Psi_3(\mu_1,\mu_2,\mu_3) = &\,\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\mathcal{B}(\mu_3)\Omega_+ + b_3^{(1)}(\mu_3;\mu_2,\mu_1)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+\\
&+ b_3^{(1)}(\mu_1;\mu_2,\mu_3)\mathcal{B}(\mu_2)\mathcal{B}(\mu_3)\Omega_+ + b_3^{(1)}(\mu_2;\mu_1,\mu_3)\mathcal{B}(\mu_1)\mathcal{B}(\mu_3)\Omega_+\\
&+ b_3^{(2)}(\mu_1,\mu_2;\mu_3)\mathcal{B}(\mu_3)\Omega_+ + b_3^{(2)}(\mu_1,\mu_3;\mu_2)\mathcal{B}(\mu_2)\Omega_+\\
&+ b_3^{(2)}(\mu_2,\mu_3;\mu_1)\mathcal{B}(\mu_1)\Omega_+ + b_3^{(3)}(\mu_1,\mu_2,\mu_3)\Omega_+,
\end{aligned}
\tag{5.50}
$$

where the coefficient functions $b_3^{(1)}(\mu_1;\mu_2,\mu_3)$, $b_3^{(2)}(\mu_1,\mu_2;\mu_3)$ and $b_3^{(3)}(\mu_1,\mu_2,\mu_3)$ are given by

$$b_3^{(1)}(\mu_1;\mu_2,\mu_3)=\left(-\frac{\psi^+}{\kappa^+}\right)\left(\prod_{i=2}^{3}\frac{\sinh(\mu_1+\mu_i)\sinh(\mu_1-\mu_i-\eta)}{\sinh(\mu_1-\mu_i)\sinh(\mu_1+\mu_i+\eta)}\times\right.$$
$$\times\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\,e^{-(\xi^+-\mu_1)}\,\alpha(\mu_1)$$
$$\left.-\prod_{i=2}^{3}\frac{\sinh(\mu_1-\mu_i+\eta)\sinh(\mu_1+\mu_i+2\eta)}{\sinh(\mu_1-\mu_i)\sinh(\mu_1+\mu_i+\eta)}\,e^{-(\xi^++\mu_1+\eta)}\,\widehat{\delta}(\mu_1)\right), \tag{5.51}$$

$$b_3^{(2)}(\mu_1,\mu_2;\mu_3)=\frac{1}{1+e^{2\eta}}\Big(b_3^{(1)}(\mu_1;\mu_2,\mu_3)\,b_2^{(1)}(\mu_2;\mu_3)$$
$$+b_3^{(1)}(\mu_2;\mu_1,\mu_3)\,b_2^{(1)}(\mu_1;\mu_3)\Big), \tag{5.52}$$

$$b_3^{(3)}(\mu_1,\mu_2,\mu_3)=\frac{1}{1+2e^{2\eta}+2e^{4\eta}+e^{6\eta}}\Big(b_3^{(1)}(\mu_1;\mu_2,\mu_3)\,b_2^{(1)}(\mu_2;\mu_3)\,b_1(\mu_3)$$
$$+b_3^{(1)}(\mu_1;\mu_2,\mu_3)\,b_2^{(1)}(\mu_3;\mu_2)\,b_1(\mu_2)$$
$$+b_3^{(1)}(\mu_2;\mu_1,\mu_3)\,b_2^{(1)}(\mu_1;\mu_3)\,b_1(\mu_3)$$
$$+b_3^{(1)}(\mu_2;\mu_1,\mu_3)\,b_2^{(1)}(\mu_3;\mu_1)\,b_1(\mu_1)$$
$$+b_3^{(1)}(\mu_3;\mu_1,\mu_2)\,b_2^{(1)}(\mu_1;\mu_2)\,b_1(\mu_2)$$
$$+b_3^{(1)}(\mu_3;\mu_1,\mu_2)\,b_2^{(1)}(\mu_2;\mu_1)\,b_1(\mu_1)\Big). \tag{5.53}$$

It is important to notice that the coefficient functions $b_3^{(2)}(\mu_1,\mu_2;\mu_3)$ and $b_3^{(3)}(\mu_1,\mu_2,\mu_3)$ are defined above in terms of the function $b_3^{(1)}(\mu_1;\mu_2,\mu_3)$ and the functions $b_2^{(1)}(\mu_1;\mu_2)$ and $b_1(\mu)$ already given in (5.47) and (5.33), respectively. The action of $t(\lambda)$ (5.21) on $\Psi_3(\mu_1,\mu_2,\mu_3)$, obtained using evident generalization of the formulas (B.8), (B.9) and (B.10) and subsequent rearranging of terms, reads

$$t(\lambda)\Psi_3(\mu_1,\mu_2,\mu_3)=\Lambda_3(\lambda,\{\mu_i\})\Psi_3(\mu_1,\mu_2,\mu_3)+\sum_{i=1}^{3}\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\times$$
$$\times\ \kappa^+\sinh(\xi^+-\mu_i)\ F_3(\mu_i;\{\mu_j\}_{j\neq i})\ \Psi_3(\lambda,\{\mu_j\}_{j\neq i}), \tag{5.54}$$

where the eigenvalue is given by

$$\Lambda_3(\lambda,\{\mu_i\})=\kappa_1(\lambda)\,\alpha(\lambda)\prod_{i=1}^{3}\frac{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i-\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}$$
$$+\kappa_2(\lambda)\,\widehat{\delta}(\lambda)\prod_{i=1}^{3}\frac{\sinh(\lambda-\mu_i+\eta)\sinh(\lambda+\mu_i+2\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \tag{5.55}$$

and the three unwanted terms in (5.54) are canceled by the Bethe equations which follow from

$$F_3(\mu_i;\{\mu_j\}_{j\neq i}) = \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)}\,\alpha(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{3}\frac{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j-\eta)}{\sinh(\mu_i-\mu_j)\sinh(\mu_i+\mu_j+\eta)}$$
$$- \frac{\sinh(\xi^+ +\mu_i+\eta)}{\sinh(\xi^+ -\mu_i)}\,\widehat{\delta}(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{3}\frac{\sinh(\mu_i-\mu_j+\eta)\sinh(\mu_i+\mu_j+2\eta)}{\sinh(\mu_i-\mu_j)\sinh(\mu_i+\mu_j+\eta)}$$
$$=0, \tag{5.56}$$

with $i=\{1,2,3\}$. Therefore the Bethe equations are

$$\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{\sinh(2(\mu_i+\eta))\,\kappa_2(\mu_i)}{\sinh(2\mu_i)\,\kappa_1(\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{3}\frac{\sinh(\mu_i-\mu_j+\eta)\sinh(\mu_i+\mu_j+2\eta)}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j-\eta)}, \tag{5.57}$$

where $i=\{1,2,3\}$. Thus, as expected, we have obtained the strikingly simple expression for the off-shell action of the transfer matrix of the XXZ Heisenberg chain with the upper triangular reflection matrices on the Bethe vector $\Psi_3(\mu_1,\mu_2,\mu_3)$, which is by definition (5.50) symmetric function of its arguments $\{\mu_i\}_{I=1}^3$. As before, $\Psi_3(\mu_1,\mu_2,\mu_3)$ is a special case of the more general Bethe vector $\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3,C_1,C_2,C_3)$ we have found along the lines similar to the $M=1$ and $M=2$ cases, for details see the Appendix C, where we also give the generalized form of the Bethe vector for arbitrary M. The most significant advantage of this particular form of the Bethe vector is that it is defined by the recurrence procedure which is analogous to the one proposed in the case of the XXX Heisenberg chain [12]. Notice the right-hand-side of the equations (5.52) and (5.53) differ only by the multiplicative factors from the analogous equations (V.32) and (V.34) in [12].

We readily proceed to define $\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)$ as a sum of 2^M terms, for an arbitrary positive integer M, and as a symmetric function of its arguments by the recurrence procedure

$$\begin{aligned}
&\Psi_M(\mu_1,\mu_2,\ldots,\mu_M) = \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_M)\Omega_+\\
&+ b_M^{(1)}(\mu_M;\mu_1,\mu_2,\ldots,\mu_{M-1})\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_{M-1})\Omega_+\\
&+\cdots+ b_M^{(2)}(\mu_{M-1},\mu_M;\mu_1,\mu_2,\ldots,\mu_{M-2})\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\cdots\mathcal{B}(\mu_{M-2})\Omega_+\\
&\vdots\\
&+ b_M^{(M-1)}(\mu_1,\mu_2,\ldots,\mu_{M-1};\mu_M)\mathcal{B}(\mu_M)\Omega_+ + b_M^{(M)}(\mu_1,\mu_2,\ldots,\mu_M)\Omega_+,
\end{aligned} \tag{5.58}$$

where the first coefficient function is explicitly given by

$$\begin{aligned}
&b_M^{(1)}(\mu_1;\mu_2,\mu_3,\ldots,\mu_M)\\
&= \left(-\frac{\psi^+}{\kappa^+}\right)\left(\prod_{i=2}^{M}\frac{\sinh(\mu_1+\mu_i)\sinh(\mu_1-\mu_i-\eta)}{\sinh(\mu_1-\mu_i)\sinh(\mu_1+\mu_i+\eta)}\,\frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\times\right.\\
&\left.\times\; e^{-(\xi^+-\mu_1)}\,\alpha(\mu_1) - \prod_{i=2}^{M}\frac{\sinh(\mu_1-\mu_i+\eta)\sinh(\mu_1+\mu_i+2\eta)}{\sinh(\mu_1-\mu_i)\sinh(\mu_1+\mu_i+\eta)}\,e^{-(\xi^++\mu_1+\eta)}\,\widehat{\delta}(\mu_1)\right),
\end{aligned} \tag{5.59}$$

and all the other coefficient functions are given by the following recurrence formulae

$$\begin{aligned}
&b_M^{(2)}(\mu_1,\mu_2;\mu_3,\ldots,\mu_M)\\
&=\frac{q^{-1}}{[2]_q!}\Big(b_M^{(1)}(\mu_1;\mu_2,\mu_3,\ldots,\mu_M)b_{M-1}^{(1)}(\mu_2;\mu_3,\ldots,\mu_M)\\
&+b_M^{(1)}(\mu_2;\mu_1,\mu_3,\ldots,\mu_M)b_{M-1}^{(1)}(\mu_1;\mu_3,\ldots,\mu_M)\Big),
\end{aligned}\tag{5.60}$$

$$\vdots$$

$$\begin{aligned}
&b_M^{(M-1)}(\mu_1,\mu_2,\ldots,\mu_{M-1};\mu_M)\\
&=\frac{q^{-\frac{(M-1)(M-2)}{2}}}{[M-1]_q!}\sum_{\rho\in S_{M-1}}b_M^{(1)}(\mu_{\rho(1)};\mu_{\rho(2)},\ldots,\mu_M)\times\\
&\times b_{M-1}^{(1)}(\mu_{\rho(2)};\mu_{\rho(3)},\ldots,\mu_M)b_{M-2}^{(1)}(\mu_{\rho(3)};\mu_{\rho(4)},\ldots,\mu_M)\cdots b_2^{(1)}(\mu_{\rho(M-1)};\mu_M)
\end{aligned}\tag{5.61}$$

$$\begin{aligned}
&b_M^{(M)}(\mu_1,\mu_2,\ldots,\mu_M)\\
&=\frac{q^{-\frac{M(M-1)}{2}}}{[M]_q!}\sum_{\sigma\in S_M}b_M^{(1)}(\mu_{\sigma(1)};\mu_{\sigma(2)},\ldots,\mu_{\sigma(M)})\,b_{M-1}^{(1)}(\mu_{\sigma(2)};\mu_{\sigma(3)},\ldots,\mu_{\sigma(M)})\times\\
&\times b_{M-2}^{(1)}(\mu_{\sigma(3)};\mu_{\sigma(4)},\ldots,\mu_{\sigma(M)})\cdots b_2^{(1)}(\mu_{\sigma(M-1)};\mu_{\sigma(M)})\,b_1(\mu_{\sigma(M)}),
\end{aligned}\tag{5.62}$$

where, for a positive integer N, $[N]_q=\frac{q^N-q^{-N}}{q-q^{-1}}$ and $[N]_q!=[N]_q\cdot[N-1]_q\cdots[2]_q\cdot[1]_q$, with $q=e^\eta$ and S_{M-1} and S_M are the symmetric groups of degree $M-1$ and M, respectively. As is the case $M=3$, the formulae (5.60)–(5.62) are deformation of the corresponding relations (V.32)–(V.35) in the case of the XXX Heisenberg chain [12].

A straightforward calculation based on evident generalization of the formulas (B.8), (B.9) and (B.10) and subsequent rearranging of terms, yields the off-shell action of the transfer matrix on the Bethe vector $\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)$

$$\begin{aligned}
t(\lambda)\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)&=\Lambda_M(\lambda,\{\mu_i\})\Psi_M(\mu_1,\mu_2,\ldots,\mu_M)\\
&+\sum_{i=1}^M\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\times\\
&\times\ \kappa^+\sinh(\xi^+-\mu_i)\ F_M(\mu_i;\{\mu_j\}_{j\neq i})\ \Psi_M(\lambda,\{\mu_j\}_{j\neq i}),
\end{aligned}\tag{5.63}$$

where the corresponding eigenvalue is given by

$$\begin{aligned}
\Lambda_M(\lambda,\{\mu_i\})&=\kappa_1(\lambda)\,\alpha(\lambda)\prod_{i=1}^M\frac{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i-\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\\
&+\kappa_2(\lambda)\,\widehat{\delta}(\lambda)\prod_{i=1}^M\frac{\sinh(\lambda-\mu_i+\eta)\sinh(\lambda+\mu_i+2\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}
\end{aligned}\tag{5.64}$$

and the M unwanted terms on the right hand side of (5.63) are canceled by the Bethe equations which follow from

$$F_M(\mu_i;\{\mu_j\}_{j\neq i}) = \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)}\,\alpha(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j-\eta)}{\sinh(\mu_i-\mu_j)\sinh(\mu_i+\mu_j+\eta)}$$

$$-\frac{\sinh(\xi^+ +\mu_i+\eta)}{\sinh(\xi^+ -\mu_i)}\,\widehat{\delta}(\mu_i)\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{\sinh(\mu_i-\mu_j+\eta)\sinh(\mu_i+\mu_j+2\eta)}{\sinh(\mu_i-\mu_j)\sinh(\mu_i+\mu_j+\eta)}$$

$$=0, \tag{5.65}$$

with $i=\{1,2,\dots,M\}$. Therefore the Bethe equations are

$$\frac{\alpha(\mu_i)}{\widehat{\delta}(\mu_i)} = \frac{\sinh(2(\mu_i+\eta))\,\kappa_2(\mu_i)}{\sinh(2\mu_i)\,\kappa_1(\mu_i)}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{\sinh(\mu_i-\mu_j+\eta)\sinh(\mu_i+\mu_j+2\eta)}{\sinh(\mu_i+\mu_j)\sinh(\mu_i-\mu_j-\eta)}, \tag{5.66}$$

where $i=\{1,2,\dots,M\}$. The Bethe vector $\Psi_M(\mu_1,\mu_2,\dots,\mu_M)$ we have defined in (5.58) yields the strikingly simple expression (5.63) for the off-shell action of the transfer matrix $t(\lambda)$ (5.21). Thus we have fully implemented the algebraic Bethe ansatz for the XXZ Heisenberg spin chain with the triangular reflection matrices. In the following section, we will explored further these results through the so-called quasi-classical limit in order to investigate the corresponding Gaudin model [52].

6. Corresponding Gaudin model

As it is well known [12,52,54,55], the study of the open Gaudin model requires that the parameters of the reflection matrices on the left and on the right end of the chain are the same. Thus, we impose

$$\lim_{\eta\to 0}\Big(K^+(\lambda)K^-(\lambda)\Big) = \kappa^2\sinh(\xi-\lambda)\sinh(\xi+\lambda)\mathbb{1}. \tag{6.1}$$

Notice that in general this not the case in the study of the open spin chain. However, this condition is essential for the Gaudin model. Therefore we will write

$$K^-(\lambda)\equiv K(\lambda) = \begin{pmatrix}\kappa\sinh(\xi+\lambda) & \psi\sinh(2\lambda)\\ 0 & \kappa\sinh(\xi-\lambda)\end{pmatrix} \tag{6.2}$$

so that

$$K^+(\lambda) = K(-\lambda-\eta) = \begin{pmatrix}\kappa\sinh(\xi-\lambda-\eta) & -\psi\sinh(2(\lambda+\eta))\\ 0 & \kappa\sinh(\xi+\lambda+\eta)\end{pmatrix}. \tag{6.3}$$

In [55] we have derived the generating function of the trigonometric Gaudin Hamiltonians with boundary terms following the approach of Sklyanin in the periodic case [28,56]. Analogously to the rational case [52,12], our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXZ chain and the central element, the so-called Sklyanin determinant. Finally, the expansion reads [55]

$$
\begin{aligned}
t(\lambda) - \frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)} &= \kappa^2 \sinh(\xi+\lambda)\sinh(\xi-\lambda)\mathbb{1} \\
&+ \eta\,\frac{\kappa^2}{2}\left(\cosh(2\xi)\coth(2\lambda) - \frac{\cosh(4\lambda)}{\sinh(2\lambda)}\right)\mathbb{1} \\
&+ \frac{\eta^2}{2}\,\kappa^2\left(\sinh(\xi+\lambda)\sinh(\xi-\lambda)\,(\tau(\lambda)+\mathbb{1}) - \frac{1}{2}\cosh(2\lambda)\mathbb{1}\right) + \mathcal{O}(\eta^3),
\end{aligned}
\tag{6.4}
$$

where $\tau(\lambda)$ is the generating function of the trigonometric Gaudin Hamiltonians with boundary terms

$$
\tau(\lambda) = \mathrm{tr}_0\,\mathcal{L}_0^2(\lambda), \tag{6.5}
$$

where

$$
\mathcal{L}_0(\lambda) = L_0(\lambda) - K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda), \tag{6.6}
$$

with the Gaudin Lax matrix defined by

$$
L_0(\lambda) = \sum_{m=1}^{N}\left(\sigma_0^3 \otimes \coth(\lambda-\alpha_m)S_m^3 + \frac{\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+}{2\sinh(\lambda-\alpha_m)}\right), \tag{6.7}
$$

and $K_0(\lambda)$ the upper triangular reflection matrix given in (6.2). The trigonometric Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function $\tau(\lambda)$ (6.5) at poles $\lambda = \pm\alpha_m$:

$$
\operatorname*{Res}_{\lambda=\alpha_m} \tau(\lambda) \;=\; 4\,H_m \quad \text{and} \quad \operatorname*{Res}_{\lambda=-\alpha_m} \tau(\lambda) \;=\; (-4)\,H_m \tag{6.8}
$$

where

$$
\begin{aligned}
H_m = &\sum_{n\neq m}^{N}\left(\coth(\alpha_m-\alpha_n)\; S_m^3 S_n^3 + \frac{S_m^+ S_n^- + S_m^- S_n^+}{2\sinh(\alpha_m-\alpha_n)}\right) + \sum_{n=1}^{N}\coth(\alpha_m+\alpha_n)\,\frac{S_m^3 S_n^3 + S_n^3 S_m^3}{2} \\
&+ \frac{\psi}{\kappa}\,\frac{\sinh(2\alpha_m)}{\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^3 S_n^+ + S_n^+ S_m^3}{2\sinh(\alpha_m+\alpha_n)} + \frac{\sinh(\xi-\alpha_m)}{2\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^- S_n^+ + S_n^+ S_m^-}{2\sinh(\alpha_m+\alpha_n)} \\
&- \frac{\psi}{\kappa}\,\frac{\sinh(2\alpha_m)}{\sinh(\xi-\alpha_m)}\sum_{n=1}^{N}\coth(\alpha_m+\alpha_n)\;\frac{S_m^+ S_n^3 + S_n^3 S_m^+}{2} \\
&+ \frac{\sinh(\xi+\alpha_m)}{2\sinh(\xi-\alpha_m)}\sum_{n=1}^{N}\frac{S_m^+ S_n^- + S_n^- S_m^+}{2\sinh(\alpha_m+\alpha_n)} \\
&- \frac{\psi^2}{\kappa^2}\,\frac{\sinh^2(2\alpha_m)}{2\sinh(\xi-\alpha_m)\sinh(\xi+\alpha_m)}\sum_{n=1}^{N}\frac{S_m^+ S_n^+ + S_n^+ S_m^+}{2\sinh(\alpha_m+\alpha_n)}.
\end{aligned}
\tag{6.9}
$$

Since the central element $\Delta\left[\mathcal{T}(\lambda)\right]$ can be expressed in form (4.10) it is evident that the vector Ω_+ (5.2) is its eigenvector

$$
\Delta\left[\mathcal{T}(\lambda)\right]\Omega_+ = \sinh(2\lambda)\;\alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\,\Omega_+. \tag{6.10}
$$

Moreover, it follows from (5.23) and (6.10) that Ω_+ (5.2) is an eigenvector of the difference

$$\left(t(\lambda) - \frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)}\right)\Omega_+ = \left(\Lambda_0(\lambda) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right)\Omega_+. \tag{6.11}$$

We can expand the eigenvalue on the right hand side of the equation above in powers of η, taking into account that $\phi = 0$,

$$\begin{aligned}&\left(\kappa_1(\lambda)\alpha(\lambda) + \kappa_2(\lambda)\widehat{\delta}(\lambda) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right) = \kappa^2\sinh(\xi+\lambda)\sinh(\xi-\lambda)\\&+\eta\,\frac{\kappa^2}{2}\left(\cosh(2\xi)\coth(2\lambda) - \frac{\cosh(4\lambda)}{\sinh(2\lambda)}\right)\\&+\frac{\eta^2}{2}\,\kappa^2\left(\sinh(\xi+\lambda)\sinh(\xi-\lambda)\,(\chi_0(\lambda)+1) - \frac{1}{2}\cosh(2\lambda)\right) + \mathcal{O}(\eta^3).\end{aligned} \tag{6.12}$$

Substituting the expansion above into the right hand side of (6.11) and using (6.4) to expand the left hand side, it follows that the vector Ω_+ (5.2) is an eigenvector of the generating function of the Gaudin Hamiltonians

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+, \tag{6.13}$$

with

$$\begin{aligned}\chi_0(\lambda) = 2\sum_{m,n=1}^{N}&\left(s_m s_n + \frac{\sinh(\xi+\alpha_m)\sinh(\xi-\alpha_m)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\,s_m\delta_{mn}\right)\times\\&\times\left(\coth(\lambda-\alpha_m)\coth(\lambda-\alpha_n) + 2\coth(\lambda-\alpha_m)\coth(\lambda+\alpha_n)\right.\\&\left.+\coth(\lambda+\alpha_m)\coth(\lambda+\alpha_n)\right).\end{aligned} \tag{6.14}$$

Moreover we can obtain the spectrum of the generating function of the Gaudin Hamiltonians through the expansion

$$\begin{aligned}&\left(\Lambda_M(\lambda,\{\mu_j\}_{j=1}^M) - \alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right) = \kappa^2\sinh(\xi+\lambda)\sinh(\xi-\lambda)\\&+\eta\,\frac{\kappa^2}{2}\left(\cosh(2\xi)\coth(2\lambda) - \frac{\cosh(4\lambda)}{\sinh(2\lambda)}\right)\\&+\frac{\eta^2}{2}\,\kappa^2\left(\sinh(\xi+\lambda)\sinh(\xi-\lambda)\left(\chi_M(\lambda,\{\mu_j\}_{j=1}^M)+1\right) - \frac{1}{2}\cosh(2\lambda)\right) + \mathcal{O}(\eta^3),\end{aligned} \tag{6.15}$$

where

$$\begin{aligned}\chi_M(\lambda,\{\mu_j\}_{j=1}^M) = &\frac{-2\sinh(2\lambda)}{\sinh(\xi-\lambda)\sinh(\xi+\lambda)}\sum_{j=1}^{M}\frac{\sinh(2\lambda)}{\sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j)}\\&+4\sum_{j=1}^{M-1}\sum_{k=j+1}^{M}\frac{\sinh(2\lambda)}{\sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j)}\,\frac{\sinh(2\lambda)}{\sinh(\lambda-\mu_k)\sinh(\lambda+\mu_k)}\\&-4\sum_{m=1}^{N}\frac{s_m\,\sinh(2\lambda)}{\sinh(\lambda-\alpha_m)\sinh(\lambda+\alpha_m)}\sum_{j=1}^{M}\frac{\sinh(2\lambda)}{\sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j)}\end{aligned} \tag{6.16}$$

$$+2\sum_{m,n=1}^{N}\left(s_m s_n+\frac{\sinh(\xi+\alpha_m)\sinh(\xi-\alpha_m)}{\sinh(\xi+\lambda)\sinh(\xi-\lambda)}\,s_m\delta_{mn}\right)\times$$
$$\times\left(\coth(\lambda-\alpha_m)\coth(\lambda-\alpha_n)+2\coth(\lambda-\alpha_m)\coth(\lambda+\alpha_n)\right.$$
$$\left.+\coth(\lambda+\alpha_m)\coth(\lambda+\alpha_n)\right).$$

As our next important step toward obtaining the formulas of the algebraic Bethe ansatz for the corresponding Gaudin model we observe that the first term in the expansion of the function $F_M(\mu_1;\mu_2,\ldots,\mu_M)$ (5.65) in powers of η is

$$F_M(\mu_1;\mu_2,\ldots,\mu_M)=\eta f_M(\mu_1;\mu_2,\ldots,\mu_M)+\mathcal{O}(\eta^2), \tag{6.17}$$

where

$$\begin{aligned}f_M(\mu_1;\mu_2,\ldots,\mu_M)=\kappa\sinh(2\mu_1)\Bigg(&\frac{1}{\sinh(\xi-\mu_1)}-2\sinh(\xi+\mu_1)\times\\&\times\sum_{j=2}^{M}\frac{1}{\sinh(\mu_1-\mu_j)\sinh(\mu_1+\mu_j)}\\&+2\sinh(\xi+\mu_1)\sum_{m=1}^{N}\frac{s_m}{\sinh(\mu_1-\alpha_m)\sinh(\mu_1+\alpha_m)}\Bigg).\end{aligned} \tag{6.18}$$

Along the lines developed in [12,52,55], we have used the formulas (5.32) and (5.33) as well as (5.12), (5.16) and (5.18) in order to expand the Bethe vector $\Psi_1(\mu)$ of the XXZ Heisenberg spin chain in powers of η and obtained the Bethe vector $\varphi_1(\mu)$ of the corresponding trigonometric Gaudin model

$$\Psi_1(\mu)=\eta\,\varphi_1(\mu)+\mathcal{O}(\eta^2), \tag{6.19}$$

where

$$\begin{aligned}\varphi_1(\mu)=\kappa\sinh(2\mu)\Bigg(&\sum_{m=1}^{N}\frac{\sinh(\xi-\alpha_m)\,S_m^-}{\sinh(\mu-\alpha_m)\sinh(\mu+\alpha_m)}\\&+\frac{\psi}{\kappa}\left(1+\sum_{m=1}^{N}s_m\,\frac{e^{-2\xi}+\sinh(2\alpha_m)-\cosh(2\mu)}{\sinh(\mu-\alpha_m)\sinh(\mu+\alpha_m)}\right)\Bigg)\Omega_+.\end{aligned} \tag{6.20}$$

The off-shell action of the difference of the transfer matrix of the XXX chain and the central element, the so-called Sklyanin determinant, on the Bethe vector $\Psi_1(\mu)$ (5.32) is obtained from (4.10) and (5.31) as follows

$$\begin{aligned}\left(t(\lambda)-\frac{\Delta\left[\mathcal{T}(\lambda)\right]}{\sinh(2\lambda)}\right)\Psi_1(\mu)=&\left(\Lambda_1(\lambda,\mu)-\alpha(\lambda+\eta/2)\,\widehat{\delta}(\lambda-\eta/2)\right)\Psi_1(\mu)\\&+\kappa\sinh(\xi-\mu)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)}F_1(\mu)\Psi_1(\lambda).\end{aligned} \tag{6.21}$$

Finally, the off-shell action of the generating function the Gaudin Hamiltonians on the vector $\varphi_1(\mu)$ can be obtained from the equation above by using the expansion (6.4) and (6.19) on the left hand side as well as the expansion (6.15), (6.17) and (6.19) on the right hand side

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{2}{\kappa}\,\frac{\sinh(\xi-\mu)}{\sinh(\xi-\lambda)\sinh(\xi+\lambda)} \times \\ \times \frac{\sinh(2\lambda)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\, f_1(\mu)\varphi_1(\lambda). \tag{6.22}$$

Therefore $\varphi_1(\mu)$ (6.20) is the Bethe vector of the corresponding Gaudin model, i.e. the eigenvector of the generating function the Gaudin Hamiltonians, with the eigenvalue $\chi_1(\lambda,\mu)$ (6.16), once the unwanted term is canceled by imposing the corresponding Bethe equation

$$f_1(\mu) = \kappa \sinh(2\mu)\left(\frac{1}{\sinh(\xi-\mu)} + 2\sinh(\xi+\mu)\sum_{m=1}^{N}\frac{s_m}{\sinh(\mu-\alpha_m)\sinh(\mu+\alpha_m)}\right) = 0. \tag{6.23}$$

To obtain the Bethe vector $\varphi_2(\mu_1,\mu_2)$ of the Gaudin model and the action of the generating function $\tau(\lambda)$ of the Gaudin Hamiltonians on $\varphi_2(\mu_1,\mu_2)$ we basically follow the steps we have done when studying the action of $\tau(\lambda)$ on $\varphi_1(\mu)$. The first term in the expansion of the Bethe vector $\Psi_2(\mu_1,\mu_2)$ (5.46) in powers of η yields the corresponding Bethe vector of the Gaudin model

$$\Psi_2(\mu_1,\mu_2) = \eta^2\varphi_2(\mu_1,\mu_2) + \mathcal{O}(\eta^3), \tag{6.24}$$

where

$$\varphi_2(\mu_1,\mu_2) = \kappa^2\sinh(2\mu_1)\sinh(2\mu_2)\left(\sum_{m,n=1}^{N}\frac{\sinh(\xi-\alpha_m)\,S_m^-}{\sinh(\mu_1-\alpha_m)\sinh(\mu_1+\alpha_m)} \times \right. \\ \times \frac{\sinh(\xi-\alpha_n)\,S_n^-}{\sinh(\mu_2-\alpha_n)\sinh(\mu_2+\alpha_n)} + \frac{\psi}{\kappa}\sum_{m=1}^{N}\frac{\sinh(\xi-\alpha_m)\,S_m^-}{\sinh(\mu_2-\alpha_m)\sinh(\mu_2+\alpha_m)} \times \\ \times \left(1+\sum_{n=1}^{N}\frac{e^{-2\xi}+\sinh(2\alpha_n)-\cosh(2\mu_1)}{\sinh(\mu_1-\alpha_n)\sinh(\mu_1+\alpha_n)}\,(s_n-\delta_{mn})\right) \\ + \frac{\psi}{\kappa}\sum_{m=1}^{N}\frac{\sinh(\xi-\alpha_m)\,S_m^-}{\sinh(\mu_1-\alpha_m)\sinh(\mu_1+\alpha_m)}\left(3+\sum_{n=1}^{N}\frac{e^{-2\xi}+\sinh(2\alpha_n)-\cosh(2\mu_2)}{\sinh(\mu_2-\alpha_n)\sinh(\mu_2+\alpha_n)}\,s_n\right) \\ + e^{-2\xi}\frac{\psi^2}{\kappa^2}\sum_{m=1}^{N}\frac{-e^{\xi-\alpha_m}\cosh(2\mu_1)+\cosh(\xi+\alpha_m)}{\sinh(\mu_1-\alpha_m)\sinh(\mu_1+\alpha_m)}\,\frac{\sinh(\xi-\alpha_m)}{\sinh(\mu_2-\alpha_m)\sinh(\mu_2+\alpha_m)}(2s_m) \\ + \frac{\psi^2}{\kappa^2}\left(1+\sum_{n=1}^{N}\frac{e^{-2\xi}+\sinh(2\alpha_m)-\cosh(2\mu_1)}{\sinh(\mu_1-\alpha_m)\sinh(\mu_1+\alpha_m)}s_m\right)\times \\ \left.\times\left(3+\sum_{n=1}^{N}\frac{e^{-2\xi}+\sinh(2\alpha_n)-\cosh(2\mu_2)}{\sinh(\mu_2-\alpha_n)\sinh(\mu_2+\alpha_n)}s_n\right)\right)\Omega_+. \tag{6.25}$$

Expressing Gaudin Bethe vectors by using creation operators is in accordance with the results in the rational case [12]. There the creation operator was introduced (cf. formula (6.32) in [12]), but here it is necessary to define the family of operators

$$\begin{aligned}
c_K(\mu) &= \kappa \sinh(2\mu) \left(\sum_{m=1}^{N} \frac{\sinh(\xi - \alpha_m)\, S_m^-}{\sinh(\mu - \alpha_m)\sinh(\mu + \alpha_m)} + \frac{\psi}{\kappa} ((-1 + 2K) \right. \\
&\quad + \sum_{m=1}^{N} \frac{e^{-2\xi} + \sinh(2\alpha_m) - \cosh(2\mu)}{\sinh(\mu - \alpha_m)\sinh(\mu + \alpha_m)}\, S_m^3 \Bigg) \\
&\quad \left. + e^{-2\xi} \frac{\psi^2}{\kappa^2} \sum_{m=1}^{N} \frac{\cosh(\xi + \alpha_m) - e^{\xi - \alpha_m}\cosh(2\mu)}{\sinh(\mu - \alpha_m)\sinh(\mu + \alpha_m)}\, S_m^+ \right),
\end{aligned} \tag{6.26}$$

for any natural number K. Thus the Bethe vectors (6.20) and (6.25) can be expressed as

$$\varphi_1(\mu) = c_1(\mu)\Omega_+ \qquad \text{and} \qquad \varphi_2(\mu_1, \mu_2) = c_1(\mu_1)c_2(\mu_2)\Omega_+. \tag{6.27}$$

Although in general the operators (6.26) do not commute, it is straightforward to check that the Bethe vector $\varphi_2(\mu_1, \mu_2)$ is a symmetric function

$$\varphi_2(\mu_1, \mu_2) = c_1(\mu_1)c_2(\mu_2)\Omega_+ = c_1(\mu_2)c_2(\mu_1)\Omega_+ = \varphi_2(\mu_2, \mu_1). \tag{6.28}$$

It is of interest to study the action of the difference of the transfer matrix $t(\lambda)$ and the so-called Sklyanin determinant $\Delta[\mathcal{T}(\lambda)]$ on the Bethe vector $\Psi_2(\mu_1, \mu_2)$ using (4.10) and (5.45)

$$\begin{aligned}
&\left(t(\lambda) - \frac{\Delta[\mathcal{T}(\lambda)]}{\sinh(2\lambda)} \right) \Psi_2(\mu_1, \mu_2) \\
&= \left(\Lambda_2(\lambda, \{\mu_i\}_{i=1}^2) - \alpha(\lambda + \eta/2)\, \widehat{\delta}(\lambda - \eta/2) \right) \Psi_2(\mu_1, \mu_2) \\
&+ \frac{\sinh(\eta)\sinh(2(\lambda + \eta))}{\sinh(\lambda - \mu_1)\sinh(\lambda + \mu_1 + \eta)} \kappa \sinh(\xi - \mu_1)\ F_2(\mu_1; \mu_2)\Psi_2(\lambda, \mu_2) \\
&+ \frac{\sinh(\eta)\sinh(2(\lambda + \eta))}{\sinh(\lambda - \mu_2)\sinh(\lambda + \mu_2 + \eta)} \kappa \sinh(\xi - \mu_2)\ F_2(\mu_2; \mu_1)\Psi_2(\lambda, \mu_1).
\end{aligned} \tag{6.29}$$

The off-shell action of the generating function of the Gaudin Hamiltonians on the Bethe vector $\varphi_2(\mu_1, \mu_2)$ is obtained from the equation above using the expansions (6.4) and (6.24) on the left hand side and (6.15), (6.24) and (6.17) on the right hand side. Then, by comparing the terms of the fourth power in η on both sides of (6.29) we obtain

$$\begin{aligned}
\tau(\lambda)\varphi_2(\mu_1, \mu_2) &= \chi_2(\lambda, \mu_1, \mu_2)\varphi_2(\mu_1, \mu_2) + \frac{2}{\kappa} \frac{\sinh(2\lambda)}{\sinh(\xi - \lambda)\sinh(\xi + \lambda)} \times \\
&\quad \times \left(\frac{\sinh(\xi - \mu_1)}{\sinh(\lambda - \mu_1)\sinh(\lambda + \mu_1)} f_2(\mu_1; \mu_2)\varphi_2(\lambda, \mu_2) \right. \\
&\quad \left. + \frac{\sinh(\xi - \mu_2)}{\sinh(\lambda - \mu_2)\sinh(\lambda + \mu_2)} f_2(\mu_2; \mu_1)\varphi_2(\lambda, \mu_1) \right).
\end{aligned} \tag{6.30}$$

The two unwanted terms on the right hand side of the equation above are annihilated by the following Bethe equations

$$\begin{aligned}
f_2(\mu_1; \mu_2) &= \kappa \sinh(2\mu_1) \left(\frac{1}{\sinh(\xi - \mu_1)} - \frac{2\sinh(\xi + \mu_1)}{\sinh(\mu_1 - \mu_2)\sinh(\mu_1 + \mu_2)} \right. \\
&\quad \left. + 2\sinh(\xi + \mu_1) \sum_{m=1}^{N} \frac{s_m}{\sinh(\mu_1 - \alpha_m)\sinh(\mu_1 + \alpha_m)} \right) = 0,
\end{aligned} \tag{6.31}$$

$$f_2(\mu_2;\mu_1)=\kappa\sinh(2\mu_2)\left(\frac{1}{\sinh(\xi-\mu_2)}-\frac{2\sinh(\xi+\mu_2)}{\sinh(\mu_2-\mu_1)\sinh(\mu_2+\mu_1)}\right.$$
$$\left.+2\sinh(\xi+\mu_2)\sum_{m=1}^{N}\frac{s_m}{\sinh(\mu_2-\alpha_m)\sinh(\mu_2+\alpha_m)}\right)=0. \tag{6.32}$$

In general, we have that the first term in the expansion of the Bethe vector $\Psi_M(\mu_1,\mu_2,\dots,\mu_M)$ (5.58) in powers of η is

$$\Psi_M(\mu_1,\mu_2,\dots,\mu_M)=\eta^M\varphi_M(\mu_1,\mu_2,\dots,\mu_M)+\mathcal{O}(\eta^{M+1}), \tag{6.33}$$

where M is a natural number and

$$\varphi_M(\mu_1,\mu_2,\dots,\mu_M)=c_1(\mu_1)c_2(\mu_2)\cdots c_M(\mu_M)\Omega_+, \tag{6.34}$$

and the operator $c_K(\mu_K)$, $K=1,\dots,M$, are given in (6.26).

Although the operators $c_K(\mu_K)$ do not commute, the Bethe vector of the Gaudin model $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ is a symmetric function of its arguments, since a straightforward calculation shows that the operators $c_K(\mu)$ satisfy the following identity,

$$c_K(\mu)c_{K+1}(\mu')-c_K(\mu')c_{K+1}(\mu)=0, \tag{6.35}$$

for $K=1,\dots,M-1$. The action of the generating function $\tau(\lambda)$ (6.5) on the Bethe vector $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ can be derived as in the two previous cases when $M=1$ (6.22) and $M=2$ (6.30). In the present case we use the expansions (6.15), (6.17) and (6.33) to obtain

$$\begin{aligned}\tau(\lambda)\varphi_M(\mu_1,\mu_2,\dots,\mu_M)&=\chi_M(\lambda,\{\mu_i\}_{i=1}^M)\varphi_M(\mu_1,\mu_2,\dots,\mu_M)\\&\quad+\frac{2}{\kappa}\frac{\sinh(2\lambda)}{\sinh(\xi-\lambda)\sinh(\xi+\lambda)}\sum_{i=1}^{M}\frac{\sinh(\xi-\mu_i)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i)}\times\\&\quad\times f_M(\mu_i;\{\mu_j\}_{j\neq i})\varphi_M(\lambda,\{\mu_j\}_{j\neq i}),\end{aligned} \tag{6.36}$$

where $\chi_M(\lambda,\{\mu_i\}_{i=1}^M)$ is given in (6.16) and the unwanted terms on the right hand side of the equation above are canceled by the following Bethe equations

$$\begin{aligned}f_M(\mu_i;\{\mu_j\}_{j\neq i}^M)&=\kappa\sinh(2\mu_i)\left(\frac{1}{\sinh(\xi-\mu_i)}-2\sinh(\xi+\mu_i)\times\right.\\&\quad\times\sum_{j=2}^{M}\frac{1}{\sinh(\mu_i-\mu_j)\sinh(\mu_i+\mu_j)}\\&\quad\left.+2\sinh(\xi+\mu_i)\sum_{m=1}^{N}\frac{s_m}{\sinh(\mu_i-\alpha_m)\sinh(\mu_i+\alpha_m)}\right)=0,\end{aligned} \tag{6.37}$$

for $i=1,2,\dots M$. As expected, due to our definition of the Bethe vector $\varphi_M(\mu_1,\mu_2,\dots,\mu_M)$ (6.34), the quasi-classical limit has yielded the above simple formulae for the off-shell action of the generating function $\tau(\lambda)$.

An alternative approach to the implementation of the algebraic Bethe ansatz for the trigonometric $s\ell(2)$ Gaudin model, with the triangular K-matrix (6.2), is based on the corresponding non-unitary classical r-matrix. This study will be reported in [55].

7. Conclusions

We have implemented fully the off-shell algebraic Bethe ansatz for the XXZ Heisenberg spin chain in the case when both boundary matrices have the upper-triangular form. As opposed to the case of the XXX Heisenberg spin chain where the general reflection matrices could be put into the upper triangular form without any loss of generality [10,12], here the triangularity of the reflection matrices has to be imposed as extra conditions on the respective parameters. A suitable realization for the Sklyanin monodromy matrix is obtained as a direct consequence of the identity satisfied by the Lax operator. This realization led to the action of the entries of the Sklyanin monodromy matrix on the vector Ω_+ and consequently to the observation that Ω_+ is an eigenvector of the transfer matrix of the chain.

The essential step of the algebraic Bethe ansatz is the definition of the corresponding Bethe vectors. Initially we have obtained the Bethe vectors $\widetilde{\Psi}_M(\mu_1, \mu_2, \ldots, \mu_M)$, for $M = 1, 2, 3, 4$, by requiring that their scaling limit corresponds to the Bethe vectors of the XXX Heisenberg chain. We gave a step by step presentation of the $M = 1, 2, 3$ Bethe vectors, including the formulae for the action of $t(\lambda)$, the corresponding eigenvalues and Bethe equations. In this way we have exposed the property of these vectors to make the off shell action of the transform matrix as simple as possible. We did not present here all the necessary formulae of the Bethe vector $\widetilde{\Psi}_4(\mu_1, \mu_2, \mu_3, \mu_4)$, as they are cumbersome. More importantly, they do not admit any compact closed form for an arbitrary natural number M. However, we have noticed the identities (C.11) and (C.12) which enabled the general form of the Bethe vectors for a fixed M. The general form of Bethe vectors can be expressed as a sum of a particular one and a linear combination of lower order Bethe vectors that correspond to the same eigenvalue (C.13). This is indeed the case with Bethe vectors of any order, for details see Appendix C. A careful analysis reveals that there exists a particular form of the Bethe vector $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ which, for an arbitrary natural number M, can be defined by the suitable recurrence procedure analogous to the one proposed in the case of the XXX Heisenberg chain [12]. Actually, the recurrence relations defining the relevant coefficient functions differ only in the multiplicative factors from the respective ones in the case of the XXX Heisenberg chain. As expected, the action of $t(\lambda)$ on the Bethe vector $\Psi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is again very simple. Actually, the action of the transfer matrix is as simple as it could possible be since it almost coincides with the corresponding action in the case when the two boundary matrices are diagonal [6,40].

As in the case of the XXX Heisenberg chain [52], the quasi-classical expansion of the linear combination of the transfer matrix of the XXZ Heisenberg spin chain and the central element, the so-called Sklyanin determinant yields the generating function of the trigonometric Gaudin Hamiltonians with boundary terms [55]. Based on this result, and the appropriate definition of the corresponding Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, we showed how the quasi-classical limit yields the off-shell action of the generating function of the Gaudin Hamiltonians as well as the spectrum and the Bethe equations. As opposed to the rational case where the Gaudin Bethe vectors were defined by the action of the creation operator [12], here it was necessary to define the family of operators. As in the case of the spin chain, the off-shell action of the generating function $\tau(\lambda)$ on the Bethe vectors $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$ is strikingly simple. It is as simple as it can be since it practically coincide with the corresponding formula in the case when the boundary matrix is diagonal [40].

It would be of interest to establish a relation between Bethe vectors of the Gaudin model and solutions to the corresponding generalized Knizhnik–Zamolodchikov equations, along the lines it was done in the case when the boundary matrix is diagonal [40], as well as to study possible relations between Bethe vectors of XXZ chain obtained in the Section 5 and the solutions to the boundary quantum Knizhnik–Zamolodchikov equations [70–72].

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Appendix A. Basic definitions

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{A.1}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$. The operators S^α with $\alpha = +, -, 3$, act in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations [60,63,66]

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{\sinh(2\eta S^3)}{\sinh(\eta)} = [2S^3]_q, \tag{A.2}$$

with $q = e^\eta$, and Casimir operator

$$c_2 = \frac{q + q^{-1}}{2}[S^3]_q^2 + \frac{1}{2}(S^+S^- + S^-S^+) = \frac{q + q^{-1}}{2}[S^3]_q^2 + \frac{1}{2}[2S^3]_q + S^-S^+. \tag{A.3}$$

In the space $\mathbb{C}^{2s+1}$ these operators admit the following matrix representation

$$S^3 = \sum_{i=1}^{2s+1} a_i e_{i\,i}, \quad S^+ = \sum_{i=1}^{2s+1} b_i e_{i\,i+1}, \quad S^- = \sum_{i=1}^{2s+1} b_i e_{i+1\,i} \quad \text{and} \quad c_2 = [s+1]_q\,[s]_q\,\mathbb{1}, \tag{A.4}$$

where

$$(e_{ij})_{kl} = \delta_{i\,k}\delta_{j\,l}, \quad a_i = s + 1 - i, \quad b_i = \sqrt{[i]_q\,[2s+1-i]_q} \quad \text{and} \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{A.5}$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

Appendix B. Commutation relations

The equation (4.6) yields the exchange relations between the operators $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$, $\mathcal{C}(\lambda)$ and $\widehat{\mathcal{D}}(\lambda)$. The relevant relations are

$$\mathcal{B}(\lambda)\mathcal{B}(\mu) = \mathcal{B}(\mu)\mathcal{B}(\lambda), \qquad \mathcal{C}(\lambda)\mathcal{C}(\mu) = \mathcal{C}(\mu)\mathcal{C}(\lambda), \tag{B.1}$$

$$
\begin{aligned}
\mathcal{A}(\lambda)\mathcal{B}(\mu) &= \frac{\sinh(\lambda+\mu)\sinh(\lambda-\mu-\eta)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)}\mathcal{B}(\mu)\mathcal{A}(\lambda) \\
&+ \frac{\sinh(\eta)\sinh(2\mu)}{\sinh(\lambda-\mu)\sinh(2\mu+\eta)}\mathcal{B}(\lambda)\mathcal{A}(\mu) \\
&- \frac{\sinh(\eta)}{\sinh(\lambda+\mu+\eta)}\mathcal{B}(\lambda)\widehat{\mathcal{D}}(\mu),
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\widehat{\mathcal{D}}(\lambda)\mathcal{B}(\mu) &= \frac{\sinh(\lambda-\mu+\eta)\sinh(\lambda+\mu+2\eta)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)}\mathcal{B}(\mu)\widehat{\mathcal{D}}(\lambda) \\
&- \frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu)\sinh(2\lambda+\eta)}\mathcal{B}(\lambda)\widehat{\mathcal{D}}(\mu) \\
&+ \frac{\sinh(\eta)\sinh(2\mu)\sinh(2(\lambda+\eta))}{\sinh(2\lambda+\eta)\sinh(2\mu+\eta)\sinh(\lambda+\mu+\eta)}\mathcal{B}(\lambda)\mathcal{A}(\mu),
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
[\mathcal{C}(\lambda), \mathcal{B}(\mu)] &= \frac{\sinh(\eta)\sinh(2\lambda)\sinh(\lambda-\mu+\eta)}{\sinh(\lambda-\mu)\sinh(2\lambda+\eta)\sinh(\lambda+\mu+\eta)}\mathcal{A}(\mu)\mathcal{A}(\lambda) \\
&- \frac{\sinh^2(\eta)\sinh(2\lambda)}{\sinh(\lambda-\mu)\sinh(2\lambda+\eta)\sinh(2\mu+\eta)}\mathcal{A}(\lambda)\mathcal{A}(\mu) \\
&+ \frac{\sinh(\eta)\sinh(\lambda+\mu)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu+\eta)}\mathcal{A}(\mu)\widehat{\mathcal{D}}(\lambda) \\
&- \frac{\sinh(\eta)\sinh(2\lambda)}{\sinh(\lambda-\mu)\sinh(2\lambda+\eta)}\mathcal{A}(\lambda)\widehat{\mathcal{D}}(\mu) \\
&- \frac{\sinh^2(\eta)}{\sinh(2\mu+\eta)\sinh(\lambda+\mu+\eta)}\widehat{\mathcal{D}}(\lambda)\mathcal{A}(\mu) \\
&- \frac{\sinh(\eta)}{\sinh(\lambda+\mu+\eta)}\widehat{\mathcal{D}}(\lambda)\widehat{\mathcal{D}}(\mu).
\end{aligned} \tag{B.4}
$$

For completeness we include the following commutation relations

$$
[\mathcal{A}(\lambda), \mathcal{A}(\mu)] = \frac{\sinh(\eta)}{\sinh(\lambda+\mu+\eta)}\left(\mathcal{B}(\mu)\mathcal{C}(\lambda) - \mathcal{B}(\lambda)\mathcal{C}(\mu)\right) \tag{B.5}
$$

$$
\left[\mathcal{A}(\lambda), \widehat{\mathcal{D}}(\mu)\right] = \frac{\sinh(\eta)\sinh(2(\mu+\eta))}{\sinh(\lambda-\mu)\sinh(2\mu+\eta)}\left(\mathcal{B}(\lambda)\mathcal{C}(\mu) - \mathcal{B}(\mu)\mathcal{C}(\lambda)\right) \tag{B.6}
$$

$$
\left[\widehat{\mathcal{D}}(\lambda), \widehat{\mathcal{D}}(\mu)\right] = \frac{\sinh(\eta)\sinh(2(\lambda+\eta))\sinh(2(\mu+\eta))}{\sinh(2\lambda+\eta)\sinh(2\mu+\eta)\sinh(\lambda+\mu+\eta)}\left(\mathcal{B}(\lambda)\mathcal{C}(\mu) - \mathcal{B}(\mu)\mathcal{C}(\lambda)\right) \tag{B.7}
$$

The implementation of the algebraic Bethe ansatz presented in Section 5 is based on the above relations. For convenience, we also include the following three relations which follow from the ones above and are essential in the derivation of the off-shell action (5.37) of the transfer matrix of the inhomogeneous XXZ chain (5.21) on the Bethe vector $\widetilde{\Psi}_2(\mu_1, \mu_2)$ (5.34)

$$
\begin{aligned}
&\mathcal{A}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&= \prod_{i=1}^{2} \frac{\sinh(\lambda+\mu_i)\sinh(\lambda-\mu_i-\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \alpha(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&+ \sum_{i=1}^{2} \frac{\sinh(\eta)\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)\sinh(\lambda-\mu_i)} \frac{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \times \\
&\times \alpha(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+ \\
&- \sum_{i=1}^{2} \frac{\sinh(\eta)}{\sinh(\lambda+\mu_i+\eta)} \frac{\sinh(\mu_i-\mu_{3-i}+\eta)\sinh(\mu_i+\mu_{3-i}+2\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \times \\
&\times \widehat{\delta}(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+,
\end{aligned}
\tag{B.8}
$$

analogously,

$$
\begin{aligned}
&\widehat{\mathcal{D}}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&= \prod_{i=1}^{2} \frac{\sinh(\lambda-\mu_i+\eta)\sinh(\lambda+\mu_i+2\eta)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \widehat{\delta}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ \\
&- \sum_{i=1}^{2} \frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(2\lambda+\eta)\sinh(\lambda-\mu_i)} \frac{\sinh(\mu_i-\mu_{3-i}+\eta)\sinh(\mu_i+\mu_{3-i}+2\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_1+\mu_{3-i}+\eta)} \times \\
&\times \widehat{\delta}(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+ \\
&+ \sum_{i=1}^{2} \frac{\sinh(\eta)\sinh(2\mu_i)\sinh(2(\lambda+\eta))}{\sinh(2\lambda+\eta)\sinh(2\mu_i+\eta)\sinh(\lambda+\mu_i+\eta)} \times \\
&\times \frac{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \alpha(\mu_i)\mathcal{B}(\lambda)\mathcal{B}(\mu_{3-i})\Omega_+,
\end{aligned}
\tag{B.9}
$$

and finally,

$$
\begin{aligned}
&\mathcal{C}(\lambda)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ = \sum_{i=1}^{2} \Bigg(\frac{\sinh(\eta)\sinh(2\mu_i)\sinh(2\lambda)}{\sinh(2\lambda+\eta)\sinh(2\mu_i+\eta)\sinh(\lambda+\mu_i+\eta)} \times \\
&\times \frac{\sinh(\lambda+\mu_{3-i})\sinh(\lambda-\mu_{3-i}-\eta)}{\sinh(\lambda-\mu_{3-i})\sinh(\lambda+\mu_{3-i}+\eta)} \frac{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \alpha(\lambda)\alpha(\mu_i) \\
&- \frac{\sinh(\eta)\sinh(2\lambda)}{\sinh(\lambda-\mu_i)\sinh(2\lambda+\eta)} \frac{\sinh(\lambda+\mu_{3-i})\sinh(\lambda-\mu_{3-i}-\eta)}{\sinh(\lambda-\mu_{3-i})\sinh(\lambda+\mu_{3-i}+\eta)} \times \\
&\times \frac{\sinh(\mu_i-\mu_{3-i}+\eta)\sinh(\mu_i+\mu_{3-i}+2\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \alpha(\lambda)\widehat{\delta}(\mu_i) \\
&+ \frac{\sinh(\eta)\sinh(2\mu_i)}{\sinh(\lambda-\mu_i)\sinh(2\mu_i+\eta)} \frac{\sinh(\lambda-\mu_{3-i}+\eta)\sinh(\lambda+\mu_{3-i}+2\eta)}{\sinh(\lambda-\mu_{3-i})\sinh(\lambda+\mu_{3-i}+\eta)} \times \\
&\times \frac{\sinh(\mu_i+\mu_{3-i})\sinh(\mu_i-\mu_{3-i}-\eta)}{\sinh(\mu_i-\mu_{3-i})\sinh(\mu_i+\mu_{3-i}+\eta)} \alpha(\mu_i)\widehat{\delta}(\lambda) \\
&- \frac{\sinh(\eta)}{\sinh(\lambda+\mu_i+\eta)} \frac{\sinh(\lambda-\mu_{3-i}+\eta)\sinh(\lambda+\mu_{3-i}+2\eta)}{\sinh(\lambda-\mu_{3-i})\sinh(\lambda+\mu_{3-i}+\eta)} \times
\end{aligned}
$$

$$\times \frac{\sinh(\mu_i - \mu_{3-i} + \eta) \sinh(\mu_i + \mu_{3-i} + 2\eta)}{\sinh(\mu_i - \mu_{3-i}) \sinh(\mu_i + \mu_{3-i} + \eta)} \widehat{\delta}(\lambda)\widehat{\delta}(\mu_i) \Big) \mathcal{B}(\mu_{3-i})\Omega_+$$
$$+ \Bigg(\frac{\sinh^2(\eta) \sinh(2\mu_1) \sinh(2\mu_2) \sinh(\mu_1 + \mu_2)}{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) \sinh(2\mu_1 + \eta) \sinh(2\mu_2 + \eta)} \times$$
$$\times \frac{\sinh(\lambda + \mu_1) \sinh(\lambda - \mu_2 + \eta) + \sinh(\lambda - \mu_1) \sinh(\lambda + \mu_2 + \eta)}{\sinh(\lambda + \mu_1 + \eta) \sinh(\lambda + \mu_2 + \eta) \sinh(\mu_1 + \mu_2 + \eta)} \alpha(\mu_1)\alpha(\mu_2)$$
$$- \frac{\sinh^2(\eta) \sinh(2\mu_1) \sinh(\mu_1 - \mu_2 - \eta)}{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) \sinh(2\mu_1 + \eta) \sinh(\mu_1 - \mu_2)} \times$$
$$\times \frac{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) + \sinh(\lambda + \mu_1) \sinh(\lambda + \mu_2 + 2\eta)}{\sinh(\lambda + \mu_1 + \eta) \sinh(\lambda + \mu_2 + \eta)} \alpha(\mu_1)\widehat{\delta}(\mu_2)$$
$$- \frac{\sinh^2(\eta) \sinh(2\mu_2) \sinh(\mu_2 - \mu_1 - \eta)}{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) \sinh(2\mu_2 + \eta) \sinh(\mu_2 - \mu_1)} \times$$
$$\times \frac{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) + \sinh(\lambda + \mu_1) \sinh(\lambda + \mu_2 + 2\eta)}{\sinh(\lambda + \mu_1 + \eta) \sinh(\lambda + \mu_2 + \eta)} \alpha(\mu_2)\widehat{\delta}(\mu_1)$$
$$- \frac{\sinh^2(\eta) \sinh(\mu_1 + \mu_2 + 2\eta)}{\sinh(\lambda - \mu_1) \sinh(\lambda - \mu_2) \sinh(\mu_1 + \mu_2 + \eta)} \times$$
$$\times \frac{\sinh(\lambda + \mu_1 + 2\eta) \sinh(-\lambda + \mu_2 + \eta) + \sinh(-\lambda + \mu_1) \sinh(\lambda + \mu_2 + \eta)}{\sinh(\lambda + \mu_1 + \eta) \sinh(\lambda + \mu_2 + \eta)} \times$$
$$\times \widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2) \Big) \mathcal{B}(\lambda)\Omega_+. \tag{B.10}$$

Appendix C. Bethe vectors

With the aim of pursuing the general case in this appendix we present the Bethe vector $\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3)$, which in the scaling limit corresponds to the corresponding Bethe vector of the XXX chain [12],

$$\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3)$$
$$= \mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\mathcal{B}(\mu_3)\Omega_+ + \widetilde{b}_3^{(1)}(\mu_3; \mu_2, \mu_1)\mathcal{B}(\mu_1)\mathcal{B}(\mu_2)\Omega_+ + \widetilde{b}_3^{(1)}(\mu_1; \mu_2, \mu_3) \times$$
$$\times \mathcal{B}(\mu_2)\mathcal{B}(\mu_3)\Omega_+ + \widetilde{b}_3^{(1)}(\mu_2; \mu_1, \mu_3)\mathcal{B}(\mu_1)\mathcal{B}(\mu_3)\Omega_+ + \widetilde{b}_3^{(2)}(\mu_1, \mu_2; \mu_3)\mathcal{B}(\mu_3)\Omega_+$$
$$+ \widetilde{b}_3^{(2)}(\mu_1, \mu_3; \mu_2)\mathcal{B}(\mu_2)\Omega_+ + \widetilde{b}_3^{(2)}(\mu_2, \mu_3; \mu_1)\mathcal{B}(\mu_1)\Omega_+ + \widetilde{b}_3^{(3)}(\mu_1, \mu_2, \mu_3)\Omega_+, \tag{C.1}$$

where the coefficient functions $\widetilde{b}_3^{(1)}(\mu_1; \mu_2, \mu_3)$, $\widetilde{b}_3^{(2)}(\mu_1, \mu_2; \mu_3)$ and $\widetilde{b}_3^{(3)}(\mu_1, \mu_2, \mu_3)$ are explicitly given by

$$\widetilde{b}_3^{(1)}(\mu_1; \mu_2, \mu_3) = \left(-\frac{\psi^+}{\kappa^+} \right) \left(\prod_{i=2}^{3} \frac{\sinh(\mu_1 + \mu_i) \sinh(\mu_1 - \mu_i - \eta)}{\sinh(\mu_1 - \mu_i) \sinh(\mu_1 + \mu_i + \eta)} \frac{\sinh(2\mu_1)}{\sinh(2\mu_1 + \eta)} \times \right.$$
$$\times \cosh(\xi^+ - \mu_1)\, \alpha(\mu_1) - \prod_{i=2}^{3} \frac{\sinh(\mu_1 - \mu_i + \eta) \sinh(\mu_1 + \mu_i + 2\eta)}{\sinh(\mu_1 - \mu_i) \sinh(\mu_1 + \mu_i + \eta)} \times$$
$$\left. \times \cosh(\xi^+ + \mu_1 + \eta)\, \widehat{\delta}(\mu_1) \right), \tag{C.2}$$

$$
\begin{aligned}
\widetilde{b}_3^{(2)}(\mu_1,\mu_2;\mu_3) = \left(\frac{\psi^+}{\kappa^+}\right)^2 & \Bigg(\prod_{i=1}^{2} \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)} \frac{\sinh(\mu_i+\mu_3)\sinh(\mu_i-\mu_3-\eta)}{\sinh(\mu_i-\mu_3)\sinh(\mu_i+\mu_3+\eta)} \times \\
& \times \frac{\sinh(\mu_1+\mu_2)}{\sinh(\mu_1+\mu_2+\eta)} \cosh(2\xi^+ - \mu_1 - \mu_2 + \eta)\, \alpha(\mu_1)\alpha(\mu_2) \\
& - \frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)}\ \frac{\sinh(\mu_1+\mu_3)\sinh(\mu_1-\mu_3-\eta)}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3+\eta)} \times \\
& \times \frac{\sinh(\mu_2-\mu_3+\eta)\sinh(\mu_2+\mu_3+2\eta)}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3+\eta)}\ \frac{\sinh(\mu_1-\mu_2-\eta)}{\sinh(\mu_1-\mu_2)} \times \\
& \times \cosh(2\xi^+ - \mu_1 + \mu_2 + 2\eta)\, \alpha(\mu_1)\widehat{\delta}(\mu_2) \\
& - \frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)}\ \frac{\sinh(\mu_2+\mu_3)\sinh(\mu_2-\mu_3-\eta)}{\sinh(\mu_2-\mu_3)\sinh(\mu_2+\mu_3+\eta)} \times \\
& \times \frac{\sinh(\mu_1-\mu_3+\eta)\sinh(\mu_1+\mu_3+2\eta)}{\sinh(\mu_1-\mu_3)\sinh(\mu_1+\mu_3+\eta)} \frac{\sinh(\mu_2-\mu_1-\eta)}{\sinh(\mu_2-\mu_1)} \times \\
& \times \cosh(2\xi^+ + \mu_1 - \mu_2 + 2\eta)\, \widehat{\delta}(\mu_1)\alpha(\mu_2) \\
& + \frac{\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1+\mu_2+\eta)} \prod_{i=1}^{2} \frac{\sinh(\mu_i-\mu_3+\eta)\sinh(\mu_i+\mu_3+2\eta)}{\sinh(\mu_i-\mu_3)\sinh(\mu_i+\mu_3+\eta)} \times \\
& \times \cosh(2\xi^+ + \mu_1 + \mu_2 + 3\eta)\, \widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)\Bigg),
\end{aligned}
\tag{C.3}
$$

and

$$
\begin{aligned}
& \widetilde{b}_3^{(3)}(\mu_1,\mu_2,\mu_3) = \left(-\frac{\psi^+}{\kappa^+}\right)^3 \Bigg(\prod_{i=1}^{3} \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)}\ \prod_{j>i}^{3} \frac{\sinh(\mu_i+\mu_j)}{\sinh(\mu_i+\mu_j+\eta)} \times \\
& \times \cosh(3\xi^+ - \mu_1 - \mu_2 - \mu_3 + 3\eta)\ \times \alpha(\mu_1)\alpha(\mu_2)\alpha(\mu_3) \\
& - \frac{\sinh(\mu_1+\mu_2)}{\sinh(\mu_1+\mu_2+\eta)} \prod_{i=1}^{2} \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)}\ \frac{\sinh(\mu_i-\mu_3-\eta)}{\sinh(\mu_i-\mu_3)} \times \\
& \times \cosh(3\xi^+ - \mu_1 - \mu_2 + \mu_3 + 4\eta)\, \alpha(\mu_1)\alpha(\mu_2)\widehat{\delta}(\mu_3) \\
& - \frac{\sinh(\mu_1+\mu_3)}{\sinh(\mu_1+\mu_3+\eta)} \prod_{\substack{i=1\\ i\neq 2}}^{3} \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)} \frac{\sinh(\mu_i-\mu_2-\eta)}{\sinh(\mu_i-\mu_2)} \\
& \times \cosh(3\xi^+ - \mu_1 + \mu_2 - \mu_3 + 4\eta)\, \alpha(\mu_1)\alpha(\mu_3)\widehat{\delta}(\mu_2) - \frac{\sinh(\mu_2+\mu_3)}{\sinh(\mu_2+\mu_3+\eta)} \times \\
& \times \prod_{i=2}^{3} \frac{\sinh(2\mu_i)}{\sinh(2\mu_i+\eta)} \frac{\sinh(\mu_i-\mu_1-\eta)}{\sinh(\mu_i-\mu_1)} \times \\
& \times \cosh(3\xi^+ + \mu_1 - \mu_2 - \mu_3 + 4\eta)\alpha(\mu_2)\alpha(\mu_3)\widehat{\delta}(\mu_1) \\
& + \frac{\sinh(2\mu_1)}{\sinh(2\mu_1+\eta)} \prod_{i=2}^{3} \frac{\sinh(\mu_1-\mu_i-\eta)}{\sinh(\mu_1-\mu_i)}\ \frac{\sinh(\mu_2+\mu_3+2\eta)}{\sinh(\mu_2+\mu_3+\eta)}
\end{aligned}
$$

$$\times\ \cosh(3\xi^+ - \mu_1 + \mu_2 + \mu_3 + 5\eta)\alpha(\mu_1)\widehat{\delta}(\mu_2)\widehat{\delta}(\mu_3)$$
$$+\ \frac{\sinh(2\mu_2)}{\sinh(2\mu_2+\eta)} \prod_{\substack{i=1\\ i\neq 2}}^{3} \frac{\sinh(\mu_2-\mu_i-\eta)}{\sinh(\mu_2-\mu_i)}\ \frac{\sinh(\mu_1+\mu_3+2\eta)}{\sinh(\mu_1+\mu_3+\eta)} \times$$
$$\times \cosh(3\xi^+ + \mu_1 - \mu_2 + \mu_3 + 5\eta)\ \alpha(\mu_2)\widehat{\delta}(\mu_1)\widehat{\delta}(\mu_3)$$
$$+\ \frac{\sinh(2\mu_3)}{\sinh(2\mu_3+\eta)} \prod_{i=1}^{2} \frac{\sinh(\mu_3-\mu_i-\eta)}{\sinh(\mu_3-\mu_i)} \frac{\sinh(\mu_1+\mu_2+2\eta)}{\sinh(\mu_1+\mu_2+\eta)} \times$$
$$\times \cosh(3\xi^+ + \mu_1 + \mu_2 - \mu_3 + 5\eta)\ \alpha(\mu_3)\widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)$$
$$\left. - \prod_{i=1}^{3}\prod_{j>i}^{3} \frac{\sinh(\mu_i+\mu_j+2\eta)}{\sinh(\mu_i+\mu_j+\eta)}\ \cosh(3\xi^+ + \mu_1 + \mu_2 + \mu_3 + 6\eta)\ \widehat{\delta}(\mu_1)\widehat{\delta}(\mu_2)\widehat{\delta}(\mu_3) \right). \tag{C.4}$$

The action of $t(\lambda)$ (5.21) on $\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3)$, obtained by a straightforward calculations using evident generalization of the formulas (B.8), (B.9) and (B.10) and subsequent rearranging of terms, is give by

$$t(\lambda)\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3) = \Lambda_3(\lambda,\{\mu_i\})\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3) + \sum_{i=1}^{3} \frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \times$$
$$\times\ \kappa^+ \sinh(\xi^+ - \mu_i)\ F_3(\mu_i;\{\mu_j\}_{j\neq i})\ \widetilde{\Psi}_3(\lambda,\{\mu_j\}_{j\neq i}), \tag{C.5}$$

where the eigenvalue $\Lambda_3(\lambda, \{\mu_i\})$ is given in (5.55) and the function $F_3(\mu_i; \{\mu_j\}_{j\neq i})$ in (5.56).

With the aim of adding some extra terms, multiplied by some arbitrary coefficients and in this sense generalizing $\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3)$ in such a way that the action of $t(\lambda)$ (C.5) is preserved, we observe the following six identities. The first three identities, which are straightforward generalization of the identities (5.41) and (5.42) relevant in the $M = 2$ case, are given by

$$\Lambda_2(\lambda,\{\mu_j\}_{j\neq i}^{3}) - \Lambda_3(\lambda,\{\mu_j\}_{j=1}^{3}) = \kappa^+\sinh(\xi^+-\lambda)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} \times$$
$$\times\ F_3(\lambda;\{\mu_j\}_{j\neq i}^{3}), \tag{C.6}$$

here $i = 1, 2, 3$ and the other three identities, which are generalization of the identity (5.43) in the $M = 2$ case, are

$$\frac{F_3(\mu_j;\{\mu_k\}_{k\neq j}^3)\ F_2(\mu_i;\{\mu_k\}_{k\neq i,j}^3) - F_3(\mu_i;\{\mu_k\}_{k\neq i}^3)\ F_3(\mu_j;\lambda,\{\mu_k\}_{k\neq i,j}^3)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)} +$$
$$+\ \frac{F_3(\mu_i;\{\mu_k\}_{k\neq i}^3)\ F_2(\mu_j;\{\mu_k\}_{k\neq i,j}^3) - F_3(\mu_j;\{\mu_k\}_{k\neq j}^3)\ F_3(\mu_i;\lambda,\{\mu_k\}_{k\neq i,j}^3)}{\sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j+\eta)} = 0, \tag{C.7}$$

here $i < j$, $i = 1, 2$, and $j = 2, 3$. Therefore the general form of the Bethe vector $\widetilde{\Psi}_3(\mu_1, \mu_2, \mu_3, C_1, C_2, C_3)$ is given by

$$\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3,C_1,C_2,C_3)=\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3)+C_3\,\frac{\psi^+}{\kappa^+}\times\sum_{i=1}^{3}\sinh(\xi^+-\mu_i)F_3(\mu_i;\{\mu_j\}_{j\neq i})\widetilde{\Psi}_2(\{\mu_j\}_{j\neq i},C_1,C_2),\tag{C.8}$$

where C_3 does not depend on $\{\mu_i\}_{i=1}^3$ and $\widetilde{\Psi}_2(\lambda,\mu_i,C_1,C_2)$ is given in (5.44). Due to (C.5) and the above identities (C.6)–(C.7) it is straightforward to check that the off-shell action of transfer matrix $t(\lambda)$ on $\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3,C_1,C_2,C_3)$ is

$$\begin{aligned}&t(\lambda)\widetilde{\Psi}_3(\mu_1,\mu_2,\mu_3,C_1,C_2,C_3)\\&=\Lambda_3(\lambda,\{\mu_i\})\Psi_3(\mu_1,\mu_2,\mu_3,C_1,C_2,C_3)+\sum_{i=1}^{3}\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}\times\\&\times\kappa^+\sinh(\xi^+-\mu_i)\;F_3(\mu_i;\{\mu_j\}_{j\neq i})\;\Psi_3(\lambda,\{\mu_j\}_{j\neq i},C_1,C_2,C_3).\end{aligned}\tag{C.9}$$

By setting $C_1=\dfrac{1-2e^{2\eta}-2e^{4\eta}-e^{6\eta}}{1-e^{6\eta}}$, $C_2=-\tanh(\eta)$ and $C_3=1$ in (C.8) we obtain the corresponding Bethe vector $\Psi_3(\mu_1,\mu_2,\mu_3)$ (5.50), i.e.

$$\Psi_3(\mu_1,\mu_2,\mu_3)=\widetilde{\Psi}_3\Big(\mu_1,\mu_2,\mu_3,C_1=\frac{1-2e^{2\eta}-2e^{4\eta}-e^{6\eta}}{1-e^{6\eta}},C_2=-\tanh(\eta),C_3=1\Big).\tag{C.10}$$

Although it would be natural to continue this approach and present here the Bethe vector $\widetilde{\Psi}_4(\mu_1,\mu_2,\mu_3,\mu_4)$, which in the scaling limit corresponds to the Bethe vector of the XXX chain [12], it turns out that the expressions for the coefficients functions $\widetilde{b}_4^{(i)}(\mu_1,\ldots,\mu_i;\mu_{i+1},\ldots,\mu_4)$ are cumbersome, not admitting any compact form. For this reason we have decided not present them here.

Indeed, the main obstacle in this approach is the lack of the closed form for the coefficients functions $\widetilde{b}_M^{(i)}(\mu_1,\ldots,\mu_i;\mu_{i+1},\ldots,\mu_M)$ of the Bethe vector $\widetilde{\Psi}_M(\mu_1,\ldots,\mu_M)$, whose scaling limit corresponds to the Bethe vector of the XXX chain, for an arbitrary natural number M. All the necessary identities are know, the M identities of the first type

$$\begin{aligned}&\Lambda_{M-1}(\lambda,\{\mu_j\}_{j\neq i}^M)-\Lambda_M(\lambda,\{\mu_j\}_{j=1}^M)\\&=\kappa^+\sinh(\xi^+-\lambda)\frac{\sinh(\eta)\sinh(2(\lambda+\eta))}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}F_M(\lambda;\{\mu_j\}_{j\neq i}^M),\end{aligned}\tag{C.11}$$

here $i=1,\ldots,M$ and the $\frac{M(M-1)}{2}$ identities of the second type

$$\begin{aligned}&\frac{F_M(\mu_j;\{\mu_k\}_{k\neq j}^M)\;F_{M-1}(\mu_i;\{\mu_k\}_{k\neq i,j}^M)-F_M(\mu_i;\{\mu_k\}_{k\neq i}^M)\;F_M(\mu_j;\lambda,\{\mu_k\}_{k\neq i,j}^M)}{\sinh(\lambda-\mu_i)\sinh(\lambda+\mu_i+\eta)}+\\&+\frac{F_M(\mu_i;\{\mu_k\}_{k\neq i}^M)\;F_{M-1}(\mu_j;\{\mu_k\}_{k\neq i,j}^M)-F_M(\mu_j;\{\mu_k\}_{k\neq j}^M)\;F_M(\mu_i;\lambda,\{\mu_k\}_{k\neq i,j}^M)}{\sinh(\lambda-\mu_j)\sinh(\lambda+\mu_j+\eta)}=0,\end{aligned}\tag{C.12}$$

here $i<j$, $i=1,2,\ldots,M-1$, and $j=2,3,\ldots,M$. The most general form of the Bethe vector, for an arbitrary positive integer M, is given as a sum of a particular vector and a linear combination of lower order Bethe vectors that correspond to the same eigenvalue

$$
\begin{aligned}
&\widetilde{\Psi}_M(\{\mu_i\}_{i=1}^M, \{C_j\}_{j=1}^M) \\
&= \widetilde{\Psi}_M(\mu_1, \ldots, \mu_M) + C_M \frac{\psi^+}{\kappa^+} \sum_{i=1}^{M} \sinh(\xi^+ - \mu_i) F_M(\mu_i; \{\mu_j\}_{j\neq i}^M) \times \\
&\times \widetilde{\Psi}_{M-1}(\{\mu_j\}_{j\neq i}^M, \{C_k\}_{k=1}^{M-1}).
\end{aligned}
\tag{C.13}
$$

Unfortunately, this approach cannot be used in general case due to the lack of the closed form for the coefficients functions of the Bethe vector $\widetilde{\Psi}_M(\mu_1, \ldots, \mu_M)$. On the other hand, as it is evident form the formulae (5.60)–(5.62), the recurrence procedure we propose is clearly advantages providing basically the same formulae, up to the multiplicative factors, like in the case of the XXX Heisenberg spin chain [12], for the coefficients functions $b_M^{(i)}(\mu_1, \ldots, \mu_i; \mu_{i+1}, \ldots, \mu_M)$ of the Bethe vector $\Psi_M(\mu_1, \ldots, \mu_M)$, besides $b_M^{(1)}(\mu_1; \mu_2, \ldots, \mu_M)$ which is given explicitly in (5.59).

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$O(6)$ algebraic approach to three bound identical particles in the hyperspherical adiabatic representation

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ABSTRACT

We construct the three-body permutation symmetric $O(6)$ hyperspherical harmonics and use them to solve the non-relativistic three-body Schrödinger equation in three spatial dimensions. We label the states with eigenvalues of the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras, and we present the $K \leq 4$ harmonics and tables of their matrix elements. That leads to closed algebraic form of low-K energy spectra in the adiabatic approximation for factorizable potentials with square-integrable hyper-angular parts. This includes homogeneous pairwise potentials of degree $\alpha \geq -1$. More generally, a simplification is achieved in numerical calculations of non-adiabatic approximations to non-factorizable potentials by using our harmonics.

1. Introduction

The three-body bound-state problem has been addressed by a huge literature, see e.g. Refs. [1–7], in which the hyperspherical harmonics (H.H.) provide one of the most firmly established theoretical tools. All three-body calculations conducted thus far have been numerical, suggesting that perhaps there are no quantum-mechanical three-body bound state problems that can be solved in closed form.

Very little is known about the general structure of the three-body bound-state spectrum, such as the ordering of states, even in the (simplest) normal case of three identical particles interacting with a two-body interaction strong enough to bind two particles, i.e., in the non-Borromean regime. In comparison, the two-body bound state problem is much better understood, see Refs. [8–11], where theorems controlling the ordering of bound states in convex two-body potentials were proven more than 30 years ago. In this paper we make the first significant advance in the problem of three-body bound state ordering after the 1990 paper by Taxil & Richard, Ref. [12].

The basic difficulty lay in the absence of a systematic construction of permutation-symmetric three-body wave functions. Classification of wave functions into distinct classes under permutation symmetry in the three-body system, should be a matter of course, and yet permutation symmetric three-body hyperspherical harmonics in three dimensions were known explicitly only in a few special cases, such as those with total orbital angular momentum $L = 0$, see Refs. [5,13]. Instead, mathematically unjustified bases for hyperspherical harmonics were routinely used in the literature, thus leading to significant computational difficulties. This is reflected already at the level of quantum numbers used for labelling of the harmonics, that often feature two sets, (l_ρ, m_ρ) and (l_λ, m_λ), of $SO(3)$ quantum numbers, related to separate rotations of the two Jacobi vectors, $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$, e.g. Refs. [3,4,6].[1]

The main goal of this paper is to point out the recent progress in the construction and application of permutation symmetric three-body hyperspherical harmonics [14,15]. Rather than going into the technical details of the construction of these harmonics, we here restrict ourselves to simply listing their explicit forms for $K \leq 4$ in Ref. [14] and concentrate on their application to the quantum mechanical three-body problem.

The hyperspherical harmonics we use are permutation-symmetric three-body $O(6)$ HH based on the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras, where $U(1)$ is the "democracy transformation", or "kinematic rotation" group for three particles, $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. This particular chain was recently suggested in Ref. [18], but also by the previous discovery of the dynamical $O(2)$ symmetry of the Y-string potential, Ref. [19]: this $O(2) = U(1)$ symmetry has the permutation group $S_3 \subset O(2)$ as its (discrete) subgroup. The close

[1] Permutation symmetric N-body (with $N \geq 4$) hyperspherical harmonics had only been constructed by means of a numerical recursive procedure that symmetrizes non-permutation-symmetric hyperspherical harmonics, see Refs. [16,17], which, to our knowledge, has not been applied to the three-body problem.

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relation of $U(1)$ kinematic rotations to permutations, on one hand, and the fact that this is the only subgroup of the full $O(6)$ hyperspherical symmetry that commutes with rotations, on the other, imply that the corresponding quantum number must appear in any mathematically justified and permutationally symmetric basis of hyperspherical harmonics.

In two-dimensional space, this requirement strongly suggested an $O(4)$ algebraic approach, Ref. [20] to solve the three-body bound state problem. An independent study of "universal states" using $O(4)$ permutation-symmetric three-body harmonics in two dimensions has appeared recently, Ref. [21]. In three dimensions (3D) the (maximal) hyperspherical symmetry is $O(6)$, however, and thus requires a new set of permutation-symmetric three-body hyperspherical harmonics, that were lacking hitherto, and which we present here.

Then, we apply the new harmonics to the three-identical-particles Schrödinger equation, as written in the so-called hyperspherical adiabatic representation, defined in Refs. [21–24] which simplifies the resulting equations significantly, especially in the case of factorizable (in the hyper-radius and hyper-angles) three-body potentials. Factorizable potentials, see Sect. 3.3.2, include homogeneous potentials, which, in turn, include pairwise sums of two-body power-law potentials, such as the Coulomb one, and the confining "Δ-string", as well as the genuinely three-body "Y-string" potential and Refs. [19,20].

In the adiabatic approximation to the Schrödinger equation with this class of potentials, the energy spectra can be evaluated in closed form, for sufficiently small ($\mathrm{K} \le 7$) values of the grand angular momentum K. Inhomogeneous potentials, and non-adiabatic approximations can only be treated numerically, yet significant simplifications appear there, too, in our method, due to the maximal/optimal sparseness of the adiabatic potential matrix in the permutation-symmetric basis.

In this paper, we shall show: 1) the properties of permutation-symmetric three-body $O(6)$ hyperspherical harmonics; 2) how the Schrödinger equation for three identical particles can be reduced to a set of ordinary differential equations with coefficients determined by $O(6)$ symmetric matrix elements; 3) how, in homogeneous three-body potentials, this set of coupled equations for three identical particles reduces to a set of single decoupled differential equation with coupling strengths determined by $O(6)$ algebra; 4) that our method allows closed-form ("analytical") results in this class of potentials, for sufficiently small values (i.e. for $\mathrm{K} \le 7$) of the grand angular momentum K.

Our work is based on the recent advances in the construction of three-body wave functions with well-defined permutation symmetry, see Sects. 2.1, 2.3, and Ref. [14].

2. Three-body problem in hyper-spherical coordinates

The three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ can be transcribed from the Euclidean relative position (Jacobi) vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2})$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3})$, into hyper-spherical coordinates as $\Psi(R, \Omega_5)$, where $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyper-radius, and five angles Ω_5 that parametrize a hyper-sphere in the six-dimensional Euclidean space. Three (Φ_i; $i = 1, 2, 3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. As we saw above, one linear combination of the two variables $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$, is already taken by the hyper-radius R, so the shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere.

There are two traditional ways of parameterizing this sphere: 1) the standard Delves choice, [3], of hyper-angles (χ, θ), that somewhat obscures the full S_3 permutation symmetry of the problem; 2) the Iwai, Ref. [7], hyper-angles (α, ϕ): $(\sin\alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2-\boldsymbol{\lambda}^2}\right)$, reveal the full S_3 permutation symmetry of the problem: the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions. We shall use the latter choice, as it leads to permutation-symmetric hyperspherical harmonics, as explained in Sects. 2.1, 2.3. Specific hyperspherical harmonics with $\mathrm{K} \le 4$ are displayed in Ref. [14].

2.1. $O(6)$ Symmetry of the hyper-spherical approach

The decomposition of the three-body spatial wave functions in terms of the $O(6)$ "grand angular momentum" $K_{\mu\nu}$ eigenfunctions, or hyperspherical harmonics, is based on the fact that the equal-mass three-body kinetic energy T is $O(6)$ invariant and can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2} \tag{1}$$

where the "grand angular" momentum tensor $K_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots, 6)$

$$\begin{aligned} K_{\mu\nu} &= m\left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu\right) \\ &= \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu\right) \end{aligned} \tag{2}$$

and $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$. $K_{\mu\nu}$ has 15 linearly independent components, that contain, among themselves three components of the "ordinary" orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}}\right)$.

Apart from the hyperangular momentum K, which labels the $O(6)$ irreducible representation, all hyperspherical harmonics must carry additional labels specifying the transformation properties of the harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group. The symmetries of most three-body potentials, including the three-quark confinement ones, are: parity, rotations and permutations (spatial exchange of particles).

Therefore, the three-body hyperspherical harmonics ought to have definite transformation properties w.r.t. to these three symmetries. Parity is the simplest one to implement, as it is directly related to K: $P = (-1)^{\mathrm{K}}$. The rotation symmetry implies that the hyperspherical harmonics must carry quantum numbers L and m associated with the rotational subgroup $SO(3)_{rot}$.

2.2. Permutation-symmetric three-body hyper-spherical harmonics

We introduce the complex coordinates:

$$X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3. \tag{3}$$

Nine of 15 hermitian $SO(6)$ generators $K_{\mu\nu}$ in these new coordinates become

$$iL_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} + X_i^- \frac{\partial}{\partial X_j^-} - X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}, \tag{4}$$

$$2Q_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} - X_i^- \frac{\partial}{\partial X_j^-} + X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}. \tag{5}$$

Of these, L_{ij} is an antisymmetric tensor, with three components, corresponds to the physical angular momentum vector $\mathbf{L}$, and the symmetric tensor Q_{ij} decomposes as (5) + (1) w.r.t. rotations. The trace:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+ \frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^- \frac{\partial}{\partial X_i^-} \tag{6}$$

is the only scalar under rotations, and generates so-called democracy transformations, a special case of which are the cyclic permutations, so its eigenvalue is a natural choice for an additional label of permutation-symmetric hyperspherical harmonics. The remaining five components of the symmetric tensor Q_{ij}, together with three antisymmetric tensors L_{ij} generate the $SU(3)$ Lie algebra, which together with the single scalar Q form an $U(3)$ algebra, Ref. [18].

Therefore, labelling of the $O(6)$ hyper-spherical harmonics with labels K, Q, L and m corresponds to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$. Yet, these four quantum numbers are in general insufficient to uniquely specify an $SO(6)$ hyper-spherical harmonic: it is well known that $SU(3)$ representations in general have nontrivial multiplicity w.r.t. decomposition into $SO(3)$ subgroup representations, and such a multiplicity also appears here. In this context the operator:

$$\mathcal{V}_{LQL} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{7}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ and Q_{ij} is given by Eq. (5)) has often been used in the literature, to label the multiplicity of SU(3) states. This operator commutes both with the angular momentum L_i, and with the "democracy rotation" generator Q:

$$\left[\mathcal{V}_{LQL}, L_i\right] = 0; \quad \left[\mathcal{V}_{LQL}, Q\right] = 0$$

Therefore we demand that the hyperspherical harmonics be eigenstates of this operator:

$$\mathcal{V}_{LQL} \mathcal{Y}^{\mathrm{K}Q\nu}_{L,m} = \nu \mathcal{Y}^{\mathrm{K}Q\nu}_{L,m}.$$

Thus, ν will be the fifth label of the hyper-spherical harmonics, beside the (K, Q, L, m).

2.3. Permutation properties of $O(6)$ hyper-spherical harmonics

We seek hyperspherical harmonics with well-defined values of parity $P = (-1)^{\mathrm{K}}$, rotation-group quantum numbers (L, m), and permutation symmetry, such as the M (mixed), S (symmetric), and A (antisymmetric) ones. In the mixed (M) symmetry representation of the S_3 permutation group being two-dimensional, there are two different H.H. (state vectors) in each mixed permutation symmetry multiplet, usually denoted by M_ρ and M_λ.

Two- and three-particle permutation properties of H.H. $\mathcal{Y}^{\mathrm{K}Q\nu}_{J,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ can be inferred from the transformation properties of the coordinates $X^{\pm}_i$, as follows. Under the two-body permutations $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs of particles (1,2), (2,3) and (3,1), the Jacobi vectors $\boldsymbol{\rho}, \boldsymbol{\lambda}$ transform as:

$$\begin{aligned}
\mathcal{T}_{12}: &\quad \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho}, \\
\mathcal{T}_{23}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \\
\mathcal{T}_{31}: &\quad \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned} \tag{8}$$

This induces the following transformations of complex vectors $X^{\pm}_i$:

$$\begin{aligned}
\mathcal{T}_{12}: &\quad X^{\pm}_i \to X^{\mp}_i, \\
\mathcal{T}_{23}: &\quad X^{\pm}_i \to e^{\pm\frac{2i\pi}{3}} X^{\mp}_i, \\
\mathcal{T}_{31}: &\quad X^{\pm}_i \to e^{\mp\frac{2i\pi}{3}} X^{\mp}_i.
\end{aligned} \tag{9}$$

The quantum numbers K, L and m do not change under permutations of two particles, whereas the values of the "democracy label" Q and multiplicity label ν are inverted under transpositions: $Q \to -Q$, $\nu \to -\nu$.

In addition to the changes of labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyper-spherical harmonic. For multiplicity-free values of K, Q, L and m, the following transformation properties of H.H. hold under (two-particle) particle transpositions:

$$\begin{aligned}
\mathcal{T}_{12}: &\quad \mathcal{Y}^{\mathrm{K}Q\nu}_{L,m} \to (-1)^{\mathrm{K}-J} \mathcal{Y}^{\mathrm{K},-Q,-\nu}_{L,m}, \\
\mathcal{T}_{23}: &\quad \mathcal{Y}^{\mathrm{K}Q\nu}_{L,m} \to (-1)^{\mathrm{K}-L} e^{\frac{2Qi\pi}{3}} \mathcal{Y}^{\mathrm{K},-Q,-\nu}_{L,m}, \\
\mathcal{T}_{31}: &\quad \mathcal{Y}^{\mathrm{K}Q\nu}_{L,m} \to (-1)^{\mathrm{K}-L} e^{-\frac{2Qi\pi}{3}} \mathcal{Y}^{\mathrm{K},-Q,-\nu}_{L,m}.
\end{aligned} \tag{10}$$

In order to determine which representation of the S_3 permutation group any particular H.H. $\mathcal{Y}^{\mathrm{K}Q\nu}_{L,m}$ belongs to, one has to consider various cases, with and without multiplicity. The following linear combinations of the H.H.

$$\mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,\pm} \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m} \pm (-1)^{\mathrm{K}-L} \mathcal{Y}^{\mathrm{K},-|Q|,-\nu}_{L,m} \right) \tag{11}$$

are no longer eigenfunctions of Q operator but are eigenfunctions of the transposition $\mathcal{T}_{12}$ instead:

$$\mathcal{T}_{12}: \mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,\pm} \to \pm \mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,\pm}.$$

They are the appropriate H.H. with well-defined permutation properties:

1. $Q \not\equiv 0 \pmod 3$: the H.H. $\mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,\pm}$ belongs to the mixed representation M, where the $\pm$ sign determines which of the two components it is, M_ρ, M_λ.
2. $Q \equiv 0 \pmod 3$: the H.H. $\mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,+}$ belongs to the symmetric representation S and $\mathcal{Y}^{\mathrm{K}|Q|\nu}_{L,m,-}$ belongs to the antisymmetric representation A.

The above rules define the permutation-group representation for any given H.H.

2.3.1. Labels of K ≤ 4 $O(6)$ hyper-spherical harmonics

As an illustration, in Table 1 we give the values of "$O(6)$ indices" Q, L, m, ν for the lowest K ≤ 4 permutation-symmetric hyperspherical harmonics. The corresponding h.s. harmonics, as well as their hyper-angular matrix elements can be found in Ref. [14].

The K ≥ 4 h.s. harmonics and the corresponding $O(6)$ matrix elements can be readily evaluated using our code written in a commercially available symbolic manipulation language, Ref. [14].

Note that only in the K $= 4$ shell there appear (at most) two multiplets with equal permutation properties and equal (L, m) labels that may mix: a) the $\mathcal{Y}^{4,\pm2,\pm2}_{2,m} \simeq |[70, 2^+]\rangle$ and $\mathcal{Y}^{4,\pm4,\pm3}_{2,m} \simeq |[70', 2^+]\rangle$; and b) the $\mathcal{Y}^{4,\mp2,\pm5}_{4,m} \simeq |[70, 4^+]\rangle$ and $\mathcal{Y}^{4,\mp4,\pm10}_{4,m} \simeq |[70', 4^+]\rangle$. Note, moreover, that both of these have orbital angular momenta $L \ge 2$, as this is required for multiplicity to occur.

3. The three-body Schrödinger equation

First, we briefly explain the adiabatic hyperspherical representation of the three-body Schrödinger equation. Then, we apply the permutation-symmetric h.s. harmonics to this problem, and solve the adiabatic approximation to Schrödinger equation with homogeneous potentials.

3.1. Adiabatic hyperspherical representation

Here we follow the standard derivation of the adiabatic hyperspherical representation, Refs. [23,24]. The three-body Schrödinger equation in 3D for the scaled wave function $\psi = R^{5/2}\Psi$,

Table 1
The labels of distinct $K \le 4$ h.s. harmonics $\mathcal{Y}^{K,Q,\nu}_{L,m}$ (three-body states, with allowed orbital angular momentum value L; only $L = m$ labels are shown). The correspondence between the S_3 permutation group irreps. and $SU(6)_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$ and $M \leftrightarrow 70$.

K	(K, Q, L, m, ν)	[SU(6), L^P]	S_3 irrep.
0	(0, 0, 0, 0, 0)	$[56, 0^+]$	S
1	(1, ±1, 1, 1, ∓1)	$[70, 1^-]$	M
2	(2, ±2, 0, 0, 0)	$[70, 0^+]$	M
2	(2, 0, 2, 2, 0)	$[56, 2^+]$	S
2	(2, ∓2, 2, 2, ±3)	$[70, 2^+]$	M
2	(2, 0, 1, 1, 0)	$[20, 1^+]$	A
3	(3, ∓3, 1, 1, ±1)	$[20, 1^-]$	A
3	(3, ∓3, 1, 1, ±1)	$[56, 1^-]$	S
3	(3, ±1, 1, 1, ±3)	$[70, 1^-]$	M
3	(3, ∓1, 2, 2, ±5)	$[70, 2^-]$	M
3	(3, ∓1, 3, 3, ±2)	$[70, 3^-]$	M
3	(3, ±3, 3, 3, ∓6)	$[56, 3^-]$	S
3	(3, ±3, 3, 3, ∓6)	$[20, 3^-]$	A
4	(4, ±4, 0, 0, 0)	$[70, 0^+]$	M
4	(4, 0, 0, 0, 0)	$[56, 0^+]$	S
4	(4, ±2, 1, 1, ±2)	$[70, 1^+]$	M
4	$(4, 0, 2, 2, \mp\sqrt{105})$	$[56, 2^+]$	S
4	$(4, 0, 2, 2, \mp\sqrt{105})$	$[20, 2^+]$	A
4	(4, ±2, 2, 2, ±2)	$[70, 2^+]$	M
4	(4, ±4, 2, 2, ∓3)	$[70', 2^+]$	M
4	(4, ∓2, 3, 3, ±13)	$[70, 3^+]$	M
4	(4, 0, 3, 3, 0)	$[20, 3^+]$	A
4	(4, 0, 4, 4, 0)	$[56, 4^+]$	S
4	(4, ∓2, 4, 4, ±5)	$[70, 4^+]$	M
4	(4, ∓4, 4, 4, ±10)	$[70', 4^+]$	M

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial R^2} + H_{\rm ad}(R;\Omega_5)\right]\psi_E(R;\Omega_5) = E\psi_E(R;\Omega_5), \tag{12}$$

can be (re)formulated as an algebraic (matrix) eigenvalue problem for the "adiabatic Hamiltonian" $H_{\rm ad}(R;\Omega_5)$

$$H_{\rm ad}(R;\Omega_5) = \frac{K^2_{\mu\nu}(\Omega_5) - 1/4}{2mR^2} + V(R,\alpha,\phi), \tag{13}$$

where $K^2_{\mu\nu}(\Omega_5)$ is the grand angular momentum squared, i.e., the hyper-angular part of the kinetic energy, $V(R,\alpha,\phi)$ is the interparticle interaction potential, E is the total energy and $\Omega_5 \equiv (\gamma,\alpha,\phi)$ denotes the set of three Euler (γ) and two hyper-angles (α,ϕ). The shift of $K^2_{\mu\nu}(\Omega_5)$ by 1/4 in Eq. (13), as compared with Eq. (1), is due to the rescaling $\Psi \to \psi/R^{5/2}$ of the wave function that was implemented in order to eliminate the first derivative in R term from Eq. (12).

In the adiabatic hyperspherical representation, the scaled three-body wave function $\psi_E(R;\Omega_5)$ is expanded in terms of the "channel functions" $\Phi_\mu(R;\Omega_5)$,

$$\psi_E(R;\Omega_5) = \sum_\mu F_{\mu E}(R)\Phi_\mu(R;\Omega_5), \tag{14}$$

Here $F_{\mu E}(R)$ are the hyper-radial wave functions and the channel functions $\Phi_\mu(R;\Omega_5)$ form a complete set of orthonormal functions at each value of R being the eigenfunctions of $H_{\rm ad}$,

$$H_{\rm ad}(R;\Omega_5)\Phi_\mu(R;\Omega_5) = U_\mu(R)\Phi_\mu(R;\Omega_5) \tag{15}$$

The "channel index" μ,[2] represents all quantum numbers necessary to specify each channel and "may serve to identify new sets of approximate quantum numbers", Ref. [23]. The eigenvalue problem Eq. (15) is (still) an infinite-dimensional one (in spite of absence of hyper-radial derivatives): $H_{\rm ad}(R;\Omega_5)$ is a linear Hermitian differential operator in the hyper-angles Ω_5. In general Eq. (15) cannot be solved exactly, so that approximate and/or numerical solutions must be sought.

The eigenvalues $U_\mu(R)$ correspond to the three-body potentials in the channel specified by the set of quantum numbers μ. From the eigenvalues $U_\mu(R)$ one can define the effective three-body potentials for the hyper-radial motion in those channels.

The basic idea of the adiabatic representation/expansion, is that the "channel functions" $\Phi_\mu(R;\Omega_5)$ vary smoothly with R except in localized regions of avoided crossings. The simplest approximation is to ignore the coupling of different channels – this is called the adiabatic approximation.[3] The energies obtained by solving two slightly different adiabatic approximations form an upper- and a lower bound on the true eigenenergy, Refs. [22,24].

3.2. *O*(6) *reduction*

The presence of the hyper-angular momentum squared, $K^2_{\mu\nu}(\Omega_5)$ in $H_{\rm ad}(R;\Omega_5)$, immediately suggests the $O(6)$ hyper-spherical harmonics as the basis vectors in three-body systems. Thus we employ hyperspherical harmonics to solving the channel eigenvalue equation (15), and hence decompose the channel functions $\Phi_\nu(R;\Omega_5)$ as

$$\Phi_\mu(R;\Omega_5) = \sum_{\rm K,[m]} f^{\rm K}_{\rm [m]}(R)\mathcal{Y}^{\rm K}_{\rm [m]}(\Omega_5),$$

where [m] denotes all the labels of hyperspherical harmonics apart from K. After projecting out the $\mathcal{Y}^{\rm K'}_{\rm [m']}$ component, Eq. (15) becomes

$$\left[\frac{{\rm K}({\rm K}+4)-1/4}{2mR^2} - U^{\rm K}_\mu(R)\right] f^{\rm K}_{\rm [m]}(R) + \\ + \sum_{\rm K',[m']} V^{\rm K\,K'}_{\rm [m][m']}(R) f^{\rm K'}_{\rm [m']}(R) = 0, \tag{16}$$

where

$$V^{\rm K\,K'}_{\rm [m][m']}(R) = \left\langle\!\!\left\langle \mathcal{Y}^{\rm K}_{\rm [m]} \middle| V(R,\alpha,\phi) \middle| \mathcal{Y}^{\rm K'}_{\rm [m']}\right\rangle\!\!\right\rangle. \tag{17}$$

The double-bracket matrix element signifies that integrations are carried out only over the angular coordinates Ω_5. Eq. (16) is the (final) result of the $O(6)$ reduction of the eigenvalue equation (15) – it turns into an eigenvalue problem for an infinite-dimensional, hyper-radius dependent matrix. For arbitrary potentials it can only be solved numerically, but there are special cases, such as factorizable potentials and/or dominantly hyper-radially dependent potentials, that can be treated (semi)analytically, see below.

It is immediately clear, however, that the application of the permutation-symmetric hyperspherical harmonics simplifies this

[2] Not to be confused with the reduced mass μ, nor with the index of the grand angular momentum tensor $K_{\mu\nu}$, Eq. (2).

[3] The name is apparently due to the formal similarity to the adiabatic approximation, where solving the time-dependent Schrödinger equation is separated into two steps: first solve the ("quasi-static") eigenvalue problem (without the partial derivative in time) at each moment in time; and then insert these eigenvalue solutions into the full Schrödinger equation including the partial derivative in time and solve it, Ref. [35]. The validity of the conventional (time-dependent) "adiabatic approximation" depends on just how slowly the potential changes with time: the slower, the better. Here, we have made a similar separation, albeit with an eigenvalue problem Eq. (15), which contains no partial derivatives in the hyper-radius R. Its solutions are then "fed" into the full Schrödinger equation (12) that contains the partial derivative(s) in hyper-radius R. The name (hyper-radial) "adiabatic approximation" is a misnomer here, because the eigenvalue problem Eq. (15) always contains the $\frac{K^2_{\mu\nu}(\Omega_5)-1/4}{2mR^2}$ term, with its strong R dependence, no matter how the potential $V(R,\alpha,\phi)$ depends on R. Indeed, the only non-trivial case when this approximation is exact is with the $-1/R^2$ potential, which changes rather rapidly in R!

matrix eigenproblem substantially, as the matrix then turns into a block-diagonal form, with block sub-matrices corresponding to labels from the set [m] that are preserved by the symmetries of the potential, *viz.* rotational numbers L and m, parity P, and permutation symmetry labels A, S, M. Consequently, the channel functions $\Phi_\mu(R;\Omega_5)$ must be labelled by these four good quantum numbers, i.e., the channel index $\mu = (L^P, m, Q)$ must consist of at least these four good quantum numbers.[4]

Apart from the case of hyper-radial potentials, matrix elements Eq. (17) may be nonzero when $\mathrm{K} \neq \mathrm{K}'$ (i.e. the level crossing $\Delta\mathrm{K} \neq 0$ transitions may exist), meaning that K is not a good quantum number for labelling of the channels in general. Nevertheless, the breaking of $O(6)$ symmetry by permutation-symmetric homogeneous potentials is sufficiently small, see Table 2, so as to allow a systematic approximation scheme based on $O(6)$ symmetry.[5] Thus, in the following, K may be treated as an approximate quantum number.

3.3. Potential matrix elements

3.3.1. Hyperspherical expansion of three-body potentials

As the spatial part of any spin-independent three-body interaction potential must be invariant under overall ("ordinary $O(3)$") rotations, it is a scalar, or equivalently, it contains only the zero-angular momentum $L = m = 0$ hyperspherical components. Of course, this holds for both the permutation-symmetric and unsymmetrized hyperspherical harmonics.

So far, we have eschewed specifying the h.s. harmonics used in Sect. 3.2. Next we show the substantial advantages/simplifications in the form of the hyperspherical expansion of the three-body potential, and in the evaluation of hyper-angular matrix elements, gained by using the permutation-symmetric set.

This means choosing the set $[\mathrm{m}] = [L^P, L_z = m, Q, \nu]$ that consists of parity P, the (total orbital) angular momentum L, its projection on the z-axis $L_z = m$, the Abelian hyper-angular momentum quantum number Q conjugated with the Iwai angle ϕ, and the multiplicity label ν that distinguishes between hyperspherical harmonics with remaining four quantum numbers that are identical.

The three-body potential $V(R,\alpha,\phi)$ can be expanded in terms of $O(6)$ hyper-spherical harmonics with zero angular momenta $L = m = 0$ (due to the rotational invariance of the potential),

$$V(R,\alpha,\phi) = \sum_{\mathrm{K},Q}^{\infty} v^{\text{3-body}}_{\mathrm{K},Q}(R)\mathcal{Y}^{\mathrm{K}Q\,\nu}_{00}(\alpha,\phi) \tag{18}$$

In the present case of three identical particles (and therefore also of permutation symmetric potential) the sum runs only over double-even-order ($\mathrm{K} = 0, 4, \ldots$) $O(6)$ hyper-spherical harmonics with zero value of the democracy quantum number $G_3 = Q = 0$, as well as over $\mathrm{K} = 6, 12, 18\ldots$ $O(6)$ hyper-spherical harmonics with democracy quantum number $G_3 \equiv Q \equiv 0 \pmod 6$, always with vanishing angular momentum $L = m = 0$. There is no summation over the multiplicity index in Eq. (18), because no multiplicity arises for harmonics with $L < 2$.

Here $v^{\text{3-body}}_{\mathrm{K}Q}$ are defined as

$$v^{\text{3-body}}_{\mathrm{K},Q}(R) = \int \mathcal{Y}^{\mathrm{K},Q,\nu*}_{0,0}(\Omega_5)\, V_{\text{3-body}}(R,\alpha,\phi)\, d\Omega_5. \tag{19}$$

[4] There may be additional, approximate quantum numbers, however, depending on the specific dynamics.

[5] In exceptional cases, such as the Coulombic, or the harmonic oscillator ones, where the dynamical symmetry of the problem is larger than $O(6)$, K is not the principal quantum number; rather it is some other integer N, and K appears as the label of degenerate states within an N-multiplet, i.e., $\nu = [N, \mathrm{K}, [\mathrm{m}]]$.

Table 2
Non-vanishing expansion coefficients $v_{\mathrm{K}Q}$ of the Y- and Δ-string and the QCD Coulomb potentials in terms of $O(6)$ hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K},0,0}_{0,0}$, for $\mathrm{K} = 0$, 4, 8, respectively, and of the hyper-spherical harmonics $\mathcal{Y}^{6,\pm6,0}_{0,0}$, for $\mathrm{K} \le 11$. The last row gives the percentage of the "Parseval unity" for the potential that is accounted for by its expansion into these five harmonics, calculated as $\sum(v^{\text{3-body}}_{\mathrm{K},Q})^2/(\int(V_{\text{3-body}})^2\,d\Omega_5)$.

(K, Q)	$v^{\mathrm{Y}}_{\mathrm{K}Q}$	$v^{\Delta}_{\mathrm{K}Q}$	$v^{\text{Coulomb}}_{\mathrm{K}Q}$
(0,0)	8.18	16.04	20.04
(4,0)	−0.44	−0.44	2.95
(6,±6)	0	−0.14	1.88
(8,0)	−0.09	−0.06	1.49
$\frac{\sum(v^{\text{3-body}}_{\mathrm{K},Q})^2}{\int(V_{\text{3-body}})^2\,d\Omega_5}$	99%	99%	94%

In the special case of a factorizable three-body potential, see below, the $v^{\text{3-body}}_{\mathrm{K}Q}$ coefficients do not depend on the hyper-radius R; these coefficients are determined by the hyper-angular part $V(\alpha,\phi)$ of the potential.

The numerical values for the first four allowed (non-vanishing) $v^{\text{3-body}}_{\mathrm{K},Q}$ coefficients for $\mathrm{K} \le 11$, in the Y- and Δ-string and Coulomb potential's hyperspherical expansions are tabulated in Table 2, together with a check to which extent Parseval's identity Eq. (20) is fulfilled by the truncation of the sum. All other coefficients must vanish for $\mathrm{K} < 12$. Vanishing of the coefficient $v^{\mathrm{Y}}_{6,\pm6} = 0$ indicates (an additional) dynamical symmetry of the Y-string potential.

Note that Parseval's theorem

$$\sum_{\mathrm{K},Q}^{\infty} |v^{\text{3-body}}_{\mathrm{K},Q}|^2 = \int |V_{\text{3-body}}|^2\, d\Omega_5\,, \tag{20}$$

requires square integrability of the potential at each value of the hyper-radius R, i.e., finiteness of the right-hand side of Eq. (20), regardless of the kind of h.s. harmonics that were used. The requirement of square integrability also holds for any expansion of the potential in terms of a complete set of basis functions, whether $O(6)$ harmonics, or not.

This condition (of square integrability) eliminates all sums of two-body power-law potentials $\sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|^\alpha$, with powers $\alpha < -1$, as well as other singular potentials, such as the Dirac δ-function one. Thus, it poses a strong restriction on the class of three-body potentials that can be treated in this manner, that has not been considered so far: in particular, potentials such as the Lennard-Jones, v.d. Waals and Morse ones will have to be examined individually.

3.3.2. Factorizable potentials

Factorizable potentials satisfy

$$V(R,\alpha,\phi) = V(R)V(\alpha,\phi),$$

and form a non-negligible class that contains homogeneous potentials,[6] such as: 1) the Δ-string, $V_\Delta = \sigma_\Delta \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|$; 2) the Y-string, $V_Y = \sigma_Y \min_{\mathbf{x}_0} \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_0|$; and 3) the QCD Coulomb $V_{\text{Coulomb}} = -\alpha_C \sum_{i>j=1}^{3} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}$.

Then Eq. (17) factors into a common hyper-radial part $V(R)$ and the hyper-angular matrix $C^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}$:

$$\begin{aligned} V^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}(R) &= V(R)\langle \mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)|V(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle \\ &\equiv V(R)\, C^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}. \end{aligned} \tag{21}$$

[6] Of course, this class does not include many of the realistic potentials in molecular and nuclear physics, such as the Lennard-Jones, Morse, v.d. Waals and Yukawa potentials.

For homogeneous potentials $\sim R^{\alpha}$, with exponent $\alpha = -2$, the eigenvalue equation (16) becomes effectively independent of the hyper-radius R, which leads to conformal symmetry, Refs. [25,26], together with a substantial simplification of the problem.

3.3.3. Selection rules

Plugging the potential decomposition (18) into Eq. (17), or Eq. (21) requires the knowledge of $O(6)$ hyper-angular matrix elements of the form

$$C^{\mathrm{K}''\ \mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']} = \sum_{\mathrm{K},Q}^{\infty} v^{\text{3-body}}_{\mathrm{K},Q} \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Q\,\nu}_{00}(\alpha,\phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$$

The $O(6)$ hyper-angular matrix elements

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Q\,\nu}_{00}(\alpha,\phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$$

can be evaluated using the permutation-symmetric hyperspherical harmonics obtained in Sect. 2.2, see also Ref. [14].

Generally, the $O(6)$ matrix elements obey the following selection rules that reduce the number of non-zero values: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $m' = m''$. Moreover, Q is an Abelian (i.e. additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$. All of this reduces the sum in $C^{\mathrm{K}''\ \mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']}$ to a finite one, that depends on a finite number of coefficients $v^{\text{3-body}}_{\mathrm{K},Q}$; for small values of K, this number is also small, see Sect. 4.1.

The hyper-angular matrix element

$$\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5) | \mathcal{Y}^{\mathrm{K}Q\,\nu}_{00}(\alpha,\phi) | \mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$$

is (merely) a product of two $O(6)$ group Clebsch–Gordan coefficients that can be calculated using Ref. [14], and the physics is contained in the three-body potential expansion coefficients $v^{\text{3-body}}_{\mathrm{K},Q}$.

3.3.4. Advantages of the permutation-symmetric basis

Of course, Eq. (16) must also hold with any other complete set of three-body hyperspherical harmonics, including the permutation non-symmetric ones, such as those based on the Delves choice of hyper-angles, see Ref. [34]. Note, however, that the Delves-type h.s. harmonics do not have a well-defined set of labels ("quantum numbers"): besides the three standard/obvious quantum numbers K, L, m there is an ambiguity as to what one ought to use for the rest, see Sects. 2.3.2 and 2.3.3 in Ref. [34] and Sect. 5. in Ref. [18].

The permutation-symmetric basis is the optimal one in so far as it maximally observes the symmetries of the permutation-symmetric three-body problem and leads to a minimal number of h.s. components in the decomposition of the potential and of non-vanishing off-diagonal matrix elements. Using the Table 2 as an example, we note the following: there are overall $(\mathrm{K}+3)!(\mathrm{K}+2)/(12\mathrm{K}!) = 2366$ hyperspherical harmonics in the $\mathrm{K} \leq 11$ shells, and this number is independent of the choice of h.s. basis. However, it is a unique feature of the permutation-symmetric basis that the decomposition of *any* permutation symmetric potential has no more than four distinct nonvanishing coefficients out of 2366 possible ones! This "sparseness" is even more marked when one considers (of the order of) 10^6 off-diagonal $\mathrm{K} \leq 11$ matrix elements, all of which depend only on these four coefficients, see Sect. 4.1.

The sparseness of this matrix suggests that our three-body problem might be diagonalizable, at least in some circumstances – see Sect. 4. The manifest permutation symmetry of our hyperspherical harmonics, together with the complete set of commuting operators, simplifies all subsequent calculations. This simplification becomes increasingly pronounced as the value of K increases, see Ref. [15] where we applied these HH to the problem

Table 3
The values of non-vanishing off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|2\Re e\mathcal{Y}^{6,\pm 6,0}_{0,0}|[SU(6)_i, L_i^P]\rangle_{\mathrm{ang}}$, for various $\mathrm{K}=4$ states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}^{6,\pm 6,0}_{0,0}\rangle_{\mathrm{ang}}$
4	$[70, 2^+]$	$[70', 2^+]$	$\frac{6}{7}\sqrt{\frac{6}{5}}$
4	$[70, 4^+]$	$[70', 4^+]$	$\frac{8}{21}$

Table 4
The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|\mathcal{Y}^{4,0,0}_{00}|[SU(6)_i, L_i^P]\rangle_{\mathrm{ang}}$, for various $\mathrm{K}=0, 2, 4$ states (for all allowed orbital waves L).

$(\mathrm{K}_f, [SU(6)_f, L_f^P])$	$(\mathrm{K}_i, [SU(6)_i, L_i^P])$	$\pi\sqrt{\pi}\langle\mathcal{Y}^{4,0,0}_{00}\rangle_{\mathrm{ang}}$
$(0, [56, 0^+])$	$(4, [56, 0^+])$	1
$(2, [70, 2^+])$	$(4, [70, 2^+])$	$\frac{4}{5}\sqrt{\frac{6}{7}}$
$(2, [56, 0^+])$	$(4, [56, 0^+])$	$\frac{4}{5}\sqrt{\frac{2}{7}}$

Table 5
The values of non-vanishing off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|2\Re e\mathcal{Y}^{6,\pm 6,0}_{0,0}|[SU(6)_i, L_i^P]\rangle_{\mathrm{ang}}$, for various $\mathrm{K}=4$ states (for all allowed orbital waves L).

$(\mathrm{K}_f, [SU(6)_f, L_f^P])$	$(\mathrm{K}_i, [SU(6)_i, L_i^P])$	$\pi\sqrt{\pi}\langle 2\Re e\mathcal{Y}^{6,\pm 6,0}_{0,0}\rangle_{\mathrm{ang}}$
$(2, [70, 2^+])$	$(4, [70', 2^+])$	$2\sqrt{\frac{3}{35}}$

of three-quark bound states. In that display of utility of our approach, we explicitly calculated the orderings of $\mathrm{K} \leq 4$ states and showed that, thanks to the symmetry properties of our harmonics, these levels' energies can be accurately parameterized by only four potential-dependent constants. Furthermore, as a consequence of the mentioned matrix sparseness, the expressions for the energies in Ref. [15] are given in an analytic form.

4. Results

In general, the eigenvalue problem Eq. (16) has to be solved numerically, but its solution is significantly simplified by the use of permutation-symmetric h.s. harmonics basis, as the hyper-angular matrix elements are subject to the selection rules shown in Sect. 3.3.3.

The couplings of lower-K′ states to the higher-K″ ones are proportional to the higher-K valued coefficients $v^{\text{3-body}}_{\mathrm{K},Q}$, due to the $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ selection rule, which coefficients, in turn, are smaller than the lower-K ones, see Table 2. This reduction becomes increasingly pronounced as the values of $\mathrm{K}', \mathrm{K}''$ increase, see Ref. [15]. That fact leads, in the case of homogeneous potentials, to a clear ordering of off-diagonal matrix elements and allows controllable approximations to the solution, that may even be convergent in some special cases, e.g. with conformal invariance.

4.1. Off-diagonal matrix elements

The non-vanishing single-shell ($\Delta\mathrm{K} = 0$) off-diagonal matrix elements, for $\mathrm{K} = 0, 1, 2, 3, 4$ states, are shown in Table 3.

The non-vanishing two-shell off-diagonal (nonadiabatic) matrix elements, for various $\mathrm{K} = 0, 2, 4$ states, are shown in Tables 4, 5, and for $\mathrm{K} = 1, 3$ states, in Table 6.

4.2. Diagonalization

The sparseness of the hyper-angular coupling coefficients matrix $C^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}$ in the permutation-symmetric basis displayed in Sect. 4.1 suggests that we attempt an analytic diagonalization.

Table 6
The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\pi\sqrt{\pi}\langle[SU(6)_f, L_f^P]|\,\mathcal{Y}_{00}^{4,0,0}\,|[SU(6)_i, L_i^P]\rangle_{\text{ang}}$, for various $\mathrm{K}=1$, 3, 5 states (for all allowed orbital waves L).

$(\mathrm{K}_f, [SU(6)_f, L_f^P])$	$(\mathrm{K}_i, [SU(6)_i, L_i^P])$	$\pi\sqrt{\pi}\langle\mathcal{Y}_{00}^{4,0,0}\rangle_{\text{ang}}$
$(1, [70, 1^-])$	$(5, [70', 1^-])$	$\sqrt{\frac{2}{3}}$
$(1, [70, 1^-])$	$(3, [70, 1^-])$	$\frac{1}{\sqrt{3}}$

4.2.1. Adiabatic mixing ($\Delta\mathrm{K}=0$)

Inspection of the Table 2 reveals that all of the potentials considered there have coefficients $v_{00}^{\text{3-body}}$ that are one order of magnitude larger than the rest $v_{\mathrm{K}>0,Q}^{\text{3-body}}$. This fact justifies taking only the term proportional to $v_{00}^{\text{3-body}}$ in the expansion Eqs. (16), (18) as the zeroth order approximation. To this zeroth order, all the solutions with the same principal number K are degenerate, with $U_\mu(R) = U_{\mathrm{K}}(R) = \frac{\mathrm{K}(\mathrm{K}+4)-1/4}{2mR^2} + v_{00}^{\text{3-body}}(R)$. The first order corrections lift this degeneracy, i.e., that would amount to including all off-diagonal elements within the same K-shell (i.e. those with $\mathrm{K}=\mathrm{K}'$) into Eq. (17).

In a such case, the eigenvalue problem Eq. (16) splits into separate equations for each value of K. For a given K the term in the first line in Eq. (16) is proportional to a unit matrix, so it may be removed from the diagonalization. Therefore, the potential matrix $V_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}'}(R)$ is the only one that needs to be diagonalized; it can be brought into the diagonal form

$$V_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}'}(R) = \delta_{\mathrm{K},\mathrm{K}'}\delta_{[\mathrm{m}],[\mathrm{m}']}V_{[\mathrm{m}]}^{\mathrm{K}}(R),$$

due to its Hermiticity, yielding the eigenvalues of the equation Eq. (16) in the form

$$U_{[\mathrm{m}]}^{\mathrm{K}}(R) = \frac{\mathrm{K}(\mathrm{K}+4)-1/4}{2mR^2} + V_{[\mathrm{m}]}^{\mathrm{K}}(R). \tag{22}$$

The matters simplify further in the case of factorizable potentials, i.e., when $V_{[\mathrm{m}]}^{\mathrm{K}}(R) = V(R)C_{[\mathrm{m}]}^{\mathrm{K}}$. In such a case, the coefficients $f_{[\mathrm{m}]}^{\mathrm{K}}(R)$ form (mutually orthogonal) eigenvectors that effectively do not depend on the hyper-radius R, as we can choose the normalization so that $f_{[\mathrm{m}]}^{\mathrm{K}}(R) = f_{[\mathrm{m}]}^{\mathrm{K}}(0)$. This is so because the matrix $C_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}'}$ that is being diagonalized does not depend on R.

This implies that the non-adiabatic coupling terms, Eq. (3.10) in Ref. [23], or Eqs. (16), (17) in Ref. [21], vanish: $P_{[\mathrm{m}],[\mathrm{m}']}^{\mathrm{K},\mathrm{K}'}(R) = 0$ and $Q_{[\mathrm{m}],[\mathrm{m}']}^{\mathrm{K},\mathrm{K}'}(R) = 0$. In this sense, the single K-shell mixing approximation corresponds to the adiabatic one for factorizable potentials. That, in turn, leads to the explicit solution $V_{\text{eff }[\mathrm{m}]}^{\mathrm{K}}(R) = U_{[\mathrm{m}]}^{\mathrm{K}}(R)$, to the hyper-radial effective potential.

4.2.2. Non-adiabatic mixing ($\Delta\mathrm{K}\neq 0$)

Introducing higher-order corrections to Eq. (17) corresponds to taking into account the inter-shell ($\mathrm{K}\neq\mathrm{K}'$) mixings. It is then no longer possible (in general) to choose $f_{[\mathrm{m}]}^{\mathrm{K}}(R)$ as being independent of hyper-radius R. Note that in the $\mathrm{K},\mathrm{K}'\leq 4$ shells there is at most two-state mixing, see Tables 4, 5, 6. In such simple cases one can solve for the mixing angle $\Theta(R)$ in closed form.

For (smooth, monotonic) homogeneous potentials $V(R)\sim R^\alpha$, the two-state mixing angle $\Theta(R)$ changes monotonically from $\Theta(0)=0$ to its asymptotic value $\Theta_{\text{as.}}$, as $R\to\infty$.[7] The "hyper-radial functions" $f_{[\mathrm{m}]}^{\mathrm{K}}(R)\sim\cos\Theta(R)$ lead to non-vanishing non-adiabatic coupling coefficients, Eq. (3.10) in Ref. [23], or Eqs. (16), (17) in Ref. [21], $P_{[\mathrm{m}],[\mathrm{m}']}^{\mathrm{K},\mathrm{K}'}(R)\neq 0$ and $Q_{[\mathrm{m}],[\mathrm{m}']}^{\mathrm{K},\mathrm{K}'}(R)\neq 0$ because $\frac{d\Phi_{[\mathrm{m}]}^{\mathrm{K}}}{dR}\sim\left(\frac{df_{[\mathrm{m}]}^{\mathrm{K}}(R)}{dR}\right)\neq 0$. This leads to a non-vanishing non-adiabatic correction $Q_{[\mathrm{m}],[\mathrm{m}]}^{\mathrm{K},\mathrm{K}}(R)\neq 0$ to the hyper-radial effective potential.

In general, the problem has to be solved numerically, but solving Eq. (16) is significantly simplified in the permutation-symmetric h.s. harmonics basis, as the hyper-angular matrix elements are subject to the (now familiar) selection rules in Sect. 3.3.3. Couplings to higher-K,K' shells are proportional to higher values of expansion coefficients $v_{\mathrm{K},Q}^{\text{3-body}}$, which, in turn, are smaller than the lower ones; this allows a controlled/convergent approximation.

4.3. Homogeneous permutation-symmetric potentials in adiabatic approximation

The adiabatic approximation is obtained by setting the non-adiabatic coefficients equal to zero: $P_{[\mathrm{m}],[\mathrm{m}']}^{\mathrm{K},\mathrm{K}'}(R)=0$. One can argue that the adiabatic approximation is a reasonable one for confining ($\alpha>0$) three-body potentials, at least for low values of $\mathrm{K}\leq 4$. In such cases hyper-radial equations decouple, leading to solutions that depend on the (diagonalized values of) quantum numbers $[m]$ and thus lead to (slightly) different eigen-energies within the same K shell.

The ordering of states in each shell depends only on four coefficients $(v_{00}, v_{40}, v_{6\pm6}, v_{80})$, for $\mathrm{K}\leq 5$, and the largest number of states that mix is three, so the eigenvalue equations are at most cubic algebraic ones, with well-known closed form solutions.

Homogeneous confining three-body potentials, such as the Δ-string and the Y-string, have coefficients $v_{00}^{\text{3-body}}$ that are one order of magnitude larger than the rest $v_{\mathrm{K}>0,Q}^{\text{3-body}}$, see Table I in Ref. [15]. Consequently, the K expansion ought to converge quickly. In Ref. [15] we used the above-described methods to calculate the eigen-energies of various SU(6)/S_3 multiplets in the $\mathrm{K}\leq 4$ shells of the Y-, Δ-string potential spectra, with the following results.

The $\mathrm{K}=2$ shell depends only on two coefficients (v_{00}, v_{40}), so the level splittings depend only on one free parameter (the ratio v_{40}/v_{00}) and the $O(6)$ matrix elements/Clebsch–Gordan coefficients, thus confirming the "universal splitting" result of Refs. [28, 29].

In the $\mathrm{K}=3$ shell, however, there are three coefficients $(v_{00}, v_{40}, v_{6\pm6})$, leading to two free parameters, the independent ratios v_{40}/v_{00} and $v_{6\pm6}/v_{00}$, which means that the energy splittings depend on the potential, i.e., that they are not "universal".

A clear example of this difference appears between the eigen-energies in the Y-string and the Δ-string potential, as a consequence of $|v_{6\pm6}^{\mathrm{Y}}|\ll|v_{6\pm6}^{\Delta}|$. That is also the first direct consequence of the dynamical $O(2)$ symmetry of the "Y-string" potential. Numerical values of eigen-energies can be obtained from the results in Ref. [15] by using Eqs. (22), (24)–(26) in Sect. 3.3 and Eqs. (C1)–(C8) in App. C; as well as the numerical values shown in Tables 4, 5, 6 in Sect. 4.2 and Table 11 in App. C of Ref. [30]. The $\mathrm{K}=4$ shell is too complicated to be discussed here; for these results see Ref. [15] – the general conclusions agree with those from $\mathrm{K}=3$ shell.

The ordering of bound states has its most immediate application in the physics of three confined quarks, where the question was originally raised, Refs. [12,27–29], but, as time passed it has become more of a question in mathematical physics, see Refs. [8,9,11]. The above discussion ought to have made it clear

[7] For (in-homogeneous, smooth) non-monotonic potentials with a hard inner-core, $\alpha<-2$ and/or weak asymptotic tail falling off faster than $1/R^2$, the mixing angle $\Theta(R)$ behaves differently, and its two limits, $R\to\infty$, and $R\to 0$, may be "reversed". For $V(R)\simeq 1/R^2$, $\alpha=-2$, the mixing angle Θ does not depend on R, as the complete R dependence can be factored out of the eigenvalue equation (16). That is a consequence of the scale invariance of $V(R)\simeq 1/R^2$ potentials in non-relativistic dynamics, see Refs. [25,26].

that three-body analogons of two-body state-ordering theorems, Refs. [8,9,11], do not hold for realistic three-body systems at $K > 2$.

5. Summary and conclusions

In summary, we have constructed the three-body permutation-symmetric hyperspherical harmonics and then used them in a permutation symmetric version of the hyperspherical adiabatic representation to reduce the non-relativistic three-body problem to a set of coupled ordinary differential equation for the hyper-radial wave functions with effective potentials that are derived as functions of the three-body potential's hyperspherical harmonics expansion coefficients.

In the adiabatic approximation this set of equations decouples to one ordinary differential equation, that can be solved in the same manner as the one-body radial Schrödinger equation.

This transcription of the three-body problem into hyperspherical variables is possible only for three-body potentials whose hyper-angular dependence is square integrable, however. One such subset are the factorizable potentials, and more specifically homogeneous potentials, such as the pairwise sums of single-power-law terms, with the power larger than -1.

Then, we applied these methods to three homogeneous potentials that satisfy the square integrability condition. The ordering of states ("pattern") in the spectrum depends on the $O(6)$ symmetry-breaking, which in turn is determined by the hyperspherical expansion coefficients of the three-body potential. These coefficients depend on the dynamical "remnant" symmetries of the potential. Thus, for example the so-called Y-string potential has an $O(2)$ dynamical symmetry, Ref. [19], that is absent in potentials that are pairwise sums of single-power-law terms (for powers different than the second one). We used this $O(2)$ dynamical symmetry, of which the permutation group $S_3 \subset O(2)$ is a subgroup, to guide our construction of the permutation symmetric harmonics. In three dimensions (3D) the "hyper-spherical symmetry" is $O(6)$, and the residual dynamical symmetry of the potential is $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$, where $SO(3)_{rot}$ is the rotational symmetry associated with the (total orbital) angular momentum L.

Our $O(6)$ permutation-symmetric three-body hyperspherical harmonics appear to be the first of their kind in the literature. Symmetrized three-body hyper-spherical harmonics have been pursued before, albeit without emphasis on the the "kinematic rotation" $O(2)$ symmetry label. To our knowledge, aside from the special case $L = 0$ results of Simonov, Ref. [5] and $L = 1$ of Barnea and Mandelzweig, Ref. [13], several other attempts, Refs. [6,31–33], some based on so-called "tree pruning" techniques, exist in the literature, beside the recursively symmetrized N-body hyperspherical harmonics of Barnea and Novoselsky, Refs. [16,17]. The latter are based on the $O(3) \otimes S_N \subset O(3N - 3)$ chain of algebras, which does not explicitly include the "kinematic rotation"/"democracy" $O(2)$ symmetry.

The method of permutation-symmetric hyperspherical harmonics is not specific to any particular non-relativistic quantum three-body problem, i.e., it should find application in realistic 3D three-body problems in atomic, molecular and Efimov physics, three-quark problem in hadronic physics, as well as in positronium ion $P_s^-(= e^-e^+e^-)$ physics.

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SO(4) algebraic approach to the three-body bound state problem in two dimensions

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SO(4) algebraic approach to the three-body bound state problem in two dimensions

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We use the permutation symmetric hyperspherical three-body variables to cast the non-relativistic three-body Schrödinger equation in two dimensions into a set of (possibly decoupled) differential equations that define an eigenvalue problem for the hyper-radial wave function depending on an SO(4) hyper-angular matrix element. We express this hyper-angular matrix element in terms of SO(3) group Clebsch-Gordan coefficients and use the latter's properties to derive selection rules for potentials with different dynamical/permutation symmetries. Three-body potentials acting on three identical particles may have different dynamical symmetries, in order of increasing symmetry, as follows: (1) $S_3 \otimes O_L(2)$, the permutation times rotational symmetry, that holds in sums of pairwise potentials, (2) $O(2) \otimes O_L(2)$, the so-called "kinematic rotations" or "democracy symmetry" times rotational symmetry, that holds in area-dependent potentials, and (3) O(4) dynamical hyper-angular symmetry, that holds in hyper-radial three-body potentials. We show how the different residual dynamical symmetries of the non-relativistic three-body Hamiltonian lead to different degeneracies of certain states within O(4) multiplets. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4891399]

I. INTRODUCTION

The quantum three-body problem is an old one with a huge literature – the hyperspherical variables, together with the corresponding hyperspherical harmonics, form one of the best known sets of tools in the theorist's arsenal, Refs. 1–3. Classification/separation of wave functions into distinct classes under permutation symmetry is a fundamental property of (non-relativistic) quantum mechanics with non-trivial consequences in the three-body system. Permutation symmetric three-body hyperspherical harmonics in three dimensions, however, are known only in special cases such as the (small, definite values of the) total angular momentum $L = 0$, 1 ones, cf. Refs. 3 and 4. All other values of L have to be treated separately, usually by means of non-permutation symmetric hyperspherical harmonics. In that way, one loses the manifest permutation symmetry, however, as well as a certain dynamical O(2) symmetry, when the three-body potential is invariant under the so called "kinematic rotation," Ref. 2, or equivalently the "democracy," Refs. 5 and 6, transformations. This symmetry was viewed as mathematical esoterics, see Ref. 5, until recently it was shown, Refs. 7 and 19, to be a dynamical symmetry of area-dependent potentials, which class includes the so-called Y-string potential in QCD. Consequently follows the increased interest in its properties.

In two spatial dimensions, the problem of constructing permutation symmetric hyperspherical harmonics was solved in Ref. 8, however, almost as an afterthought of certain rather abstract internal geometric considerations in Refs. 9 and 10, and certain mathematical aspects of this problem were reconsidered more recently in Ref. 11. Although the need for such a theoretical tool (e.g., in anyon physics, cf. Refs. 12–17) was acute at the time of writing (mid-1990s) of Ref. 8, it never received the attention it deserves. Knowledge of three-body permutation symmetric hyperspherical harmonics in two dimensions (2D) allows one to calculate the discrete part of the energy spectrum of the three-body problem, very much as the quantum mechanical two-body problem can be solved using

SO(3) spherical harmonics in three dimensions. That line of research was not pursued in Ref. 8, nor elsewhere, to our knowledge.[41]

In the present paper, we extend the line of investigation started in Refs. 9 and 10 and continued in Refs. 8 and 11 to show how the Schrödinger equation for three-body bound states in two spatial dimensions can be reduced to an eigenvalue problem for the hyper-radial wave function, where the whole hyper-angular dependence has been reduced to an SO(4) hyperspherical harmonics matrix element that boils down to a product of two SO(3) Clebsch-Gordan coefficients. This is the basic contribution of the present paper. These results are not specific to any one particular three-body problem, i.e., they could find application in many realistic 2D three-body problems, such as the three-anyon one, Refs. 12–17, and/or other condensed matter physics problems in 2D, Ref. 18. The results of this paper have been used to study the 2D version of three-body confinement with the Δ and Y-string.[19,22]

In this way, the three-body problem in two dimensions has been effectively reduced to an SO(4) group theoretical problem (or "algebraized" in vulgate), and one eigenvalue equation for the hyper-radial wave function, a goal that was hypothesized about in three dimensions in Ref. 23 and elsewhere. In this algebraic language, one is looking for the "chain" of algebras $so(2) \oplus so_L(2) \subset so(3) \oplus so(3) \subset so(4)$ (where $so_L(2)$ is the total angular momentum part and $so(2)$ being the so-called "democracy" transformation, where the permutation group S_3 is a (discrete) subgroup of the so-called "kinematic rotations," Ref. 2, or equivalently the "democracy" transformation (continuous) group O(2), Refs. 5 and 6).

On the formal side, these SO(4) hyperspherical harmonics are directly related to the monopole harmonics of Ref. 24, as shown in Ref. 8, and to the spin-weighted spherical harmonics of Ref. 25. Our result, Eq. (31), for the hyper-angular matrix elements of SO(4) hyperspherical harmonics appears to be the first of its kind in the literature. It can be viewed as a continuation of the earlier results for the matrix elements of SO(4) hyperspherical harmonics in Refs. 26 and 27.

As an example of the utility of our results we apply our method to the three-quark confinement problem in 2D and show how we evaluated the (2D) eigen-energy splittings in the K = 2, 3 bands of the spectra of the Δ and Y-string potentials in QCD, that were presented in Refs. 19 and 22.

After defining preliminaries in Sec. II, we define our $SO(4)$ algebraic methods for solving the spectrum of the model in Sec. III. We apply our results to two classes of permutation-symmetric three-body problems in Sec. IV: (a) the three-body sum of pairwise terms, and (b) area-dependent potentials that are invariant under "democracy" O(2) transformations. Section V contains a summary and a discussion of the results.

II. PRELIMINARIES

A. Three-body variables

The $\boldsymbol{\rho}, \boldsymbol{\lambda}$ are the two Jacobi three-vectors, defined by

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}), \tag{1}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3}). \tag{2}$$

In the relations above, we assume that all three masses are equal. In two spatial dimensions (2D), the full symmetry of the three-body kinetic energy is $O(4)$ and $SO(2)$ is its rotation symmetry. The "larger" symmetry of the non-relativistic kinetic energy is the basis of the hyper-spherical variable approach to the three-body problem.

A crucial ingredient to the solution to the three-body bound state problem are the hyper-spherical coordinates/hyper-angles.[1–3] Here, instead of two Jacobi three-vectors $\boldsymbol{\rho}, \boldsymbol{\lambda}$, defined in Eqs. (1) and (2), the hyper-spherical formalism introduces the hyper-radius R,

$$R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}, \tag{3}$$

and two hyper-angles that are defined by way of three independent scalar three-body variables, e.g., $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. Then, one may use the hyper-space unit-vector $\hat{\mathbf{n}}$

$$\hat{\boldsymbol{n}} = (\boldsymbol{n}'_1, \boldsymbol{n}'_2, \boldsymbol{n}'_3) = \left(\frac{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}{R^2}, \frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{R^2}, \frac{2(\boldsymbol{\lambda} \times \boldsymbol{\rho})_3}{R^2}\right) \tag{4}$$

(apparently first introduced by Hopf, Ref. 29) to define a sphere with unit radius. The points on the equatorial unit circle correspond to collinear configurations ("triangles" with zero area). Two angles parametrize this sphere – they can be chosen at will.

The area of the triangle $\frac{\sqrt{3}}{2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| \simeq |(\mathbf{x_1} - \mathbf{x_2}) \times (\mathbf{x_1} - \mathbf{x_3})|$ and the hyper-radius R are related to the Smith-Iwai variables (α, ϕ),[2,9,10] as follows:

$$(\sin\alpha)^2 = \left(\boldsymbol{n}'^2_1 + \boldsymbol{n}'^2_2\right) = 1 - \left(\frac{2\boldsymbol{\rho} \times \boldsymbol{\lambda}}{R^2}\right)^2, \tag{5}$$

$$\phi = \tan^{-1}\left(\frac{n'_2}{n'_1}\right) = \tan^{-1}\left(\frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right). \tag{6}$$

The standard Delves-Simonov choice of hyper-angles is $(\chi_D = 2\chi, \theta)$,[1,3]

$$(\sin\chi_D)^2 = (\sin 2\chi)^2 = \left(\boldsymbol{n}'^2_2 + \boldsymbol{n}'^2_3\right) = 1 - \left(\frac{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}{R^2}\right)^2, \tag{7}$$

$$\theta = \tan^{-1}\left(\frac{n'_2}{n'_3}\right) = \tan^{-1}\left(\frac{|\boldsymbol{\rho} \times \boldsymbol{\lambda}|}{\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}\right). \tag{8}$$

One must choose the most appropriate parametrization according to the symmetry of the potential, Ref. 7.

Only one set of three-body variables, viz., (R, α, ϕ), with the hyper-angle $\phi = \arctan\left(\frac{2\rho\cdot\lambda}{\lambda^2-\rho^2}\right)$ makes the permutation symmetry manifest, see Ref. 7. That fact makes (α, ϕ) appropriate for permutation-symmetric three-body potentials. The (other) hyper-angle α describes the "scale-invariant area" of the triangle $\cos\alpha = 2R^{-2}(\boldsymbol{\rho} \times \boldsymbol{\lambda})_3$. That makes this set also appropriate for area-dependent potentials.

B. Three-body potentials

Three-body potentials acting on three identical particles can be divided into three interesting classes according to their permutation and/or dynamical symmetries (in order of increasing symmetry): (1) $S_3 \otimes O_L(2)$, the permutation times rotational symmetry, that holds in sums of pairwise potentials, (2) $O(2) \otimes O_L(2) \subset SO(4)$, the so-called "kinematic rotations" or "democracy symmetry" times rotational symmetry, that holds in area-dependent potentials, and (3) the full SO(4) dynamical hyper-angular symmetry, that holds for hyper-radial three-body potentials which do not depend on the shape of the triangle subtended by the three particles, but only on their "mean size," the hyper-radius R.

The third class has the highest symmetry, the harmonic oscillator being one example, but it is also the least realistic one: there are simply no known hyper-radial potentials in nature. Due to its highest symmetry, its energy spectra have the highest levels of degeneracy, and can be used as the starting point for the two cases (1) and (2) with lesser symmetries. For this reason, we shall spend the least amount of space on this (third) class.

The second class corresponds to a certain dynamical O(2) symmetry, when the three-body potential is invariant under the so called "kinematic rotations," Ref. 2, or, equivalently, the "democracy" transformations, Refs. 5 and 6. This (continuous) "kinematic rotations," or "democracy" symmetry is a generalization of the (discrete) permutation symmetry of three bodies. It used to be viewed as something of mathematical esoterics, see Ref. 5, until recently Refs. 7 and 19, showed it to be the dynamical symmetry of the Y-string potential among three quarks in QCD, in particular, and of all three-body potentials that depend only on the area of the triangle subtended by the three particles, in general. Analogous "kinematic rotations," or "democracy" symmetry, for four particles in three spatial dimensions, is the non-Abelian group $SO(3)$, Ref. 20. It is not yet clear what geometrical or physical quantity is kept invariant under the corresponding democracy transformations in the four-body case, Ref. 20, let alone five- or more bodies, Ref. 21.[42]

The first class corresponds to potentials symmetric under the full S_3 permutation group. Potentials with only a two-body permutation S_2 subgroup, or a trivial (S_1) permutation symmetry will not be dealt with here.

(1) In the first class, we consider the three-body sum of pairwise distances to power $\boldsymbol{\alpha}$ (here, we use the boldface greek letter $\boldsymbol{\alpha}$ to distinguish it from the hyperangle α, introduced in Eq. (5) above),

$$V_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}} \frac{1}{2} \sum_{i \neq j = 1}^{3} |\mathbf{x}_i - \mathbf{x}_j|^{\boldsymbol{\alpha}}. \tag{9}$$

Perhaps, the best known example of such a potential (albeit with different signs multiplying each term) is the ($\boldsymbol{\alpha} = -1$) Coulomb potential in atomic and molecular physics. More recently, potentials with different powers $\boldsymbol{\alpha} \neq 1$ have been used in few-body problems in 2D, in condensed matter physics, Ref. 18.

(2) The second class of potentials are the area-dependent ones that have the additional "democracy" dynamical O(2) symmetry, Refs. 6 and 7. Perhaps, the best known example of such a potential is the Y-string one $V_{\mathrm{Y-str.}}$, defined as

$$V_Y = \sigma_Y \min_{\mathbf{x}_0} \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_0|. \tag{10}$$

The exact string potential Eq. (10) consists of the so-called Y-string term,

$$V_{\mathrm{Y}} = \sigma_{\mathrm{Y}} \sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho} \times \boldsymbol{\lambda}|)}, \tag{11}$$

and three other angle-dependent two-body string, or so-called V-string terms specified in Ref. 37. Manifestly, Eq. (11) depends only on the hyper-radius $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ and on the area of the triangle $\frac{\sqrt{3}}{2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$.

III. THE *SO*(4) ALGEBRAIC METHOD

The decomposition of the three-body spatial wave functions in terms of the $SO(4)$ "grand angular momentum" eigenfunctions is appropriate for all permutation symmetric three-body potentials, including, though not limited to the Y-string. Three-body potentials with lesser permutation symmetry can be treated in this way, as well, though with additional complications. The approximations that are used to solve the three-body Schrödinger equation depend on the potential and form a separate part of the theoretical framework.

A. *SO*(4) symmetry in the hyper-spherical approach

First, we need to define several objects that are needed in subsequent developments. The "grand angular" momentum tensor $K_{\mu\nu}$, μ, $\nu = 1, 2, 3, 4$

$$\begin{aligned} K_{\mu\nu} &= m \left(\mathbf{X}_\mu \dot{\mathbf{X}}_\nu - \mathbf{X}_\nu \dot{\mathbf{X}}_\mu\right) \\ &= \left(\mathbf{X}_\mu \mathbf{P}_\nu - \mathbf{X}_\nu \mathbf{P}_\mu\right), \end{aligned} \tag{12}$$

where $\boldsymbol{X}_\mu = (\boldsymbol{\rho}, \boldsymbol{\lambda})$. In particular, $l_\rho \equiv K_{12}$ and $l_\lambda \equiv K_{34}$ generate $SO(2)$ rotation of vector $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, respectively.

Next, we introduce

$$\mathbf{M} = \frac{1}{2}\left(l_\rho + l_\lambda, K_{13} - K_{24}, K_{14} + K_{23}\right), \tag{13}$$

$$\mathbf{N} = \frac{1}{2}\left(l_\rho - l_\lambda, K_{13} + K_{24}, K_{14} - K_{23}\right). \tag{14}$$

Note that **M** and **N** commute and that each of them satisfies separate $SO(3)$ commutation rules

$$\begin{aligned} \left[\mathbf{M}^i, \mathbf{M}^j\right] &= i\varepsilon^{ijk}\mathbf{M}^k, \\ \left[\mathbf{N}^i, \mathbf{N}^j\right] &= i\varepsilon^{ijk}\mathbf{N}^k, \end{aligned} \tag{15}$$

explicitly demonstrating the $so(4) = so(3) \oplus so(3)$ decomposition. In this context (of $SO(4)$ hyper-spherical harmonics), the Casimir operator eigenvalues of the two $SO(3)$ subgroups $J = M = J' = N$ must be identical, leading to the requirement $J = J' \equiv \frac{1}{2}K$ (this is easily explicitly verified by using Eqs. (12)–(14)). This constraint significantly reduces the number of hyperangular harmonics, i.e., of $SO(4)$ representations that appear in this problem.

A natural basis in the space of an $SO(4)$ irreducible representation, labeled by the J value, is the tensor products basis

$$\left|\begin{matrix} J & J \\ m_1 & m_2 \end{matrix}\right\rangle = |Jm_1\rangle \otimes |Jm_2\rangle,$$

where we are free to choose which component of **M**, and of **N**, will be diagonalized and denoted as m_1 and m_2, respectively. We will take m_1 to be eigenvalue of M_1, thus, from Eq. (13), we read off $m_1 = \frac{1}{2}\left(l_\rho + l_\lambda\right) = \frac{L}{2}$, where L is the total angular momentum, a constant of the motion. One possibility, that is appropriate to the case of Delves-Simonov hyperangles (χ_D, θ), would be to take m_2 to be the eigenvalue of N_1, i.e., $m_2 = \frac{1}{2}\left(l_\rho - l_\lambda\right) = \frac{\Delta L}{2}$. (As the ΔL can have (only) integer values, we see that both the "half-integer" $N, M \in \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and the "integer" $N, M \in 0, 1, 2, \ldots$, representations of $SO(4)$ must appear, Refs. 26 and 27.) We shall find it more convenient to take m_2 to be the eigenvalue of the operator $G \equiv N_2 = \frac{1}{2}(\boldsymbol{\rho} \cdot \mathbf{p}_\lambda - \boldsymbol{\lambda} \cdot \mathbf{p}_\rho)$, corresponding to choice of the Iwai-Smith hyperangles (α, ϕ).

One may need to know the explicit form of the hyper-spherical harmonics. They can be constructed either directly, as in Sec. III B 1 below, or indirectly, by way of their connection with Wu-Yang monopole harmonics, as in Sec. III B, following Ref. 8.

B. *SO*(4) hyper-spherical harmonics

The symmetries of the Y-string confinement potential/hamiltonian are: parity, rotation, and permutation/spatial exchange of particles, or its "generalization" the "democracy group" O(2). Therefore, only wave functions with the same $P = (-1)^{l_\rho + l_\lambda}$, L, and permutation symmetry M (mixed), S (symmetric), and A (antisymmetric) may mix with each other. There are two different states with mixed permutation symmetry: the M_ρ and M_λ. If P_{ij} is the ijth particle permutation/spatial exchange operator, then the permutation symmetry can be examined using the following transposition matrices:

$$P_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

$$P_{13} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \tag{17}$$

operating on the transposed "four-vector" $\boldsymbol{X}_\mu^{\mathrm{T}} = (\boldsymbol{\rho}, \boldsymbol{\lambda})^{\mathrm{T}}$ that furnish a basis for the two-dimensional (mixed) irrep. M of S_3.

Mitchell and Littlejohn, Ref. 8, have developed a general theory of $SO(4)$ hyperspherical harmonics for the planar three-body problem. They have shown, *inter alia*, that the two sets (Delves-Simonov and Iwai-Smith) of hyper-angles are related by a (hyper-)rotation through $\frac{\pi}{2}$ about the y-hyper-axis. Next, we shall briefly review that subject as we shall need it for subsequent developments.

1. Iwai-Smith variables SO(4) hyper-spherical harmonics

The general theory of symmetrized $SO(4)$ hyper-spherical harmonics in the Iwai-Smith basis has been developed in Ref. 8 on the basis of monopole harmonics, Ref. 24, or spin-weighted spherical harmonics, Ref. 25. They show an explicit formula, Eq. (5.9) in Ref. 8, for the "symmetric representation" of planar three-body wave functions $\psi^{S}_{\lambda mn}(\alpha, \beta)$ in terms of what they call the "principal axes gauge" $SO(4)$ hyperspherical harmonics, or what we call the Iwai-Smith hyper-angles (α, ϕ), where $\beta = \phi$, which formula reads

$$\begin{aligned} \psi^{S}_{\lambda mn}(\alpha, \beta, \theta) &= \frac{2}{\sqrt{2}\pi} e^{im\theta}\, Y^{\mathrm{PA}}_{m/2,\lambda/2,n/2}(\alpha, \beta) \\ &= \frac{\sqrt{1+\lambda}}{\sqrt{2}\pi} \mathcal{D}^{\lambda/2}_{n/2,-m/2}(-\beta, \alpha, 2\theta), \end{aligned} \tag{18}$$

where $Y^{\mathrm{PA}}_{m/2,\lambda/2,n/2}(\alpha, \beta)$ are related to the Wu-Yang (magnetic) monopole spherical harmonics, Ref. 24, or spin-weighted spherical harmonics, Ref. 25, in the "north regular" gauge, cf. Eq. (2.12) in Ref. 8,

$$Y^{\mathrm{NR}}_{ql\mu}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}^{l}_{\mu,-q}(-\phi, \theta, \phi), \tag{19}$$

where $\mathcal{D}^{l}_{\mu,-q}(-\phi, \theta, \phi)$ are the Wigner $SO(3)$ rotation matrices defined by

$$\mathcal{D}^{l}_{m,m'}(\alpha, \beta, \gamma) = \langle lm| \exp(-i\alpha J_z) \exp(-i\beta J_y) \exp(-i\gamma J_z) |lm'\rangle. \tag{20}$$

In other words, we may identify $K = \lambda$, $L = m$, $G = n/2$, and write the $SO(4)$ hyper-spherical harmonics ($\mathcal{Y}^{K/2}_{L/2,G}$ in our notation)

$$\begin{aligned} \mathcal{Y}^{K/2}_{L/2,G}(\alpha, \phi, \Phi) &= \frac{2}{\sqrt{2}\pi} e^{iL\Phi}\, Y^{\mathrm{PA}}_{L/2,K/2,G}(\alpha, \phi) \\ &= \frac{\sqrt{1+K}}{\sqrt{2}\pi} \mathcal{D}^{K/2}_{G,-L/2}(-\phi, \alpha, 2\Phi). \end{aligned} \tag{21}$$

Note that when the total angular momentum L vanishes ($L = 0$) it leads to a particular simplification, because then the $SO(4)$ hyper-spherical harmonics reduce to ordinary $SO(3)$ spherical harmonics (modulo a multiplicative constant) in the shape-space hyper-angles (α, ϕ), due to the defining relation (cf. Eq. (1) in Sec. 4.17 of Ref. 28)

$$Y^{*}_{lm}(\beta, \alpha) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}^{l}_{m,0}(\alpha, \beta, \gamma). \tag{22}$$

Therefore,

$$\begin{aligned} \psi^{S}_{\frac{K}{2}0G}(\alpha, \phi, \Phi) &= \mathcal{Y}^{K/2}_{0,G}(\alpha, \phi, \Phi) \\ &= \frac{2}{\sqrt{2}\pi}\, Y^{\mathrm{PA}}_{0,K/2,G}(\alpha, \phi) \\ &= \sqrt{\frac{2}{\pi}}\, Y^{G}_{K/2}(\alpha, \phi). \end{aligned} \tag{23}$$

As any three-body (spatial part of) potential must be invariant under overall (ordinary) rotations, it is a scalar, or equivalently, it contains only zero-angular momentum hyperspherical components.

Thus, we have shown that for $L = 0$ one may use an ordinary $SO(3)$ spherical harmonic expansion of the potential to recover the full $SO(4)$ hyperspherical harmonic expansion.

C. The Schrödinger equation in hyper-spherical variables

An important property of the hyper-spherical formalism is that the three-body Schrödinger equation of the three-body systems with factorizable potentials, viz., $V(R, \alpha, \phi) = V(R)V(\alpha, \phi)$, turns into a set of infinitely many (mutually) coupled equations, that reduce to a common hyper-radial Schrödinger equation,

$$-\frac{1}{2m}\left[\frac{d^2}{dR^2} + \frac{3}{R}\frac{d}{dR} - \frac{K(K+2)}{R^2} + 2mE\right]\psi_c(R) + V_{\text{eff.}}(R)\sum_{c'} C_{c,c'}\psi_{c'}(R) = 0 \tag{24}$$

albeit with different hyper-angular coupling coefficients $C_{c,c'}$. The coupling matrix $C_{c,c'}$ is defined as the proportionality coefficient in the hyper-angular matrix element, Eq. (25)

$$\begin{aligned} V_{\text{eff.}}(R)C_{[K'],[K]} &= \langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|V(R, \alpha, \phi)|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle \\ &= V(R)\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|V(\alpha, \phi)|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle, \end{aligned} \tag{25}$$

when the three-body potential can be factored into a hyper-radial $V_{3-\text{body}}(R)$ and hyper-angular part $V_{3-\text{body}}(\alpha, \phi)$. The latter can be expanded in $SO(3)$ (hyper-)spherical harmonics,

$$V_{3-\text{body}}(\alpha, \phi) = \sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} Y_{JM}(\alpha, \phi). \tag{26}$$

As a consequence of Eq. (23), this is related to the $L = 0$ $SO(4)$ hyper-spherical harmonics $\mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi)$ as follows:

$$V_{3-\text{body}}(\alpha, \phi) = \sqrt{\frac{\pi}{2}}\sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} \mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi) \tag{27}$$

leading to

$$V_{\text{eff.}}(R)C_{[K'],[K]} = V(R)\sqrt{\frac{\pi}{2}}\sum_{J,M\geq 0}^{\infty} v_{JM}^{3-\text{body}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle. \tag{28}$$

We separate out the $J = 0$ term

$$\begin{aligned} V_{\text{eff.}}(R)C_{[K'],[K]} = V(R)\Bigg(&\delta_{[K'],[K]}\frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}} \\ &+ \sqrt{\frac{\pi}{2}}\sum_{J>0,M}^{\infty} v_{JM}^{3-\text{body}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle\Bigg) \end{aligned} \tag{29}$$

and absorb the factor $\frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}}$ into the definition of $V_{\text{eff.}}(R) = \frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}}V(R)$ to find

$$C_{[K'],[K]} = \delta_{[K'],[K]} + \pi\sqrt{2}\sum_{J>0,M}^{\infty}\frac{v_{JM}^{3-\text{body}}}{v_{00}^{3-\text{body}}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle. \tag{30}$$

Here, and in the following $\delta_{[K'],[K]} = \delta_{K',K}\delta_{L',L}\delta_{G_3',G}$. Thus, the problem has been reduced to one of evaluating the SO(4) hyperspherical harmonic matrix elements $\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}_{0M}^{J}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle$.

In Table I we show, for reference purposes, the first six coefficients v_{J0}^{Δ} and $\sqrt{\frac{3}{2}}v_{J0}^{\mathrm{Y}}$ in the expansion (27) of the Δ and Y-string potentials, respectively. For a derivation see the Appendix. One can see that the series is alternating (in sign) and converging, though fairly slowly, after the initial rapid drop-off from $J = 0$ to $J = 2$. This dominance of the v_{00} coefficient (the one corresponding to

TABLE I. Expansion coefficients v^{Δ}_{J0} and $\sqrt{\frac{3}{2}}v^{\mathrm{Y}}_{J0}$ of the Δ- and Y-string three-body potentials V_{Δ}, V_{Y}, respectively, in terms of the hyper-spherical O(3) harmonics $Y_{J0}(\alpha, \phi)$, for $J = 0, 2, \ldots, 10$.

J	v^{Δ}_{J0}	$\sqrt{\frac{3}{2}}v^{\mathrm{Y}}_{J0}$
0	10.0265	5.29221
2	0.320285	0.494019
4	0.232132	−0.129813
6	0.0158003	0.0599748
8	−0.00699939	−0.0345825
10	0.00369641	0.0225086

the $SO(4)$-invariant, or "hyper-spherical" component of the three-body potential), which is illustrated in Table I, is a general property of most conventional three-body potentials that demonstrates why the $SO(4)$ symmetry is generally a good starting point for most conventional three-body calculations in two dimensions.

D. The hyperspherical harmonic matrix element

The SO(4) hyperspherical harmonic matrix element in Eq. (30) can be evaluated using the definition of $SO(4)$ hyper-spherical harmonics Eq. (19) and the standard formula for the angular integral over the product of three Wigner D-functions, see, e.g., Eq. (5) in Sec. 4.11.1 of Ref. 28, as the product of the reduced (hyper-angular) $SO(4)$ matrix element $\langle J' || \mathcal{Y}^{J}_{0M} || J'' \rangle$ and the corresponding $SO(4)$ Clebsch-Gordan coefficient, which equals the product of two $SO(3)$ Clebsch-Gordan coefficients. That leads to

$$C_{[K'],[K]} = \delta_{[K'],[K]} + \sum_{J>0,M}^{\infty} \left(\frac{v^{3-\mathrm{body}}_{JM}}{v^{3-\mathrm{body}}_{00}} \right) \sqrt{\frac{(K'+1)(2J+1)}{(K+1)}}\; C^{\frac{K}{2}\frac{L}{2}}_{J0,\frac{K'}{2}\frac{L'}{2}} C^{\frac{K}{2}G}_{JM,\frac{K'}{2}G'_3}, \tag{31}$$

where $C^{J''m''}_{Jm,J'm'}$ are the $SO(3)$ Clebsch-Gordan coefficients in the notation of Ref. 28. This is our main algebraic result: it allows one to evaluate the discrete part of the (energy) spectrum of a three-body potential as a function of its shape-sphere harmonic expansion coefficients $v^{3-\mathrm{body}}_{JM}$.

The matrix ordinary differential equation (ODE) (24) can be diagonalized (in the hyper-angular sub-space of the Hilbert space) before being solved, because the potential $V(R)$ is common to all (matrix) components. That, in turn, is a consequence of the homogeneity of the potential, $V(\lambda R) \to \lambda^{\alpha} V(R)$, under dilations, $R \to \lambda R$, where α is the power of R in the potential[43] that simplifies the solution. Diagonalization of such a matrix is generally not a big problem numerically, but analytic diagonalization has its intrinsic limitations: if the matrix exceeds the 4×4 "size," then the secular equation becomes of the fifth order and thus generally not solvable in closed form. Due to the smallness of the ratio $\frac{v^{3-\mathrm{body}}_{JM}}{v^{3-\mathrm{body}}_{00}} \ll 1$ of the potential's expansion coefficients $v^{3-\mathrm{body}}_{00}$, $v^{3-\mathrm{body}}_{JM}$, the diagonalization of this usually small correction may be adequately dealt with the first-order perturbation theory. This is because the value of v_{00} is usually higher than v_{JM}, for any other $J \neq 0$ value for most conventional three-body potentials, see Table I. This is not to say that one cannot construct three-body potentials with a smaller, or even vanishing value of v_{00}, however.

The $SO(3)$ Clebsch-Gordan coefficients are subject to the usual "triangular" conditions $J' + J'' \geq J \geq |J' - J''|$. As $J = \frac{1}{2}K'$ and $J'' = \frac{1}{2}K$ we find constraints on the values of $J' = J$, where J is the "multipole order" of the interaction: $K' + K \geq 2J \geq |K' - K|$. So, for example, with the lowest non-trivial multipole order $J = 2$, we have additional constraints, discussed in Sec. IV A below. This leads to some remarkable simplifications, for example, the facts (a) that the three lowest K bands eigen-energies are entirely determined by two numbers: the expansion coefficients v_{00} and v_{20}, which has been known at least since Refs. 30 and 31 and (b) that the $K = 3$ band of states introduces just one new free parameter, the $v_{3\pm3}$, which has been known at least since Refs. 32–34.

Moreover, due to angular momentum conservation reflected in the first Clebsch-Gordan coefficient of Eq. (31), $K/2$ and $K'/2$ values must be either both integer, or both half-integer. In turn, this has consequence that all non-diagonal terms with $[K] = [K'] \pm n$, where n is an odd integer, are forbidden.

IV. APPLICATION TO THREE-QUARK CONFINEMENT

Our own interest in this matter stems from the three-quark confinement by the Δ and/or Y-strings, Refs. 7 and 37–39. Lattice QCD appears to demand one of two confining potentials: either the so-called Y-junction string three-quark potential, Eq. (10), as suggested in Refs. 35 and 36, or the sum of two-body ("Δ-string") potentials

$$V_{\Delta-\text{str.}} = \sigma_\Delta \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|. \tag{32}$$

The Y-string potential contains certain two-body terms when one of the angles in the triangle subtended by the quarks is greater than $\frac{2}{3}\pi$, cf. Subsection 2 of the Appendix. In the present paper, we shall ignore such terms, which generally make (very) small contributions to the energies of low-lying states, as shown explicitly in Ref. 37. We were led to the permutation symmetric hyperangles in the process of our study of the dynamical symmetry of the Y-string Refs. 7 and 37–39: the residual $\mathrm{O}_L(2) \otimes \mathrm{O}(2) \subset \mathrm{SO}(4)$ dynamical symmetry of the non-relativistic Y-string Hamiltonian is best visualized in terms of permutation-adapted hyper-angles.

Present methods can relate the regularities in the spectrum to the permutation, or dynamical symmetry properties of the potential. Moreover, one can use this method in systems with "only" two (rather than three) identical particles, i.e., in potentials that are only partially permutation symmetric, such as the Coulomb bound state(s) of two electrons and one positron (or vice versa).

A. The string potential's hyper-angular matrix elements

The S_3 permutation group symmetrized hyper-spherical harmonics correspond to different $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry multiplets (Young diagrams/tableaux) of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$, and $M \leftrightarrow 70$. For more about the $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry multiplets and their relation to the S_3 permutation group, see Ref. 40. Thus, we may use the "democracy" index G to classify the wave functions, i.e., the symmetrized hyper-spherical harmonics, according to their S_3 permutation group, or equivalently to their $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry properties.

1. The Y-string and other area-dependent potentials

In the case of area-dependent potentials,

$$\begin{aligned} V_{\text{area-dep.}}(\alpha, \phi = 0) &= \sum_{n=0,2,\ldots}^{\infty} v_{n0}^{\text{area-dep.}} Y_{n0}(\alpha, \phi = 0) \\ &= \sqrt{\frac{\pi}{2}} \sum_{J=K/2=0,2,\ldots:G=0}^{\infty} v_{J,0}^{\text{area-dep.}} \mathcal{Y}_{0,0}^{J}(\alpha, \phi = 0, \Phi = 0) \end{aligned} \tag{33}$$

the expansion coefficients $v_{n0}^{\text{area-dep.}}$, corresponding to $v_{JM}^{3-\text{body}}$ in Eq. (27), have non-vanishing values only for the zero value of the "hyper-magnetic" quantum number $M = 0$, due to the independence of the area-dependent potentials on the azimuthal angle ϕ, see Subsection 2 of the Appendix.

Therefore, the hyper-angular matrix $C_{[K'],[K]}$ of an area-dependent three-body potential becomes

$$\begin{aligned} C_{[K'],[K]} &= \delta_{[K'],[K]} + \frac{v_{20}}{v_{00}} \sqrt{4\pi} \langle \mathcal{Y}_{[K']}(\alpha, \phi) | Y_{20}(\alpha, \phi) | \mathcal{Y}_{[K]}(\alpha, \phi) \rangle + \cdots \\ &= \delta_{[K'],[K]} + \frac{v_2}{v_0} \sqrt{4\pi} \sqrt{\frac{\pi}{2}} \langle \mathcal{Y}_{\frac{1}{2}L',G'}^{\frac{1}{2}K'}(\alpha, \phi) | \mathcal{Y}_{00}^{2}(\alpha, \phi) | \mathcal{Y}_{\frac{1}{2}L,G}^{\frac{1}{2}K}(\alpha, \phi) \rangle + \cdots . \end{aligned} \tag{34}$$

We use the potential's $SO(4)$ transformation properties to express its matrix element in terms of general $SO(4)$ Clebsch-Gordan coefficients; to that end we note that such area-dependent potentials are eigenfunctions of two $SO(4)$ generators: $\mathbf{M}^1 = \frac{1}{2}L$ and $\mathbf{N}^2 = G$ with both eigenvalues being equal to zero. Thus, a residual dynamical $SO_L(2) \otimes SO(2)$ symmetry of ordinary rotations (in the physical/geometric configuration space) and hyper-rotations (in the shape space) remains in this system. In addition to this, space parity transformation and permutation of two particles Eq. (16), which are also symmetries of this potential, extend the residual symmetry to $O_L(2) \otimes O(2)$.

Therefore, the hyper-angular matrix element of the Y-string and other area-dependent potentials is, in the lowest order, proportional to the product of the reduced matrix element $\langle J' || \mathcal{Y}^2 || J \rangle$, that is explicitly determined in Eq. (31), and the corresponding $SO(4)$ Clebsch-Gordan coefficient

$$\left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \mathcal{Y}^2_{00} \middle| \begin{matrix} J & J \\ m_1 & m_2 \end{matrix} \right\rangle = \langle J' \, || \mathcal{Y}^2 | \, | J \rangle \left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \begin{matrix} 2 & 2; & J & J \\ 0 & 0; & m_1 & m_2 \end{matrix} \right\rangle .$$

This $SO(4)$ Clebsch-Gordan coefficient is the product of two $SO(3)$ Clebsch-Gordan coefficients

$$\begin{aligned} \left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \begin{matrix} 2 & 2; & J & J \\ 0 & 0; & m_1 & m_2 \end{matrix} \right\rangle &= (J, m_1, 2, 0 | J', m_1')(J, m_2, 2, 0 | J', m_2') \\ &= \delta_{m_1 m_1'} \delta_{m_2 m_2'} (J, m_1, 2, 0 | J', m_1)(J, m_2, 2, 0 | J', m_2). \end{aligned} \tag{35}$$

The corresponding (non-vanishing) $SO(3)$ Clebsch-Gordan coefficients are those with: $J' = J$, $J' = J \pm 1$, and $J' = J \pm 2$. It is clear, however, that some of these matrix elements are often not necessary. For example, the product $(J, m_1, 2, 0 | J \pm 1, m_1)(J, m_2, 2, 0 | J \pm 1, m_2)$ vanishes, due to symmetries of Clebsch-Gordan coefficients, when either $m_1 = 0$ (angular momentum of the state is zero) or $m_2 = 0$ (this also holds for higher order corrections from Eq. (33) – when either one of m_1 and m_2 is zero, the difference $|J - J'|$ must be even, i.e., $|K - K'|$ is a multiple of 4). Even when neither is the case, the $(J, m_2, 2, 0 | J + 1, m_2)$ Clebsch-Gordan coefficient connects states with values of K that differ by two units, which is important only when the $(\mathrm{K} + 2)$ band energy is degenerate with some K-band hyper-radially excited state, which happens only in the harmonic oscillator and $1/R$ hyper-Coulomb potentials.

Moreover, the Clebsch-Gordan coefficient $(J, m_2, 2, 0 | J \pm 2, m_2)$ is physically significant in situations when the absolute value of the difference of K's for the two states equals four: $|K - K'| = 4$ and the unperturbed levels are degenerate, something that only happens in the higher shells/bands of the harmonic oscillator and the $1/R$ hyper-Coulomb potential. Thus, for most practical purposes, we shall only need the $J' = J$ terms.

2. The Δ-string potential

The Δ-string potential contains all of the "ordinary" multipoles present in the area-dependent potentials, though not in the same proportion. The first distinctly "two-body potential" contribution transforms as $Y_{3\pm3}(\alpha, \phi) = \sqrt{\frac{\pi}{2}} \mathcal{Y}^3_{0\pm3}$. The corresponding coefficient for the Δ-string potential is $v^{\Delta}_{3\pm3} = 0.141232$, see Subsection 1 of the Appendix. This breaks the residual dynamical $O(2) \otimes O_L(2)$ symmetry down to $S_3 \otimes O_L(2)$. Consequently, the Clebsch-Gordan coefficients appearing in Eq. (31) are different as well, so they bring about different selection rules: the $v_{3\pm3}$ term can only contribute to $K \geq 3$ matrix elements.

B. Results

The numerical results have been discussed in Refs. 19 and 22, so here we shall only discuss the "missing steps" in their derivation.

1. K = 2 band results

We have calculated the hyper-angular matrix elements $\langle \mathcal{Y}^2_{00} \rangle_{\mathrm{ang}}$ for the SU(6) multiplets (states with the same permutation group S_3 properties) of the four lowest K(=0,1,2,3) bands: as explained

TABLE II. The values of the three-body potential hyper-angular matrix elements $\pi\sqrt{2}\langle \mathcal{Y}^2_{00} \rangle_{\rm ang}$, for various K = 2 states (for all allowed orbital waves L). The correspondence between the S_3 permutation group irreps. and $\mathrm{SU(6)}_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$, and $M \leftrightarrow 70$.

K	$[SU(6), L^P]$	$\pi\sqrt{2}\langle \mathcal{Y}^2_{00} \rangle_{\rm ang}$
2	$[70, 0^+]$	$-\frac{1}{\sqrt{5}}$
2	$[56, 2^+]$	$-\frac{1}{\sqrt{5}}$
2	$[70, 2^+]$	$\frac{1}{2\sqrt{5}}$
2	$[20, 0^+]$	$\frac{2}{\sqrt{5}}$

earlier, the $K = 0$, 1 bands are affected only by the v_{00} coefficient. The calculated energy splittings of $K = 2$ band states depend only on the Clebsch-Gordan coefficient $(J, m_2, 2, 0|J, m_2)$ with various values of $m_1 = G$ and $m_2 = L/2$ belonging to different SU(6) multiplets being listed in Table II. Our main concern is the energy splitting pattern among the states within the K = 2 hyper-spherical $SO(4)$ multiplet. The hyper-radial matrix elements of the linear hyper-radial potential are identical for all the (hyper-radial ground) states in one K band. Therefore, the 2D energy differences among various sub-states of a particular K band multiplet are integer multiples of the energy splitting "unit" Δ_K, just as they are in 3D.

Note that the two K = 2 $SU(6)$, or S_3 multiplets $[70, 0^+]$ and $[56, 2^+]$ are degenerate in 2D, as opposed to 3D, where they are split. Moreover, the 3D $[20, 1^+]$ multiplet has $L = 0$ in 2D. Otherwise, the 2D and 3D states coincide and their energy splitting patterns agree. This is but another manifestation of the Bowler-Tynemouth theorem[32,33] for this class of three-body potentials.

The 2D splitting pattern is similar, but not identical to the 3D one: the 2D multiplets $[20, 0^+]$, $[70, 2^+]$, $[20, 0^+]$, $[56, 2^+]$, are split just like the 3D multiplets $[20, 1^+]$, $[70, 2^+]$, $[20, 0^+]$, $[56, 2^+]$, but the $[70, 0^+]$ and the $[56, 2^+]$ are degenerate in 2D, whereas they are split in 3D. This indicates the differences between 2D and 3D in this problem.

2. K = 3 band results

The calculated energies of states with of K = 3 and various values L are listed in Table II and displayed in Fig. 2 of Ref. 22. With an area-dependent (i.e., ϕ-independent) potential in 2D, we find that the K = 3 band $SU(6)$, or S_3 multiplets have one of (only) two possible energies: the ($[70, 1^-]$, $[56, 3^-]$, $[20, 3^-]$) are degenerate, as are ($[70, 3^-]$, $[56, 1^-]$, $[20, 1^-]$) (at a different energy) (Tables III and IV). Note that the 3D $[70, 2^-]$ multiplet has no analogon in 2D. Upon introduction of a ϕ-dependent ("two-body") potential component proportional to $v^{\Delta}_{3\pm3}$, and upon diagonalization of the $C_{[K'],[K]}$ matrix, one finds further splittings among the previously degenerate states $[70, 1^-]$,

TABLE III. The values of the three-body potential hyper-angular diagonal matrix elements $\langle Y_{20}(\alpha, \phi)\rangle_{\rm ang}$, for various K = 3 states (for all allowed orbital waves L).

K	$[SU(6), L^P]$	$\pi\sqrt{2}\langle \mathcal{Y}^2_{00} \rangle_{\rm ang}$
3	$[20, 1^-]$	$-\frac{1}{\sqrt{5}}$
3	$[56, 1^-]$	$-\frac{1}{\sqrt{5}}$
3	$[70, 1^-]$	$\frac{1}{\sqrt{5}}$
3	$[56, 3^-]$	$\frac{1}{\sqrt{5}}$
3	$[70, 3^-]$	$-\frac{1}{\sqrt{5}}$
3	$[20, 3^-]$	$\frac{1}{\sqrt{5}}$

TABLE IV. The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\langle \mathcal{Y}^{K}_{\frac{1}{2}L_f,G_{3f}} | \mathcal{Y}^{3}_{0\pm 3} | \mathcal{Y}^{K}_{\frac{1}{2}L_i,G_{3i}} \rangle_{\rm ang}$, for various K = 3 states (for all allowed orbital waves L).

K	$[SU(6)_f, L^P_f]$	$[SU(6)_i, L^P_i]$	$\pi\sqrt{2}\langle \mathcal{Y}^{K}_{\frac{1}{2}L_f,G_{3f}} \| \mathcal{Y}^{3}_{0\pm 3} \| \mathcal{Y}^{K}_{\frac{1}{2}L_i,G_{3i}} \rangle_{\rm ang}$
3	$[20, 1^-]$	$[56, 1^-]$	$-\frac{6}{\sqrt{35}}$
3	$[56, 1^-]$	$[20, 1^-]$	$-\frac{6}{\sqrt{35}}$
3	$[70, 1^-]$	$[\text{all}, 1^-]$	0
3	$[56, 3^-]$	$[20, 3^-]$	$\frac{2}{\sqrt{35}}$
3	$[70, 3^-]$	$[\text{all}, 1^-]$	0
3	$[20, 3^-]$	$[56, 3^-]$	$\frac{2}{\sqrt{35}}$

$[56, 3^-]$, and $[20, 3^-]$, as well as among the $[70, 3^-]$, $[56, 1^-]$, and $[20, 1^-]$

$$
\begin{aligned}
&[20, 1^-] \quad v_{00} - \frac{1}{\sqrt{5}} v_{20} + \frac{2}{\sqrt{35}} v_{33},\\
&[56, 1^-] \quad v_{00} - \frac{1}{\sqrt{5}} v_{20} - \frac{2}{\sqrt{35}} v_{33},\\
&[70, 1^-] \quad v_{00} + \frac{1}{\sqrt{5}} v_{20},\\
&[70, 3^-] \quad v_{00} - \frac{1}{\sqrt{5}} v_{20},\\
&[20, 3^-] \quad v_{00} + \frac{1}{\sqrt{5}} v_{20} + \frac{6}{\sqrt{35}} v_{33},\\
&[56, 3^-] \quad v_{00} + \frac{1}{\sqrt{5}} v_{20} - \frac{6}{\sqrt{35}} v_{33}.
\end{aligned}
\tag{36}
$$

For the K = 3 band in 3D, the energy splittings have been calculated by Bowler and Tynemouth[32,33] for two-body anharmonic potentials perturbing the harmonic oscillator and confirmed and clarified by Richard and Taxil, Ref. 34, in the hyper-spherical formalism with linear two-body potentials (the Δ-string). One should compare the above results, Eqs. (36), with Eq. (45) in Ref. 34, in particular. Comparing the sizes of the $v^{\Delta}_{3\pm 3}$-induced splittings in 3D and 2D, one finds comparable values: 1/3 in 2D vs. 2/7 in 3D.

In hindsight, Richard and Taxil's, Ref. 34, separation of $V_4(R)$ and $V_6(R)$ potentials' contributions is particularly illuminating (prescient?): the former corresponds precisely to our "area-dependent" term v_{20} and the latter to the "two-body" contribution $v_{3\pm 3}$. As both the Y- and Δ strings contain the former, whereas only the Δ string contains the latter, we see that the latter to be the source of different degeneracies/splittings in the spectra of these two types of potentials. This was not noted by Richard and Taxil, Ref. 34, however, so our contribution here is the (first) demonstration of this fact in 2D. The 3D case ought to be analogous, but has not been worked out in detail, yet.

V. SUMMARY AND DISCUSSION

In summary, we have reduced the non-relativistic three-body bound state problem in a permutation symmetric potential in two dimensions to a single ordinary differential equation for the hyper-radial wave function with coefficients determined by $SO(4)$ group-theoretical arguments multiplying the (homogenous) hyper-radial potential. That one equation can be solved just like the radial Schrödinger equation in 3D. The breaking of the $SO(4)$ symmetry determines the spectrum.

In 2D, the "hyper-spherical symmetry" is $SO(4)$, and the residual dynamical symmetry of the potential is $O(2) \otimes O_L(2) \subset SO(4)$, where $O_L(2)$ is the rotational symmetry associated with the (total) angular momentum. Due to the fact that $so(4) \simeq so(3) \oplus so(3)$, one may use many of the $so(3)$ algebra results, such as the $SO(3)$ Clebsch-Gordan coefficients. Thus, we looked at the "algebra chain" $so(2) \oplus so_L(2) \subset so(3) \oplus so_L(2) \subset so(4)$.

We formulated the problem in terms of $SO(4)$-group covariant three-body variables and then brought the Schrödinger equation into a form that can be (exactly) solved. More specifically, we expanded the three-body Schrödinger equation and its eigen-functions, as well as the potential in $SO(4)$ hyperspherical harmonics. Then we showed how the energy eigenvalues (the energy spectrum) can be calculated as functions of the three-body potential's (hyper-)spherical harmonics expansion coefficients, of the $SO(4)$ reduced matrix element(s) and of the $SO(4)$ Clebsch-Gordan coefficients, that are related to the ordinary $SO(3)$ Clebsch-Gordan coefficients.

The dynamical $O(2)$ symmetry of the Y-string potential was discovered in Ref. 7, with the permutation group $S_3 \subset O(2)$ as its subgroup. The existence of an additional dynamical symmetry strongly suggested an algebraic approach, such as those used in Refs. 32 and 33, which were not based on the Y-string, however. A careful perusal of Refs. 32 and 33 showed that an $O(2)$ group had been used as an enveloping structure for the permutation group $S_3 \subset O(2)$, but was not interpreted as a (possible) dynamical symmetry. With this in mind we started an algebraic study of the Y-string-like ("collective") area-dependent potentials. We first established the basic consequences of this dynamical symmetry of the Y-string potential. The 3D case is substantially more complicated than the 2D one: for this reason we have limited ourselves to the two dimensions in this paper.

We have used these results to calculate the energy splittings of various SU(6)/S_3 multiplets in the K = 2 and K = 3 bands of the Y- and Δ string spectra, and found close correspondence with the splittings calculated by other methods in three dimensions. It is only in the K = 3 band that a difference appears between the spectra of these two confining models. That is, the first explicit consequence of the dynamical $O(2)$ symmetry of the Y-string.

Our results can be used in other three-body problems in two dimensions, such as the three-anyon problem, Refs. 13–17, and some other condensed matter physics problems, Ref. 18. There is also hope that one can extend these methods to three dimensions and thus simplify the hyper-spherical harmonics approach to the three-body problem in general.

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APPENDIX: THREE-BODY POTENTIALS IN TERMS OF HYPER-SPHERICAL VARIABLES

1. The sum of two-body α-power potentials

The Δ-string potential $V_{\Delta-\mathrm{str.}}$, Eq. (32), is proportional to the sum of pairwise distances between the bodies. It can be viewed as a special case ($\boldsymbol{\alpha} = 1$) of the three-body sum of pairwise distances to power $\boldsymbol{\alpha}$ Eq. (9).

In terms of Iwai-Smith hyper-angles, Eq. (32) reads

$$V_{\Delta-\mathrm{str.}}(R, \alpha, \phi) = \sigma_\Delta R \left(\sqrt{1 + \sin(\alpha) \sin\left(\frac{\pi}{6} - \phi\right)} + \sqrt{1 + \sin(\alpha) \sin\left(\phi + \frac{\pi}{6}\right)} + \sqrt{1 - \sin(\alpha) \cos(\phi)} \right). \tag{A1}$$

In order to find the general hyper-spherical harmonic expansion of the sum of α-power two-body potentials, we note that it factors into the hyper-radial $V_{\boldsymbol{\alpha}}(R) = \sigma_{\boldsymbol{\alpha}} R^{\boldsymbol{\alpha}}$ and the hyper-angular part

$V_{\alpha}(\alpha, \phi)$

$$V_{\alpha}(R, \alpha, \phi) = V_{\alpha}(R) V_{\alpha}(\alpha, \phi) = V_{\alpha}(R) \sum_{J,M}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi), \tag{A2}$$

where

$$v_{JM}^{\alpha} = \int_0^{2\pi} d\phi \int_0^{\pi} V_{\alpha}(\alpha, \phi)\, Y_{JM}^{*}(\alpha, \phi) \sin(\alpha)\, d\alpha. \tag{A3}$$

We note that any S^3 permutation symmetric sum of two-body potentials (with the sole exception of the harmonic oscillator) has a specific "triple-periodic" azimuthal ϕ hyper-angular dependence with the angular period of $\frac{2}{3}\pi$. That provides additional selection rules for the magnetic quantum number M dependent terms in this expansion, besides the $J = 0, 2, \dots$ rule for $M = 0$ terms discussed below in Subsection 2

$$\sum_{JM}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) = \sum_{J=0,2,\dots}^{\infty} v_{J0}^{\alpha} Y_{J0}(\alpha, \phi) + \sum_{J;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) + \sum_{J;M=\pm 6}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) + \cdots . \tag{A4}$$

Reality of the potential $V = \Re e(V)$ and the "azimuthal parity" under $(\phi \to -\phi)$ lead to the fact that only the "zonal harmonics" coefficients, Eq. (A5) survive, whereas the "sectorial harmonics" coefficients, Eq. (A6) vanish

$$v_{JM} = \frac{1}{2}\left(v_{JM} + v_{JM}^{*}\right), \tag{A5}$$

$$0 = \frac{1}{2}\left(v_{JM} - v_{JM}^{*}\right). \tag{A6}$$

The aforementioned reflection symmetry with respect to the "hyper-equatorial plane" $(\cos(\alpha) \to -\cos(\alpha))$, adds new selection rules for each of the new sub-series. For example,

$$\sum_{J;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) = \sum_{J=3,5,7,\dots;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi). \tag{A7}$$

The first such non-vanishing coefficient for the Δ-string potential is $v_{3\pm 3}^{\Delta} = 0.141232$. Thus, we see that the number of non-vanishing coefficients in the Iwai-Smith parametrization of the shape sphere is decimated, as compared with the number of the Delves-Simonov parametrization coefficients which fact ought to improve the speed of convergence of corresponding numerical calculations.

2. Area-dependent potentials and their dynamical symmetry

The Y-string potential $V_{\mathrm{Y-str.}}$ is defined in Eq. (10). The complexity of the Y-string potential is perhaps best seen when expressed in terms of three-body Jacobi (relative) coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$: The exact string potential Eq. (10) consists of the so-called Y-string term, Eq. (11), which is valid when

$$\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}, \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}, \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}, \end{cases}$$

and three other angle-dependent two-body string, or so-called V-string terms (for their explicit functional form, see Ref. 37). These additional terms are relevant only when the above angular conditions are not met – which occurs only in a small part of the configuration space – and contribute to the same "sub-leading" multipoles $v_{J\pm 3}$ (the lowest one being $v_{3\pm 3}$) as the sum of two-body terms in Eq. (A7). In this sense, the V-string terms are indistinguishable from the Δ-string contributions, except by the size of their contributions, which is smaller than the Δ-string's. Of course, they contribute to the "leading" multipoles υ_{J0}, as well. Thus, their effect can be thought of as one of slightly changing the values of the Y-string multipoles. For this reason, we shall ignore these two-body V-string terms hereafter.

The $|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$ term in Eq. (11) is proportional to the area of the triangle subtended by the three quarks. Next, we show that $V_{\rm Y}$ is a function of both Delves-Simonov hyper-angles (χ, θ),

$$V_{\rm Y}(R, \chi, \theta) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + \sin 2\chi |\sin\theta|\right)}, \tag{A8}$$

whereas it is a function of only one Smith-Iwai hyper-angle – the "polar angle" α

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + |\cos\alpha|\right)}. \tag{A9}$$

This independence of the "azimuthal" Smith-Iwai hyper-angle ϕ means that the associated component G of the hyper-angular momentum is a constant-of-the-motion of the Y-string; this result holds in all area-dependent potentials, Ref. 7.

Equation (A9) can be further re-written as a (non-polynomial) function of (the absolute value of) only one $SO(3)$ (hyper-)spherical harmonic in the shape (hyper-)space: using the following formula for $Y_{10}(\alpha, \phi)$:

$$\cos\alpha = \sqrt{\frac{4\pi}{3}} Y_{10}(\alpha, \phi), \tag{A10}$$

that leads to

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + \sqrt{\frac{4\pi}{3}} |Y_{10}(\alpha, \phi)|\right)}. \tag{A11}$$

Now, the absolute value of $|Y_{10}(\alpha, \phi)|$ can be expressed as $\sqrt{Y^*_{10}(\alpha, \phi) Y_{10}(\alpha, \phi)}$ and the $SO(3)$ Clebsch-Gordan expansion can be applied to $Y^*_{10}(\alpha, \phi) Y_{10}(\alpha, \phi)$, which contains only the (even) values of $J = 0, 2$

$$\begin{aligned} |\cos\alpha| &= \sqrt{\frac{4\pi}{3}} \sqrt{Y^*_{10}(\alpha, \phi) Y_{10}(\alpha, \phi)} \\ &= \sqrt{\frac{4\pi}{3}} \sqrt{Y^2_{00}(\alpha, \phi) + \frac{2}{\sqrt{5}} Y_{00}(\alpha, \phi) Y_{20}(\alpha, \phi)} \\ &= \sqrt{\frac{1}{3}} \sqrt{1 + \frac{2}{\sqrt{5}} \frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)}}. \end{aligned} \tag{A12}$$

The square root in Eq. (A12) can be expanded in a Taylor-like series, the first two terms of which coincide with the expansion in Legendre polynomials, or $SO(3)$ spherical harmonics, and in $SO(4)$ hyper-spherical harmonics. Therefore, the exact Legendre polynomial expansion of Eq. (A12) runs over even-order $J = 0, 2, 4, \ldots$, zero "hyper-magnetic" quantum number $G = M = 0$ $SO(3)$ (hyper-)spherical harmonics. This is not an accident: all three-body potentials are invariant under the reflection symmetry with respect to the "hyper-equator" $\cos(\alpha) \to -\cos(\alpha)$, which together with the independence of $V_{\rm Y}$ on the azimuthal hyper-angle ϕ leads to the fact that this series cannot depend on the "hyper-magnetic quantum number" $G = M$ and consequently to the aforementioned "selection rule": it is a sum over even values of J only

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}} \sum_{J=0,2,\ldots}^{\infty} \upsilon^{\rm Y}_{J0} Y_{J0}(\alpha, \phi), \tag{A13}$$

where $\upsilon^{\rm Y}_{J0}$, $J = 0, 2, \ldots$.

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[42] It has been argued by the referee of this paper that these "democracy" transformations are identical, in fluid mechanics, i.e., in the $N \to \infty$ limit, to the indistinguishability of fluid particles, which, in turn has been shown to lead to Kelvins circulation theorem.
[43] Clearly, this does not hold for non-polynomial potentials, e.g., in the Yukawa potential, so this is a particular property of the homogenous polynomial hyper-radial potentials.

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$SL(n,R)$ IN PARTICLE PHYSICS AND GRAVITY — DECONTRACTION FORMULA AND UNITARY IRREDUCIBLE REPRESENTATIONS

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$SL(n,R)$ and $Diff(n,R)$ groups play a prominent role in various particle physics and gravity theories, notably in chromogravity (that models the IR region of QCD), gauge affine generalizations of general relativity, and pD-branes. Applications of these groups require a knowledge of their features and especially rely on the unitary irreducible representation details. Lie algebra, topology and unitary representation issues of the covering groups of the $SL(n,R)$ and $Diff(n,R)$ groups with respect to their maximal compact $SO(n)$ subgroups are considered. Topological properties determining spinorial representations of these groups are reviewed. An especial attention is paid to the fact that, contrary to other classical Lie algebras, the $SL(n,R)$, $n \geq 3$ covering groups are groups of infinite matrices, as are all their spinorial representations. A notion of Lie algebra decontraction, also known as the Gell-Mann formula, that plays a role of an inverse to the Inonu–Wigner contraction, is recalled. Contrary to orthogonal type of algebras, the decontraction formula has a limited validity. The validity domain of this formula for $sl(n,R)$ algebras contracted with respect to their $so(n)$ subalgebras is outlined. A recent generalization of the decontraction formula, that applies to all $SL(n,R)$ covering group representations, as well as an explicit closed expression of all non-compact $sl(n,R)$ operators matrix elements for all representations is presented. A construction of the unitary $sl(n,R)$ representations is discussed within a framework than combines the Harish-Chandra results and a method of fulfilling the unitarity requirements in Hilbert spaces with non-trivial scalar product kernel.

Keywords: Gell-Mann decontraction formula; Lie algebra contraction; $SL(n)$ representations.

Mathematics Subject Classification 2010: 20C33, 20C40

1. Introduction

The Poincaré spacetime and internal $SU(n)$ symmetries, both global and local, played a crucial role in describing fundamental forces in nature, physical conservation laws, and the basic matter fields. These symmetries are the core essence of the Standard Model and Einstein's General Relativity Theory, the two pillars of contemporary fundamental physics. In this work we consider the $SL(n, R)$ symmetries in the content of particle physics and gravity theory. First, we recollect several prominent examples and extract the knowledge on the relevant required $SL(n, R)$ representations. Afterwards, we pose a general framework for constructing the $SL(n, R)$ unitary irreducible representations, and outline the basic facts about recent generalization of the Gell-Mann's decontraction formula that yields all matrix elements of the $sl(n, R)$ algebra elements for all representations.

Already in 1965, Gell-Mann, Dothan and Ne'eman proposed the $SL(3, R)$ symmetry to describe the Regge trajectories of hadron recurrences in a spectrum generating algebra approach [1]. The model was subsequently generalized to the relativistic $SL(4, R)$ one, describing both parent and daughter trajectories [2]. A construction of the unitary irreducible $SL(3, R)$ representations was a first step on the way to fulfill this proposal. Moreover, the spinorial representations, faithful representations of the $\overline{SL}(3, R)$ covering group, were essential in order to describe baryonic recurrences. After some confusion among researchers at the time, denying even an existence of the covering group on the basis of a wrong interpretation of certain Cartan's statement, it was soon clear that there are specific features of the $SL(3, R)$ symmetry (subsequently, all $SL(n, R)$, $n \geq 3$, symmetries) and its representations [3]. The covering $\overline{SL}(n, R)$, $n \geq 3$, groups are necessarily defined in infinite dimensional spaces (groups of infinite matrices), thus there are no finite spinorial representations, and their representations considered with respect to maximal compact Spin(n), i.e. $\overline{SO}(n)$ subgroups have as a rule non-trivial multiplicity. An explicit construction of all $\overline{SL}(3, R)$ unitary irreducible representations confirmed these facts [4].

A potential relevance of the $SL(n, R)$, $n = 3, 4$ symmetries in describing confinement of quarks was noted even at the early stage of the so-called "bag-models" featuring a volume-preserving part of the action that yields confinement. These symmetries revive on the fundamental dynamic QCD level. The adoption of QCD and its incorporation in the Standard Model were the outcome of the success of asymptotic freedom (AF) in fitting the scaling results of deep inelastic electron-nucleon scattering, coupled with the fact that color-$SU(3)$ provides an explanation for some (otherwise paradoxical) key features of the Non-Relativistic Quark Model (NRQM): "wrong" spin-statistics of the baryon (**56** in $SU(6)$) ground state, zero-triality of the entire $SU(3)$ (Eightfold-Way) physical spectrum. AF provides a successful perturbative treatment for the "ultraviolet" (UV) region, e.g., high-energy electro-weak hadronic interactions, corresponding to the current-quarks aspects of NRQM. There is also a prosperous understanding of hadronic strong interactions in the "hard" and "semi-hard" regimes. Nothing of the sort has emerged in the "infrared" (IR) frequency antipode region. After several decades, we still lack a complete proof of color-confinement.

2. Chromogravity

A chromogravity approach to the IR QCD sector [5] is based on a conjecture: (a) that gluon exchange forces (with the gluons in color-neutral combinations) make up an important component of inter-hadron interactions *in the "softest" region and in confinement*; (b) that the physical role of this component is to produce *a longer-range force*, with many of the characteristics of *gravity*, starting with the basic mathematical foundation, namely, *invariance under (pseudo) diffeomorphisms*; (c) that the simplest such n-gluon exchange, that of the two-gluon system

$$G_{\mu\nu}(x) = (\kappa)^{-2} g_{ab} B^a_\mu(x) B^b_\nu(x) \tag{1}$$

fulfills the role of an effective (pseudo) metric — "chromometric", with respect to these (pseudo) diffeomorphisms — "chromo diffeomorphisms", in the same manner that the physical metric (through its Christoffel connection) "gauges" the true diffeomorphisms. Here κ has the dimensions of mass, $\mu, \nu, \ldots$ are Lorentz 4-vector indices, $a, b, \ldots$ are $SU(3)$ adjoint representation (octet) indices, g_{ab} is the Cartan metric for the $SU(3)$ octet, and B^a_μ is a gluon field.

The gluon color-$SU(3)$ gauge field transforms under an infinitesimal local $SU(3)$ variation according to

$$\delta_\epsilon B^a_\mu = \partial_\mu \epsilon^a + B^b_\mu \{\lambda_b\}^a_c \epsilon^c = \partial_\mu \epsilon^a + i f^a_{bc} B^b \epsilon^c \tag{2}$$

(we use the adjoint representation $\{\lambda_b\}^a_c = -i f^a_{bc} = i f^a_{bc}$). To deal with the non-perturbative IR region, we expand the gauge field operator around a constant global vacuum solution N^a_μ,

$$\partial_\mu N^a_\nu - \partial_\nu N^a_\mu = i f^a_{bc} N^b_\mu N^c_\nu, \tag{3}$$

$$B^a_\mu = N^a_\mu + A^a_\mu. \tag{4}$$

Such a vacuum solution might be of the instanton type, for instance, that at large distances is required to approach a constant value.

The leading part of the color-$SU(3)$ infinitesimal gauge variation of the pseudo-metric field $G_{\mu\nu}$ in the infrared region reads [5]

$$\delta_\xi G_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\mu(\xi^\sigma G_{\sigma\nu}) + \partial_\nu(\xi^\sigma G_{\mu\sigma}), \tag{5}$$

where, $\xi_\mu = \eta_{ab} \epsilon^a N^b_\mu$, and where one can reidentify δ_ξ as a variation under a formal diffeomorphism of the R^4 manifold. This $G_{\mu\nu}$ variation simulates the infinitesimal variation of a "world tensor" $G_{\mu\nu}$ under Einstein's covariance group, $x^\sigma \to x^\sigma + \xi^\sigma$. ξ^σ thus has to be defined as a contravariant vector, and $G_{\mu\nu}$ is invertible, thanks to the constant part N^a_μ. Note that as the μ, ν indices are "true" Lorentz indices, acted upon by the physical Lorentz group, the manifold has to be Riemannian (only Riemannian manifolds, with or without torsion, have tangents with orthogonal or pseudo-orthogonal symmetry). Thus

$$D_\sigma G_{\mu\nu} = 0, \tag{6}$$

the commutator of two such variations,

$$[\delta_{\xi_1}, \delta_{\xi_2}] G_{\mu\nu} = \delta_{\xi_3} G_{\mu\nu}, \tag{7}$$

where

$$\xi_{3\mu} := (\partial_\nu \xi_{1\mu})\xi_2^\nu + (\partial_\mu \xi_{1\nu})\xi_2^\nu - (\partial_\nu \xi_{2\mu})\xi_1^\nu - (\partial_\mu \xi_{2\nu})\xi_1^\nu \tag{8}$$

indeed closes on the covariance group's commutation relations.

In the general case, the QCD "gluon-made" operators which mutually connect various hadron states are characterized by color-singlet quanta. The corresponding color-singlet n-gluon field operator has the following form

$$G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = d^{(n)}_{a_1a_2\cdots a_n} B^{a_1}_{\mu_1} B^{a_2}_{\mu_2} \cdots B^{a_n}_{\mu_n} \tag{9}$$

where

$$\begin{aligned} d^{(2)}_{a_1a_2} &= g_{a_1a_2}, \\ d^{(3)}_{a_1a_2a_3} &= d_{a_1a_2a_3}, \\ d^{(n)}_{a_1a_2\cdots a_n} &= d_{a_1a_2b_1} g^{b_1c_1} d_{c_1b_2a_3} \cdots g^{b_{n-4}c_{n-4}} d_{c_{n-4}b_{n-3}a_{n-2}} g^{b_{n-3}c_{n-3}} d_{c_{n-3}a_{n-1}a_n}, \\ & \qquad n > 3, \end{aligned} \tag{10}$$

B^a_μ is the dressed gluon field, $g_{a_1a_2}$ is the $SU(3)$ Cartan metric, and $d_{a_1a_2a_3}$ is the $SU(3)$ totally symmetric $8\times8\times8 \to 1$ tensor. The set of all $G^{(n)}_{\mu_1\mu_2\cdots\mu_n}$ operators, $n = 1, 2, \ldots,$ forms a basis of a vector space of colorless purely gluonic configurations. Again, in the infrared region approximation the infinitesimal color-$SU(3)$ variation can be rewritten in terms of effective pseudo-diffeomorphisms,

$$\delta_\epsilon G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \partial_{\{\mu_1} \xi^{(n-1)}_{\mu_2\mu_3\cdots\mu_n\}} \equiv \delta_\xi G^{(n)}_{\mu_1\mu_2\cdots\mu_n}, \tag{11}$$

where $\{\mu_1\mu_2\cdots\mu_n\}$ denotes symmetrization of indices, and

$$\xi^{(n-1)}_{\mu_1\mu_2\cdots\mu_{n-1}} \equiv d^{(n)}_{a_1a_2\cdots a_n} N^{a_1}_{\mu_1} N^{a_2}_{\mu_2} \cdots N^{a_{n-1}}_{\mu_{n-1}} \epsilon^{a_n}. \tag{12}$$

A subsequent application of two $SU(3)$-induced variations closes algebraically

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \delta_{\epsilon_3} G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \quad \text{i.e. } [\delta_{\xi_1}, \delta_{\xi_2}] G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \delta_{\xi_3} G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \tag{13}$$

thus yielding an infinitesimal nonlinear realization of the $Diff(4, R)$ Chromodiffeomorphisms group in the space of fields $\{G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \,|\, n = 2, 3, \ldots\}$.

2.1. *Matter particles and fields*

The simplest way to describe hadronic matter fields is by making use of nonlinear realizations of the $Diff(4, R)$ chromodiffeomorphisms group over its maximal linear subgroup, i.e. over the $GA(4,R) \supset SA(4,R)$ [8]. Here, $GA(4,R)$, $SA(4,R)$ are the semidirect product groups of the translation group T_4 and the $GL(4,R)$, $SL(4,R)$ groups, respectively.

In the following, we consider the relevant groups in an n-dimensional space time, i.e. the $Diff(n,R)$, T_n, $SL(n,R)$, $SO(n)$ groups, thus setting up a mathematical framework applicable to gravity and extended objects considerations in higher dimensions as well, and we focus on the $SL(n,R)$ group, since this group determines the non-Abelian features of the $GL(n,R)$ group as well.

The matter particles and the matter fields in quantum theory are described by the affine group, $SA(n,R) = T_n \wedge SL(n,R)$, representations in Hilbert spaces of states and fields, respectively.

The commutation relations of the $sa(n,R)$ algebra of the $SA(n,R)$ group read

$$\begin{aligned} [P_a, P_b] &= 0, \\ [Q_{ab}, P_c] &= ig_{ac}P_b, \\ [Q_{ab}, Q_{cd}] &= ig_{bc}Q_{ad} - ig_{ad}Q_{cb}, \end{aligned} \tag{14}$$

the structure constants g_{mn} being either $\delta_{ab} = (+1,+1,\dots,+1)$, $a,b,c,d = 1,2,\dots,n$ for the $SO(n)$ subgroup or $\eta_{ab} = (+1,-1,\dots,-1)$, $a,b,c,d = 0,1,\dots,n-1$ for the n-dimensional Lorentz subgroup $SO(1,n-1)$ of the $SL(n,R)$ group. The maximal compact $SO(n)$ subgroup of the $SL(n,R)$ group is generated by the metric preserving antisymmetric operators $J_{ab} = Q_{[ab]}$, while the remaining non-compact traceless symmetric operators $T_{ab} = Q_{(ab)}$, the shear operators, generate the (non-trivial) n-volume preserving transformations. The $SL(n,R)$ commutation relations are given as follows

$$\begin{aligned} [M_{ab}, M_{cd}] &= -i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} - i\eta_{bd}M_{ac}, \\ [M_{ab}, T_{cd}] &= -i\eta_{ac}T_{bd} - i\eta_{ad}T_{bc} + i\eta_{bc}T_{ad} + i\eta_{bd}T_{ac}, \\ [T_{ab}, T_{cd}] &= +i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} + i\eta_{bd}M_{ac}. \end{aligned} \tag{15}$$

The quantum mechanical symmetry group is given as the $U(1)$ minimal extensions of the corresponding classical symmetry group. In practice, one finds it by taking the universal covering group of the classical group (topology changes), and by solving the algebra commutation relations for possible central charges (algebra deformation). There are no non-trivial central charges of the $sa(n,R)$ and $sl(n,R)$ algebras, and the remaining important question for quantum applications is the one of the affine symmetry covering group. The translational part of the $SA(n,R)$ group is contractible to a point and thus irrelevant for the covering question. The $SL(n,R)$ subgroup is, according to the Iwasawa decomposition, given by $SL(n,R) = SO(n,R) \times A \times N$, where A is a subgroup of Abelian transformations (e.g., diagonal matrices) and N is a nilpotent subgroup (e.g., upper triangular matrices). Both A and N subgroups are contractible to point. Therefore, the covering features are determined by the topological properties of the maximal compact subgroup of the group in question. In our case, that is the $SO(n,R)$ group, i.e. more precisely its central subgroup. The universal covering group of the $SO(n)$, $D \geq 3$ group is its double covering group isomorphic to Spin(n). In other words $SO(n) \simeq \mathrm{Spin}(n)/Z_2$.

The universal covering group of a given group is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, $\overline{SL}(n,R)$, exists provided one can embed $SL(n,R)$ into a group of finite complex matrices that contain Spin(n) as subgroup. A scan of the Cartan classical algebras points to the $SL(n,C)$ groups as a natural candidate for the $SL(n,R)$ groups covering. However, there is no match of the defining dimensionalities of the $SL(n,R)$ and Spin(n) groups for $n \geq 3$,

$$\dim(SL(n,C)) = n < 2^{[\frac{n-1}{2}]} = \dim(\mathrm{Spin}(n)), \tag{16}$$

except for $n = 8$. In the $n = 8$ case, one finds that the orthogonal subgroup of the $SL(8,R)$ and $SL(8,C)$ groups is $SO(8,R)$ and not Spin(8). Thus, there are no finite dimensional covering groups of the $SL(n,R)$ groups for any $n \geq 3$. An explicit construction of all spinorial, unitary and non-unitary multiplicity-free [6], and unitary non-multiplicity-free [4], $SL(3,R)$ representations shows that they are all defined in infinite dimensional spaces.

The universal (double) covering groups of the $\overline{SL}(n,R)$ and $\overline{SA}(n,R)$, $n \geq 3$ groups are groups of infinite complex matrices. All their spinorial representations are infinite dimensional and when reduced with respect to Spin(n) subgroups contain representations of unbounded spin values.

2.2. *Representations on states*

The $\overline{SA}(n,R)$ Hilbert space representations are, owing to the semidirect product group structure, induced as in the Poincaré case from the corresponding little group (stability subgroup) representations. The correct quantum mechanical interpretation requires the little group representations to be unitary. The unitary irreducible $\overline{SA}(n,R)$ Hilbert space representations are obtained as follows: (i) determine the vectors characterized by the maximal set of labels of the Abelian translational subgroup generators, (ii) determine the corresponding little groups as subgroups of the $SL(n,R)$ groups that leave these vectors invariant, and (iii) induce the unitary irreducible $\overline{SA}(n,R)$ representations from T_n and little groups representations. In contradistinction to the Poincaré case, the little groups that describe affine particles are more complex in structure due to the fact that a orthogonal type of group is enlarged here to the linear one.

The little group of the $\overline{SA}(n,R)$ Hilbert-space particle states is of the form $T^{\sim}_{n-1} \wedge \overline{SL}(n-1,R)$, where the Abelian invariant subgroup $T^{\sim}_{n-1}$ of the little group is generated by Q_{1j}, $j = 2,3,\ldots,n$. Owing to the fact that the little group itself is given as a semidirect product, there is number of possibilities. The simplest one is when the $T^{\sim}_{n-1}$ subgroup is represented trivially, $D(T^{\sim}_{n-1}) \to 1$, i.e. $D(Q_{1j}) \to 0$, the remaining part of the little group is $\overline{SL}(n-1,R)$, and the corresponding "affine particle" is described by the unitary irreducible $\overline{SL}(n-1,R)$ representations. These representations are infinite dimensional, even in the tensorial case, due to non-compactness of the $SL(n,R)$ group.

2.3. *Representations on fields*

The representations of the $\overline{SA}(n,R)$ group generalize the known Poincaré group representations on fields and are given as follows,

$$(D(a,\bar{\Lambda})\Phi_i)(x) = (D(\bar{\Lambda}))_i^j \Phi_j(\Lambda^{-1}(x-a)) \quad (a,\bar{\Lambda}) \in T_n \wedge \overline{SL}(1,n-1), \tag{17}$$

where i,j enumerate a basis of the representation space of the field components. There are two physical requirements that have to be satisfied in the affine case in order to provide the due particle-field connection: (i) representations of the affine-particle little group $\overline{SL}(n-1,R)$ have to be unitary and thus (due to the little group's non-compactness) infinite dimensional, and (ii) representations of the Lorentz subgroup $\mathrm{Spin}(1,n-1)$ have to be finite dimensional and thus non-unitary as required by their Poincaré subgroup interpretation. This is achieved by making use of the so called "deunitarizing" automorphism of the $\overline{SL}(n,R)$ group [7]:

$$\mathcal{A} : \overline{SL}(n,R) \to \overline{SL}(n,R), \tag{18}$$

$$J_{ij}^{\mathcal{A}} = J_{ij}, \quad K_j^{\mathcal{A}} = iN_j, \quad N_j^{\mathcal{A}} = iK_j, \tag{19}$$

$$T_{ij}^{\mathcal{A}} = T_{ij}, \quad T_{00}^{\mathcal{A}} = T_{00}, \quad i,j = 1,2,\ldots,D-1, \tag{20}$$

so that (J_{ij}, iK_i) generate the new compact $\mathrm{Spin}(n)^{\mathcal{A}}$ and (J_{ij}, iN_i) generate $\mathrm{Spin}(1,D-1)^{\mathcal{A}}$. Here, the $\overline{SL}(n-1,R)$, the stability subgroup of $\overline{SA}(n,R)$, is represented unitarily, while the Lorentz subgroup is represented by finite dimensional non-unitary representations. An efficient way of constructing explicitly the $\overline{SL}(n,R)$ infinite dimensional representations is based on the so called "decontraction" formula, which is an inverse of the Wigner–Inönü contraction, and will be treated below.

3. Affine Gravity and Spinorial Wave Equations

The metric affine [9], and gauge affine [10, 11] theories of gravity are generalizations of the Poincaré gauge theory where the Lorentz group $\mathrm{Spin}(1,n-1)$ is replaced by the $\overline{SL}(n,R)$ group. The customary way to develop such a theory in a particle physics framework is to start by the Dirac equation and then gauging the relevant global symmetry. In our case that means to start by a Dirac-like equation for an infinite-component spinorial affine field $\Psi(x)$,

$$(iX^a\partial_a - M)\Psi(x) = 0, \tag{21}$$

$$\Psi(x) \sim D^{(\mathrm{spin})}(\overline{SL}(n,R)). \tag{22}$$

The X^a, $a = 0, 1, \ldots, n-1$ vector operator, acting in the space of the Ψ field components, is an appropriate generalization of the Dirac γ matrices to the affine case. The $\overline{SL}(n, R)$ affine covariance requires that the following commutation relations are satisfied

$$[M_{ab}, X_c] = i\eta_{bc}X_a - i\eta_{ac}X_b, \tag{23}$$

$$[T_{ab}, X_c] = i\eta_{bc}X_a + i\eta_{ac}X_b. \tag{24}$$

The first relation ensures Lorentz covariance, and is generally a easy one to fulfill. The second relation, required by the full affine covariance, turns out to be rather difficult to accomplish.

We focus here on the $\overline{SL}(n, R)$ representations constrains required by the group algebraic consistency of this Dirac-like equation. In order to obtain all (physically relevant) unitary irreducible $\overline{SL}(n, R)$ representations, and in particular the spinorial ones fitting the Dirac-like equation construction, one works in Hilbert spaces of square integrable functions over the maximal compact subgroup, $\mathcal{L}^2(\mathrm{Spin}(n))$. The Hilbert space basis vectors in Dirac's notation are $\{|{}^{\{J\}}_{\{k\}\{m\}}\rangle\}$, where $\{J\}$ and $\{m\}$ are the representation labels of $\mathrm{Spin}(n)$ and its subgroups $\mathrm{Spin}(n-1), \mathrm{Spin}(n-2), \ldots, \mathrm{Spin}(n)$, respectively; while $\{k\}$ are labels of $\mathrm{Spin}(n-1), \mathrm{Spin}(n-2), \ldots, \mathrm{Spin}(n)$ groups acting to the left which are used to describe eventual multiplicity of the $\mathrm{Spin}(n)$ representations within a given $\overline{SL}(n, R)$ representation. We can split an $\overline{SL}(n, R)$ representation in terms of its $\mathrm{Spin}(n)$ subrepresentations, in a symbolic notation, as follows:

$$D(\overline{SL}(n, R)) \sim \sum_{\{J\},\{k\}} D^{\{J\}}(\mathrm{Spin}(n), \{k\}). \tag{25}$$

Representations of the shear operators T_{ab} are such that their matrix elements apriory have non-trivial $\{k\}$ dependence, i.e. they are proportional, as presented below, to the $C^{\{J''\}\,\square\square\,\{J'\}}_{\{k''\}\,\{k\}\,\{k'\}}$ $\mathrm{Spin}(n)$ Clebsch–Gordan coefficients. There are two distinct cases: (i) the $\mathrm{Spin}(n)$ multiplicity free representations when all $\{k\}$ labels are zero, and (ii) representations with non-trivial multiplicity. In the first case, the zero-value $\{k\}$ labels imply that the $\{J\}$ labels are integer, and thus all these $D(\overline{SL}(n, R))$ representations are tensorial. In the second case, when there are no constraints on the $\{k\}$ labels, one can have both tensorial and spinorial $D(\overline{SL}(n, R))$ representations.

To sum up, from the considered physical examples we conclude that applications of the $\overline{SL}(n, R)$ symmetry requires knowledge of the spinorial and tensorial unitary (infinite dimensional) representations with non-trivial $\mathrm{Spin}(n)$, $\mathrm{Spin}(1, n-1)$ subgroup multiplicity. In the following, we present an effective method of constructing all $\overline{SL}(n, R)$ representations, and set up a framework that allowes one to fulfill the unitarity and irreducibility issues as well.

4. Gell-Mann Decontraction Formula

To solve the problem of finding $\overline{SL}(n,\mathbb{R})$ representations in the basis of its (pseudo)orthogonal subgroup we will employ the so called Gell-Mann (decontraction) formula [12–16]. The aim of this formula is to provide an inverse to the well-known Inönü–Wigner contraction procedure [17]. More concretely, let a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}, \tag{26}$$

and its Inönü–Wigner contraction $\mathcal{A}' = \mathcal{M} + \mathcal{U}$:

$$[\mathcal{M},\mathcal{M}] \subset M, \quad [\mathcal{M},\mathcal{U}] \subset U, \quad [\mathcal{U},\mathcal{U}] = \{0\}, \tag{27}$$

be given. Following a mathematically less rigorous definition (more strict definition can be found in [12]), the Gell-Mann formula states that, in certain cases, elements $T_\mu \in \mathcal{T}$ can be constructed as the following simple function of the contracted algebra operators $U_\mu \in \mathcal{U}$ and $M_\nu \in \mathcal{M}$:

$$T_\mu = i\frac{\alpha}{\sqrt{U_\nu U^\nu}}[C^2(\mathcal{M}), U_\mu] + i\sigma U_\mu. \tag{28}$$

Here, $C^2(\mathcal{M})$ and $U_\nu U^\nu$ denote the (positive definite) second order Casimir operators of the $\mathcal{M}$ and $\mathcal{A}'$ algebras, respectively, while α is a normalization constant and σ is an arbitrary parameter. The formula was, to our knowledge, first introduced by Dothan and Ne'eman [16], and was advocated by Hermann [13].

The importance of this formula in our case is immediate, since it is not difficult to obtain representations of the contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ (here $r_{\frac{n(n+1)}{2}-1}$ denotes $\frac{n(n+1)}{2} - 1$ dimensional Abelian algebra and $\uplus$ stands for semidirect sum).

To represent the contracted algebra we will work in the representation space of square integrable functions $\mathcal{L}^2(\mathrm{Spin}(n))$ over the maximal compact subgroup $\mathrm{Spin}(n)$, i.e. the $SO(n)$ universal covering group, with a standard invariant Haar measure. This representation space is large enough to provide for all inequivalent irreducible representations of the contracted group, and, by a theorem of Harish-Chandra [18–21], is also rich enough to contain representatives from all equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra, representations.

The generators of the contracted group are generically represented, in this space, as follows. The $so(n)$ subalgebra operators $M_{ab}, a, b = 1, 2, \ldots, n$, act in the standard way:

$$M_{ab}|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})|_{t=0}|\phi\rangle, \tag{29}$$

where action of a $\mathrm{Spin}(n)$ element g' on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(\mathrm{Spin}(n))$ is given via action from the left on basis vectors $|g\rangle$ of this space:

$$g'|\phi\rangle = g'\int \phi(g)|g\rangle dg = \int \phi(g)|g'g\rangle dg, \quad g', g \in \mathrm{Spin}(n). \tag{30}$$

The contracted non-compact Abelian operators U_μ (27) and (28), act in the same basis as multiplicative Wigner-like D-functions (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_\mu \to |u| D^{\square\square}_{w\mu}(g^{-1}) \equiv |u| \left\langle \begin{matrix} \square\square \\ w \end{matrix} \middle| \left(D^{\square\square}(g) \right)^{-1} \middle| \begin{matrix} \square\square \\ \mu \end{matrix} \right\rangle , \tag{31}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and $\square\square$ denoting (in a parallel to the Young tableaux) the symmetric second order tensor representation of $SO(n)$. The norm $|u|$ parametrizes representation of U, but will turn out to be irrelevant in our case, as it cancels with the denominator in (28). The $\left| \begin{matrix} \square\square \\ \mu \end{matrix} \right\rangle$ vector from representation $\square\square$ space is denoted by the index of the operator U_μ, whereas the vector $\left| \begin{matrix} \square\square \\ w \end{matrix} \right\rangle$ can be an arbitrary vector belonging to $\square\square$ (the choice of w determines, in Wigner terminology, the little group of the representation in question). Taking an inverse of g in (31) insures the correct transformation properties.

A natural discrete orthonormal basis in the $\mathcal{L}^2(\mathrm{Spin}(n))$ space is given by properly normalized Wigner D-functions:

$$\left\{ \left| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle \equiv \int \sqrt{\dim(J)}\, D^J_{km}(g^{-1}) dg |g\rangle \right\} , \quad \left\langle \begin{matrix} J \\ k\, m \end{matrix} \middle| \begin{matrix} J' \\ k'\, m' \end{matrix} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'} , \tag{32}$$

where dg is an (normalized) invariant Haar measure. Here, J stands for a set of $\mathrm{Spin}(n)$ irreducible representation labels, while the k and m labels numerate the representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch–Gordan coefficients of the $\mathrm{Spin}(n)$ group as follows,

$$\left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| M_{ab} \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\; C^{J \begin{smallmatrix}\square\\\square\end{smallmatrix} J'}_{m (ab) m'} . \tag{33}$$

The matrix elements of the U_μ operators in this basis are readily found to read:

$$\begin{aligned} \left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| U^{(w)}_\mu \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle &= |u| \left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| D^{-1 \square\square}_{w\mu} \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle \\ &= |u| \sqrt{\frac{\dim(J)}{\dim(J')}}\, C^{J \square\square J'}_{k\ w\ k'}\, C^{J \square\square J'}_{m\ \mu\ m'} . \end{aligned} \tag{34}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ representations is thus explicitly given in this space by (33) and (34). To obtain representations of $sl(n,\mathbb{R})$, apart from (33), we also need to know how to represent non-compact shear generators T_μ in this space. That is given by the Gell-Mann formula (28):

$$T^{(w,\sigma)}_\mu = i\alpha [C^2(so(n)), D^{\square\square}_{w\mu}] + i\sigma D^{\square\square}_{w\mu} . \tag{35}$$

Though it seems that our goal is accomplished, it unfortunately turns out that formula (35) does not hold in the entire space $\mathcal{L}^2(\mathrm{Spin}(n))$ and for arbitrary choice of vector w (in the sense that commutator of two so constructed shear generators will not yield the correct result).

In [22], we have carried out a detailed analysis of the scope of validity of Gell-Mann formula in the $sl(n,\mathbb{R})$ case. The conclusion was that the only $sl(n,\mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$, $m = 1, 2, \ldots, n-1$. The narrowing of the space from $\mathcal{L}^2(\mathrm{Spin}(n))$ to $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m))$ in the terms of basis (32) means reduction to a subspace spanned by vectors $\left|{}^{J}_{0m}\right\rangle$, where zero denotes the vector component invariant with respect to $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$. Furthermore, vector w in (35) must be chosen to be the one invariant with respect to the action of the group $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$.

With these constraints, expression (35) becomes a proper representation of shear generators. This formula then leads to explicit expression for matrix elements of shear generators in $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m))$:

$$\begin{aligned}&\left\langle \begin{matrix} J' \\ m' \end{matrix} \middle| T_\mu^{(\sigma)} \middle| \begin{matrix} J \\ m \end{matrix} \right\rangle \\ &\quad = i\sqrt{\frac{m(n-m)}{4n}}\sqrt{\frac{\dim(J)}{\dim(J')}}(C^2(J') - C^2(J) + \sigma) C^{J\,\Box\Box\,J'}_{0\;\;0\;\;0}\, C^{J\,\Box\Box\,J'}_{m\;\;\mu\;\;m'}. \end{aligned} \tag{36}$$

The zeroes in the indices of Clebsch–Gordan coefficients again denote vectors that are invariant with respect to $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$ transformations (in that spirit $\left|{}^{\Box\Box}_{w}\right\rangle = \left|{}^{\Box\Box}_{0}\right\rangle$). We also used shorthand notation $\left|{}^{J}_{0m}\right\rangle \equiv \left|{}^{J}_{m}\right\rangle$.

The expression (41), together with the action of the $\mathrm{Spin}(n)$ generators (33), provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, valid for arbitrary value of parameter σ. However, such representations are multiplicity free with respect to the maximal compact $\mathrm{Spin}(n)$ subgroup, and all of them are tensorial: multiplicity is lost with fixing of the left index of basis vectors (32) and only tensor representations of $\mathrm{Spin}(n)$ possess components invariant with respect to any $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m), m \geq 1$ subgroup.[a]

To obtain more general class of $sl(n,\mathbb{R})$ representations (and, in particular, those with multiplicity) the Gell-Mann formula had to be generalized.

5. Generalization of the Gell-Mann Formula

One of the key steps to obtain generalized Gel-Mann formula is introduction of, so called, left action generators K:

$$K_\mu \equiv g^{\nu\lambda} D^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\mu\nu} M_\lambda, \tag{37}$$

[a]In principle, some classes of spinorial multiplicity free representations can be obtained by appropriate analytic continuation of the Clebsch–Gordan coefficient in terms of the $\mathrm{Spin}(n)$ labels.

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$. The K_μ operators have the following matrix elements in the basis (32):

$$\langle K_{ab}\rangle = \left\langle \begin{matrix} J' \\ k'm' \end{matrix} \middle| K_{ab} \middle| \begin{matrix} J \\ km \end{matrix} \right\rangle = \delta_{JJ'}\sqrt{C^2(J)}\, C^{J\,\square\,J'}_{k\,(ab)\,k'}\,. \tag{38}$$

In other words, they behave exactly as the rotation generators M_μ (33), with a difference that they act on the lower left-hand side indices. The operators K_μ and M_μ mutually commute, but the corresponding Casimir operators match (in particular $\sum K_\mu^2 = \sum M_\mu^2$).

In terms of these new operators we can write down the following expression:

$$T^{\sigma_2\ldots\sigma_n}_{ab} = i\sum_{c>d}^{n}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sum_{c=2}^{n}\sigma_c D^{\square\square}_{(cc)(ab)}. \tag{39}$$

In [23, 24], we have shown that this is indeed the sought for generalization of the Gell-Mann formula, as this expression satisfies $sl(n,\mathbb{R})$ commutation relations in the entire space $\mathcal{L}^2(\mathrm{Spin}(n))$. In this expression σ_c is a set of $n-1$ arbitrary parameters that essentially (up to some discrete parameters) label $sl(n,\mathbb{R})$ irreducible representations. General validity of the new formula is reflected in the fact that there are now $n-1$ free parameters, i.e. representation labels, matching the $sl(n,\mathbb{R})$ algebra rank, compared to just one parameter of the original Gell–Mann formula.

An alternative form of (39) that looks more like the original formula (28) is:

$$T^{\sigma_2\ldots\sigma_n}_{ab} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U^{(cc)}_{ab}] + \sigma_c U^{(cc)}_{ab}, \tag{40}$$

where $C_2(so(c)_K)$ is the second order Casimir of the $so(c)$ left action subalgebra, i.e. $C_2(so(c)_K) = \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$. It is almost as simple as the original Gell-Mann formula, with a crucial advantage of being valid in the whole representation space over $\mathcal{L}^2(\mathrm{Spin}(n))$. Thus, due to Harish-Chandra theorems, the generalized Gell-Mann formula expression for the non-compact "shear" generators T_{ab} holds for all cases of $sl(n,\mathbb{R})$ irreducible representations, irrespective of their $so(n)$ subalgebra multiplicity (multiplicity free of the original Gell-Mann formula, and non-trivial multiplicity) and whether they are tensorial or spinorial. The price paid is that the generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators K_{ab} and $U^{(cd)}_{ab}$.

We also note that the very term in (40) when $c = n$ is, essentially, the original Gell-Mann formula (since $C_2(so(n)_K) = C_2(so(n)_M)$), whereas the rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some representations yielding the original formula.

The form (40) also allows us to find matrix elements of T_{ab} operators. After some calculation the following expression is obtained:

$$\begin{aligned}\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \right| T_{\{w\}} \left| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle &= \frac{i}{2}\sqrt{\frac{\dim(\{J\})}{\dim(\{J'\})}}\, C^{\{J\}\,\square\square\,\{J'\}}_{\{m\}\,\{w\}\,\{m'\}} \\ &\quad \times \sum_{c=2}^{n} \sqrt{\frac{c-1}{c}}\left(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c\right) \\ &\quad \times C^{\{J\}\,(\square\square)^{n-c+1}\,\{J'\}}_{\{k\}\qquad (0)^{c-2}\qquad \{k'\}}. \end{aligned} \tag{41}$$

(For the notation used for indices of Clebsh–Gordan coefficients please cf. [24, 25].)

The relation of the labeling of (41) and the one of (39), i.e. (40), is achieved provided $\sigma_c = \tilde{\sigma}_c + \sum_{d=2}^{c-1} \tilde{\sigma}_d/d$. The Clebsch–Gordan coefficient with indices $\{m\}, \{w\}, \{m'\}$ in (41) can be evaluated in an arbitrary basis (which is stressed by denoting the appropriate index by w instead by ab). The other Clebsch–Gordan coefficient can be evaluated in any basis labeled according to the Spin(n) $\supset$ Spin($n-1$) $\supset \cdots \supset$ Spin(2) subgroup chain (e.g., Gel'fand–Tsetlin basis) and can be, nowadays, rather easily evaluated, at least numerically.

6. Unitarity

A convenient way to parametrize any non-compact semisimple Lie group is given by means of the Iwasawa decomposition according to which the group G can be written as a product $G = NAK$, where N is a nilpotent subgroup of G, and its elements are upper triangular matrices with ones on the diagonal, A is an Abelian subgroup of G, and for $SL(n, R)$ we take its elements to be of the form $a = \mathrm{diag}(e^{\lambda}, e^{\mu}, e^{\nu}, \ldots, e^{-(\lambda+\mu+\nu+\cdots)})$, and finally K is the maximal compact subgroup $SO(n)$. An element $g \in G$ can thus be written as a product $g = nak$, where $n \in N, a \in A, k \in K$. The Iwasawa decomposition is unique and the product of some element $k \in K$ and an arbitrary element $g \in G$ is in general an arbitrary element of G which can be uniquely written as $kg = na(k, g)k \cdot g$, where $n \in N, a(k, g) \in A$ and $k \cdot g \in K$. Owing to the Iwasawa decomposition every element $g \in SL(n, R)$ can be uniquely written as $g = ne^{h}k$. The Abelian subgroup of $SL(n, R)$ has $n-1$ generators $A_1, A_2, \ldots$, and if $\lambda_1, \lambda_2, \ldots$ are the corresponding group parameters, respectively, one has $h = \lambda_1 A_1 + \lambda_2 A_2 + \cdots$. Let α be a linear, in general complex, function such that $\alpha(h) = \lambda_1 \alpha(A_1) + \lambda_2 \alpha(A_2) + \cdots$, and let us denote $\alpha(A_1), \alpha(A_2), \ldots$ by $\sigma_1, \sigma_2, \ldots$, respectively. Existence of the mapping α is guaranteed by the 1-dimensionality of the irreducible representations of the Abelian subgroup A. The mapping α can be extended in a natural way to a mapping from the group NA into the complex numbers since N is an invariant subgroup in NA.

The set of cosets $\overline{SL}(n, R)/NA$ is in one-to-one correspondence with the group $K = SO(n)$ and can be parametrized by the elements of K. In the coset space $\overline{SL}(n, R)/NA$ one has as well a measure, which we choose to be the invariant measure

dk on K. Let $H = L^2(K)$ be the separable Hilbert space of functions on K which are square integrable with respect to the invariant measure on K, i.e. $H = \{f(k) \mid k \in K\}$, such that $\int dk f^*(k) f(k) < \infty$, and let $\int dk = 1$.

Every non-trivial unitary representation of a non-compact group is necessarily infinite dimensional and this partly accounts for the complexity which occurs when one deals with unitary representations. The class of real semisimple Lie groups is especially complex. Harish-Chandra [18–21] defines a representation $U(g)$ of $G = \overline{SL}(n, R)$ on H in the following way: $U(g)$ is a homomorphic continuous mapping from G into the set of linear transformations on H given by

$$(U(g)f)(k) = e^{(h(k,g))} f(k \cdot g), \tag{42}$$

where $g \in G,\ f \in H,\ k \in K,\ e^h \in A$ and where $(U(g)f)(k)$ denotes the value of $U(g)f$ at the point k. Harish-Chandra now defines the concept of infinitesimal equivalence of two representations in the following way: Two representations are infinitesimally equivalent if there exists a similarity transformation of one representation into the other, with a non-singular, not necessarily unitary operator. In the case of equivalence there exist a unitary operator by means of which the transformation between the two representations is carried out. If both of two infinitesimally equivalent representations are unitary, then they are equivalent. Suppose now that $U(g)$ is a representation of a group G on a Hilbert space H. Suppose further that H_1 and H_2 are the two closed invariant subspaces of H, such that $H_2 \subset H_1 \subset H$, and $H_1 \neq H_2$. Then $U(g)$ induces a representation $U'(g)$ on the quotient H_1/H_2 in a natural way. The representation $U'(g)$ is said to be deducible from the representation $U(g)$. Harish-Chandra has proved that every unirrep is infinitesimally equivalent to some irreducible representation deducible from some representation $U(g)$ of the above form. Thus it is always possible to construct a bilinear form $(\tilde{f}, \tilde{g})$ in some quotient space H_1/H_2, where $\tilde{f}, \tilde{g} \in H_1/H_2$. One can extend the domain of this bilinear form to all H_1 uniquely by defining $(,)$ to vanish on H_2. Unitarity now means that $(U(g)f, U(g)f) = (f, f),\ f \in H_1,\ g \in G$, and the additional conditions that the bilinear form is a scalar product are hermiticity and positive definiteness $(f, g) = (g, f)^*$ and $(f, f) \geq 0\ \forall\, f, g \in H_1$. It is convenient to extend the domain of the scalar product to the whole space H. Being interested in obtaining all unirreps of $\overline{SL}(n, R)$, we will start with the most general scalar product: $(f, g) = \int\int dk_1 dk_2 f^*(k_1) \kappa(k_1, k_2) g(k_2),\ f, g \in H$, where $\kappa(k_1, k_2)$ is a kernel, the integration is over K, and dk is an invariant measure. The problem of finding all unitary representations of $\overline{SL}(n, R)$ becomes now the problem of finding all scalar products, i.e. kernels for which the representation $U(g)$ is unitary. We start with the most general scalar product of the Hilbert space. We find, by making use of the fact that dk is an invariant measure and of the additivity properties of Spin(n) Wigner's functions following expressions for the scalar product in terms of the matrix elements of the kernel and the expansion coefficients

$$(f, g) = \sum_{\{J\}\{k\}\{k'\}(m)} f^{\{J\}*}_{\{k'\}\{m\}} g^{\{J\}}_{\{k\}\{m\}} \kappa^{\{J\}}_{\{k'\}\{k\}}. \tag{43}$$

The hermiticity of the scalar product yields

$$\kappa^{\{J\}*}_{\{k'\}\{k\}} = \kappa^{\{J\}}_{\{k\}\{k'\}}. \tag{44}$$

Therefore κ is a hermitian matrix and can be diagonalized. Thus without any loss of generality we write κ in the form $\kappa(\{J\}; \{k\})$. The positive definiteness of the scalar product yields

$$\kappa(\{J\}; \{k\}) \geq 0. \tag{45}$$

Finally we find that the hermiticity condition of an arbitrary group generator Q, i.e. the unitarity of the representation, $(f, Qg) = (g, Qf)^*$ reads

$$\kappa(\{J'\}; \{k'\})\langle^{\{J'\}}_{\{k'\}\{m'\}}| \, Q \, |^{\{J\}}_{\{k\}\{m\}}\rangle = \kappa(\{J\}; \{k\})\langle^{\{J\}}_{\{k\}\{m\}}| \, Q \, |^{\{J'\}}_{\{k'\}\{m'\}}\rangle^*. \tag{46}$$

We now substitute in this equations the explicit expressions for the non-compact generators as given by making use of the generalized Gell-Mann formula, and allow the representation labels values to be arbitrary complex numbers, e.g., $\sigma_i = \sigma_{iR} + i\sigma_{iI}$, $i = 1, 2, \ldots, n$, and what is left is to solve above equations and determine all possible solutions for the representation labels σ_i and the corresponding kernels of the scalar products, thus determining all $\overline{SL}(n, R)$ unitary representations. The irreducibility of the representations is most effectively achieved by using the little group technique.

Let us present explicitly the simplest case when the scalar product kernel is given by the Dirac δ function. The kernel matrix elements are now trivial, i.e. $\kappa(\{J\}; \{k\}) = 1$, for all $\{J\}$, $\{k\}$), and the unitarity equations yield $\sigma_i = i\sigma_{iI}$, where σ_{iI} is an arbitrary real number for all $i = 2, 3, \ldots, n$. The corresponding $\overline{SL}(n, R)$ unitary representations constitute the principal series of representations, for which, due to the generalized Gell-Mann formula, we obtained all matrix elements of the non-compact $\overline{SL}(n, R)$ generators.

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VALIDITY OF THE GELL-MANN FORMULA FOR $sl(n,\mathbb{R})$ AND $su(n)$ ALGEBRAS

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The so-called Gell-Mann formula, a prescription designed to provide an inverse to the Inönü–Wigner Lie algebra contraction, has a great versatility and potential value. This formula has no general validity as an operator expression. The question of applicability of Gell-Mann's formula to various algebras and their representations was only partially treated. The validity constraints of the Gell-Mann formula for the case of $sl(n,\mathbb{R})$ and $su(n)$ algebras are clarified, and the complete list of representations spaces for which this formula applies is given. Explicit expressions of the $sl(n,\mathbb{R})$ generators matrix elements are obtained for all these cases in a closed form by making use of the Gell-Mann formula.

Keywords: Gell-Mann decontraction formula; Lie algebra contraction; SL(n) representations; SU(n) representations.

Mathematics Subject Classification 2010: 20C33, 20C40

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1. Introduction

The Gell-Mann formula [1–5] is a prescription aimed to serve as an "inverse" to the Inönü–Wigner contraction [6]. Let a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}, \tag{1}$$

and its Inönü–Wigner contraction $\mathcal{A}' = \mathcal{M} + \mathcal{U}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{U}] \subset \mathcal{U}, \quad [\mathcal{U},\mathcal{U}] = \{0\}, \tag{2}$$

be given. Following a definition that is mathematically less strict but closer to the original formulation, the Gell-Mann formula states that elements $T_\mu \in \mathcal{T}$ can be constructed as the following simple function of the contracted algebra operators $U_\mu \in \mathcal{U}$ and $M_\nu \in \mathcal{M}$:

$$T_\mu = i\frac{\alpha}{\sqrt{U_\nu U^\nu}}[C^2(\mathcal{M}), U_\mu] + i\sigma U_\mu. \tag{3}$$

Here, $C^2(\mathcal{M})$ and $U_\nu U^\nu$ denote the (positive definite) second-order Casimir operators of the $\mathcal{M}$ and $\mathcal{A}'$ algebras, respectively, while α is a normalization constant and σ is an arbitrary parameter. (For a mathematically more strict definition, cf. [1].) The formula was, to our knowledge, first introduced by Dothan and Ne'eman [5], and was advocated by Hermann.

This formula is of a great potential value due to its simplicity and the fact that many aspects of the representation theory are much simpler for the contracted groups/algebras (e.g. construction of representations [7], decompositions of a direct product of representations [2], etc.). However, this formula is valid, on the algebraic level, only in the case of contractions from $\mathcal{A} = so(m+1,n)$ and/or $\mathcal{A} = so(m,n+1)$ to $\mathcal{A}' = iso(m,n)$, with $\mathcal{M} = so(m,n)$ [8, 9]. Moreover, apart from this, the formula is also partially applicable in a broad class of other contractions provided one restricts to some classes of the algebra representations. The validity of Gell-Mann's formula in a weak sense, when an algebra representation requirement is imposed as well, was investigated long ago by Hermann [2, 3]. A partial set of classes of the algebra representations for which the Gell-Mann formula holds is listed [3]. No attempt to make this list exhaustive is made, deliberately concentrating "on what seems to be the simplest situation". This analysis excluded, from the very beginning, the cases of representations where the little group (in Wigner's terminology) is nontrivially represented, not claiming a complete answer even then.

The Gell-Mann formula is especially valuable as a tool in the problem of finding all unitary irreducible representations of the $sl(n,\mathbb{R})$ algebras in spaces over the $SO(n)$ and/or $\mathrm{Spin}(n)$ groups generated by their $so(n)$ subalgebras (applying the formula to contraction of $sl(n,\mathbb{R})$ with respect to subalgebra $so(n)$). Finding representations in the basis of the maximal compact subgroup $SO(n)$ of the $SL(n,\mathbb{R})$ group is mathematically superior, and it suites well various physical applications in particular in nuclear and particle physics, gravity [10], physics of p-branes [11] etc. As an example consider a gauge theory based on the Affine spacetime symmetry $SA(n,\mathbb{R}) = T_n \wedge \overline{SL}(n,\mathbb{R})$; bar denoting the covering group. The gauge covariant derivative, D_α, $\alpha = 0, 1, \ldots, n-1$, as acting on an Affine matter field $\Psi(x)$, is given by,

$$D_\alpha \Psi_A(x) = \left(\partial_\alpha - i\Gamma_\alpha^{ab}(x)\,(Q_{ab})_A^{\ B}\right)\Psi_B(x), \quad Q_{ab} \in sl(n,\mathbb{R}),$$

where $\Gamma_\alpha^{ab}(x)$ are the $sl(n,\mathbb{R})$ connections, and A, B enumerate the matter field components. The matter-gravity vertices require the knowledge of the $sl(n,\mathbb{R})$ operators matrix elements $(Q_{ab})_A^{\ B}$ in the Hilbert space of the matter field components $\{\Psi_A(x)\}$. Operators Q_{ab} naturally split into antisymmetric generators of the compact $SO(n)$ subgroup $M_{ab} = Q_{[ab]}$ and the symmetric, so-called, sheer generators $T_{ab} = Q_{\{ab\}}$. While the matrix elements of the former are well-known, it is generally difficult task to find, for a given $sl(n,\mathbb{R})$ representation, the matrix elements of the latter. In particular, for a generic spinorial $\overline{SL}(n,R)$ matter field, an explicit form of the matrix elements of the $sl(n,\mathbb{R})$ generators, with respect to the Lorentz-like $\mathrm{Spin}(1,n-1)$ subgroup, for infinite-dimensional representation corresponding to the

Ψ field is required. The Gell-Mann formula, in principle, offers a powerful method to describe various representation details (including the matrix elements) in a simple closed analytic form.

Therefore, two obvious questions arise in this context: (i) What is the scope of applicability of the Gell-Mann formula in the $sl(n,\mathbb{R})$ case (i.e. what is the subset of irreducible representations that can be obtained using the formula)? and (ii) Can the formula be somehow generalized, as to account for all $sl(n,\mathbb{R})$ irreducible representations?

Recently [12], we have successfully answered the second question by obtaining a generalized formula of a form similar to that of (3):

$$T_{ab}^{\sigma_2,\ldots,\sigma_n} = i\sum_{m=2}^{n}\frac{1}{2}[C^2(so(m)_K), U_{ab}^{(mm)}] + \sigma_m U_{ab}^{(mm)}, \tag{4}$$

where $C^2(so(m)_K)$ is the second-order Casimir of the $so(m)$ left action subalgebra, $U_{ab}^{(mm)}$ are specifically chosen representations of the Abelian part of the contracted algebra and $\sigma_2, \sigma_3, \ldots, \sigma_n$ are the $sl(n,\mathbb{R})$ representation labels (for more details cf. [12], and a previous analysis [13] of the $n=5$ case). This generalized Gell-Mann formula expression for the noncompact "shear" generators T_{ab} holds for all cases of $sl(n,\mathbb{R})$ irreducible representations.

However, the above solution of the second problem in no way diminishes importance of the first one — i.e. when is the original formula applicable. Apart from mathematical curiosity, this question is of great value since, despite the simple form of the generalization, the original formula still has a number of advantages in applications. First, the summation that appears in the generalized formula certainly renders any practical calculation more complex. More importantly, the generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators $U_{ab}^{(lm)}$ and the so called "left action rotation generators" K_{ab} appearing through $C^2(so(m)_K) = \frac{1}{2}\sum_{a,b=1}^{m}(K_{ab})^2$. Therefore, it is still of a great value to know precisely when the original formula can be applied.

The aim of this paper is to clarify the matters of the original Gell-Mann formula applicability for the class of $sl(n,\mathbb{R})$ algebras contracted with respect to their $so(n)$ maximal compact subalgebras. Note, that owing to a direct connection of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, the conclusions readily convey to the latter case.

In the following, we stick to the notation and mathematical framework of the paper [12]. We briefly restate the minimal due set of these preliminaries in the appendix.

2. Validity of the Gell-Mann Formula

The Gell-Mann formula validity problem is due to the fact that the third commutation relation of (1) is not *a priori* satisfied as an operator relation when the algebra elements are given by expressions (3). In the $sl(n,\mathbb{R})$ case, the $\mathcal{T}$ subspace is spanned by $\frac{1}{2}n(n+1)-1$ shear generators T_μ. These operators transform as a second-order symmetric tensor with respect to Spin(n) subgroup, and, in the Cartesian basis, satisfy:

$$[T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{5}$$

Generally, we use indices from the beginning of the Latin alphabet for Cartesian basis and the Greek indices whenever we want to stress that expression is basis-independent.

To investigate circumstances in which this relation holds, we evaluate the commutator of two shear generators in the framework given in the appendix. In that framework, the Gell-Mann formula (3) reads:

$$T_\mu = i\alpha[C^2(so(n)_K), D^{\square\square}_{w\mu}] + i\sigma D^{\square\square}_{w\mu}, \tag{6}$$

where $C^2(so(n)_K) = \frac{1}{2}\sum_{a,b=1}^{n}(K_{ab})^2$. By making use of this formula, a few algebraic relations and some properties of the Wigner D-functions, after some algebra we obtain:

$$\begin{aligned}
[T_\mu, T_\nu] &= -2\alpha^2\big[K_{\{i}, [K_{j\}}, D^{\square\square}_{w\nu}]\big][K_j, D^{\square\square}_{w\mu}]K_i - (\mu\leftrightarrow\nu)\\
&= -\alpha^2\sum_J\sum_{\lambda,\lambda'}\big(C^{\square\square\,\square\square\,J}_{\mu\;\;\;\nu\;\;\;\lambda} - C^{\square\square\,\square\square\,J}_{\nu\;\;\;\mu\;\;\;\lambda}\big)\\
&\quad\times\Big(2\big(C^2(J) - 2C^2(\square\square)\big)\left\langle\left\langle {}^{J}_{\lambda'}\middle|1\otimes K_i\middle|{}^{\square\square}_{w}\right\rangle\middle|{}^{\square\square}_{w}\right\rangle\\
&\quad+\left\langle\left\langle {}^{J}_{\lambda'}\middle|[1\otimes K_i, C^2(K_{(I+II)})]\middle|{}^{\square\square}_{w}\right\rangle\middle|{}^{\square\square}_{w}\right\rangle\Big)D^J_{\lambda'\lambda}K_i,
\end{aligned} \tag{7}$$

where a summation over repeated Latin indices i and j that label the K generators in any real basis (such that $C^2(K) = K_iK_i$ is assumed). The $C^2(K_{(I+II)})$ operator here denotes the second-order Casimir operator acting in the tensor product of two $\square\square$ representations, i.e. $C^2(K_{(I+II)}) = \sum_i(K_i\otimes 1 + 1\otimes K_i)^2$.

The summation index J in (7) runs over all irreducible representations of the Spin(n) group that appear in the tensor product $\square\square\otimes\square\square$, and λ, λ' count the vectors of these representations. Since all irreducible representations terms, apart those for which the Clebsch–Gordan coefficient $C^{\square\square\,\square\square\,J}_{\mu\;\;\;\nu\;\;\;\lambda}$ is antisymmetric with respect to $\mu\leftrightarrow\nu$ vanish, we are left with only two values that J takes: one corresponding to the antisymmetric second-order tensor $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and the other one corresponding to the representation that we denote as $\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}$. The fact that in the case of $sl(n,\mathbb{R})$ algebras, there is another representation term, in addition to $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, in the antisymmetric product of two $\square\square$ representations (i.e. representations that correspond to Abelian U operators), is *in the root of the Gell-Mann formula validity problem.* Note that in the case of the $so(m+1,n)\to iso(m,n)$, i.e. $so(m,n+1)\to iso(m,n)$ contractions, where the Gell-Mann formula works on the algebraic level, the contracted U operators transform as $\square$ and the antisymmetric product of two such representations certainly belongs to the $\yng(1,1)$ representation (i.e. to the representation that corresponds to $\mathcal{M} = so(m,n)$ subalgebra operators).

The $so(n)$ Casimir operator values satisfy $C^2(\yng(2,2)) = 2C^2(\yng(2)) = 4n$, implying that one of the two terms vanishes in (7) when $J = \yng(2,2)$, leaving us with:

$$\begin{aligned}
\frac{1}{2\alpha^2}[T_\mu, T_\nu] = {}& 4(n+2)\sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\;\;\nu\;\;\lambda} \left\langle\left\langle \yng(1,1)_{\lambda'} \middle| 1\otimes K_i \middle| \yng(2)_{w} \right\rangle \middle| \yng(2)_{w} \right\rangle D^{\yng(1,1)}_{\lambda'\lambda} K_i \\
& - \sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\;\;\nu\;\;\lambda} \left\langle\left\langle \yng(1,1)_{\lambda'} \middle| [1\otimes K_i, C^2(K_{(I+II)})] \middle| \yng(2)_{w} \right\rangle \middle| \yng(2)_{w} \right\rangle D^{\yng(1,1)}_{\lambda'\lambda} K_i \\
& - \sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(2,2)}_{\mu\;\;\nu\;\;\lambda} \left\langle\left\langle \yng(2,2)_{\lambda'} \middle| [1\otimes K_i, C^2(K_{(I+II)})] \middle| \yng(2)_{w} \right\rangle \middle| \yng(2)_{w} \right\rangle D^{\yng(2,2)}_{\lambda'\lambda} K_i,
\end{aligned} \tag{8}$$

where we used that $C^2(\yng(1,1)) = 2n - 4$.

As the coefficient α can be adjusted freely, all that is needed for the Gell-Mann formula to be valid is that (8) is proportional to the appropriate linear combination of the Spin(n) generators, as determined by the Wigner–Eckart theorem, i.e.:

$$[T_\mu, T_\nu] \sim \sum_{\lambda} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\;\;\nu\;\;\lambda} M_\lambda = \sum_{\lambda,i} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\;\;\nu\;\;\lambda} D^{\yng(1,1)}_{i\lambda} K_i. \tag{9}$$

We now analyze these requirements, skipping some straightforward technical details. The third term on the right-hand side in (8), containing D functions of the representation $\yng(2,2)$, is to vanish. Since it is not possible to choose vectors w so that this term vanishes identically as an operator, the remaining possibility is to restrain the space (A.3) to some subspace $V = \{|v\rangle\} \subset \mathcal{L}^2(\mathrm{Spin}(n))$. More precisely, for this term to vanish, there must exist a subalgebra $\mathbf{L} \subset so(n)_K$, spanned by some $\{K_\alpha\}$, such that $K_\alpha \in \mathbf{L} \Rightarrow K_\alpha |v\rangle = 0$. Requiring additionally that this subspace V ought to close under an action of the shear generators, and that the first two terms of (8) ought to yield (9), we arrive at the following two necessary conditions:

(1) The algebra $\mathbf{L}$, must be a symmetric subalgebra of $so(n)$, i.e.

$$[\mathbf{L}, \mathbf{N}] \subset \mathbf{N}, \quad [\mathbf{N}, \mathbf{N}] \subset \mathbf{L}; \quad \mathbf{N} = \mathbf{L}^\perp, \tag{10}$$

(2) The vector $\left|\yng(2)_{w}\right\rangle$ ought to be invariant under the L subgroup action (subgroup of Spin(n) corresponding to $\mathbf{L}$), i.e.

$$K_\alpha \in \mathbf{L} \Rightarrow K_\alpha \left| \yng(2)_{w} \right\rangle = 0. \tag{11}$$

The second necessary condition is satisfied by requiring that the space V is given by Spin$(n)/L$. In Wigner's terminology, this means that L is the little group of the contracted algebra representation, and that necessarily it is to be represented trivially. Besides, the little group is to be an invariant subgroup of the Spin(n)

group. This coincides with one class of the solutions found by Hermann [3]. However, we demonstrated here that there are no other solutions in the $sl(n,\mathbb{R})$ algebra case, in particular, there are no solutions with little group represented nontrivially.

As for the first necessary condition, an inspection of the tables of symmetric spaces, yields two possibilities: $L = \mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$, where $\mathrm{Spin}(1) \equiv 1$, and, for $n = 2k$, $L = U(k)$ (U is the unitary group). However, this second possibility certainly does not imply another solution, since it turns out that there is no vector satisfying the second above property.

Thus, *the only remaining possibility* is as follows,

$$L = \mathrm{Spin}(m) \times \mathrm{Spin}(n-m), \quad m = 1, 2, \ldots, n-1, \quad \mathrm{Spin}(1) \equiv 1. \tag{12}$$

It is rather straightforward, however somewhat lengthy, to show that proportionality of (8) and (9) really holds in this case. The vector $\left|{\square\square \atop w}\right\rangle$ exists, and it is the one corresponding to traceless diagonal $n\times n$ matrix $\mathrm{diag}(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m})$.

3. Special Case: $SL(2,\mathbb{R})$

The analysis accomplished above cannot be applied directly to the $n = 2$ case, thus the $sl(2,\mathbb{R})$ case must be treated separately. The maximal compact subgroup $SO(2)$, that is, its double cover Spin(2), has only one generator M, and therefore it has only one-dimensional irreducible representations. In this case, there are two Abelian generators $U_\pm$ of the contracted group:

$$[M, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 0. \tag{13}$$

Based on these relations, it is easy to verify that the $T_\pm$ operators obtained by the Gell-Mann construction as:

$$T_\pm = i[M^2, U_\pm] + i\sigma U_\pm \tag{14}$$

automatically satisfy the $sl(2,\mathbb{R})$ commutation relation:

$$[T_+, T_-] = -2M. \tag{15}$$

Therefore, we demonstrate that the Gell-Mann formula applies to the $sl(2,\mathbb{R})$ case as well.

4. Matrix Elements

The approach presented in this paper allows us additionally to write down explicitly the matrix elements of the $sl(n,\mathbb{R})$ generators in the cases when the Gell-Mann formula is valid. The possible cases are determined by the numbers n and m. The corresponding representation space (not irreducible in general) is the one over the coset space $\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$. The proportionality factor α is determined to be:

$$\alpha = \frac{1}{2}\sqrt{\frac{m(n-m)}{n}}, \tag{16}$$

and, in a matrix notation for $\square\square$ representation:

$$\left| \begin{matrix} \square\square \\ w \end{matrix} \right\rangle = \sqrt{\frac{m(n-m)}{n}} \mathrm{diag}\left(\frac{1}{m}, \dots, \frac{1}{m}, -\frac{1}{n-m}, \dots, -\frac{1}{n-m}\right). \tag{17}$$

The Gell-Mann formula (3), (6), and the matrix representation of the contracted Abelian generators U (A.5) yield:

$$\left\langle \begin{matrix} J' \\ m' \end{matrix} \middle| T_\mu \middle| \begin{matrix} J \\ m \end{matrix} \right\rangle$$
$$= i\sqrt{\frac{m(n-m)}{4n}}\sqrt{\frac{\dim(J)}{\dim(J')}}(C^2(J') - C^2(J) + \sigma)C^{J\,\square\square\,J'}_{0\;\;0\;\;0}\,C^{J\,\square\square\,J'}_{m\;\;\mu\;\;m'}. \tag{18}$$

The zeroes in the indices of Clebsch–Gordan coefficients here denote vectors that are invariant with respect to $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$ transformations (in that spirit $\left|\begin{matrix} \square\square \\ w \end{matrix}\right\rangle = \left|\begin{matrix} \square\square \\ 0 \end{matrix}\right\rangle$). In the formula (18), the space reduction from $\mathcal{L}^2(\mathrm{Spin}(n))$ to $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m))$ implies a reduction of the basis (A.3), i.e. $\left|\begin{matrix} J \\ 0\,m \end{matrix}\right\rangle \to \left|\begin{matrix} J \\ m \end{matrix}\right\rangle$ (only the vectors invariant with respect to left $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$ action remain).

The expression (18), together with the action of the $\mathrm{Spin}(n)$ generators (A.4) provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, that is labeled by a free parameter σ. Such representations are multiplicity-free with respect to the maximal compact $\mathrm{Spin}(n)$ subgroup, and all of them are *a priori* tensorial. One can obtain from these representations, for certain σ parameter values, the $sl(n,\mathbb{R})$ spinorial representations as well as by explicitly evaluating the Clebsch–Gordan coefficient and performing an appropriate analytic continuation in terms of the $\mathrm{Spin}(n)$ labels.

5. Conclusion

In this paper, we clarified the issue of the Gell-Mann formula validity for the $sl(n,\mathbb{R}) \to r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra contraction. We have shown that the only $sl(n,\mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$, $m = 1, 2, \dots, n-1$. Moreover, by making use of the Gell-Mann formula in these spaces, we have obtained a closed form expressions of all irreducible representations matrix elements of the noncompact operators generating $SL(n,\mathbb{R})/SO(n)$ cosets. The matrix elements of both compact and noncompact operators of the $sl(n,\mathbb{R})$ algebra are given by (A.4) and

(18), respectively. In particular, it turns out that, due to Gell-Mann's formula validity conditions, no representations with $so(n)$ subalgebra representations multiplicity can be obtained in this way. Moreover, the matrix expressions of the noncompact operators as given by (18) do not account *a priori* for the $sl(n,\mathbb{R})$ spinorial representations. An explicit construction of spinorial representations requires an additional analytic continuation of the matrix elements explicit expressions to half-integer values of the representation labels. Due to mutual connection of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, the results of this paper apply to the corresponding $su(n)$ case as well. The $SU(n)/SO(n)$ generators differ from the corresponding $sl(n,\mathbb{R})$ operators by the imaginary unit multiplicative factor, while the spinorial representations issue in the $su(n)$ case is pointless due to the fact that the $SU(n)$ is a simply connected (there exists no double cover) group.

In many physics applications (e.g. those in [18]) one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present work, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(\mathrm{Spin}(n),\kappa)$ of square integrable functions with a scalar product given in terms of an arbitrary kernel κ, and to impose the unitarity constraints both on the scalar products itself and on the noncompact operators matrix elements in that scalar product (cf. [19]). The simplest series of the $sl(n,\mathbb{R})$ unitary irreducible representations, the Principal series, of the representations constructed above are obtained when $\sigma = i\sigma_I$, $\sigma_I \in \mathbb{R}\backslash\{0\}$, i.e. when σ takes an arbitrary nonzero pure imaginary value.

To conclude, we obtained recently a representation dependent generalization of the Gell-Mann formula for all $sl(n,\mathbb{R})$ algebras [12] to cover the cases of representations with nontrivial multiplicity. The $sl(n,\mathbb{R})$ noncompact operators representations obtained in that work together with the results of this work cover all $sl(n,\mathbb{R})$ representation cases.

Appendix A

In this paper, rather than following the approach of Hermann [3], we follow our approach of [12]. That is, we work in the representation space of square integrable functions $\mathcal{L}^2(\mathrm{Spin}(n))$, over the maximal compact subgroup $\mathrm{Spin}(n)$, i.e. the $SO(n)$ universal covering group, with a standard invariant Haar measure. This representation space is large enough to provide for all inequivalent irreducible representations of the contracted group, and, by a theorem of Harish-Chandra [14–17], is also rich enough to contain representatives from all equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra, representations.

The generators of the contracted group are generically represented, in this space, as follows. The $so(n)$ subalgebra operators act, in a standard way:

$$M_{ab}|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}|\phi\rangle,$$

where action of a Spin(n) element g' on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(\mathrm{Spin}(n))$ is given via action from the left on basis vectors $|g\rangle$ of this space:

$$g'|\phi\rangle = g' \int \phi(g)|g\rangle dg = \int \phi(g)|g'g\rangle dg, \quad g', g \in \mathrm{Spin}(n). \tag{A.1}$$

The contracted noncompact Abelian operators $U_\mu\,(2,3)$, act in the same basis as multiplicative Wigner-like D-functions (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_\mu \to |u| D^{\square\square}_{w\mu}(g^{-1}) \equiv |u| \left\langle \begin{matrix}\square\square\\ w\end{matrix} \right| \left(D^{\square\square}(g)\right)^{-1} \left| \begin{matrix}\square\square\\ \mu\end{matrix} \right\rangle, \tag{A.2}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and $\square\square$ denoting (in a parallel to the Young tableaux) the symmetric second-order tensor representation of $SO(n)$. The norm $|u|$ parametrizes representation of U, but will turn out to be irrelevant in our case, as it cancels with the denominator in (3). The $\left|\begin{matrix}\square\square\\ \mu\end{matrix}\right\rangle$ vector from representation $\square\square$ space is denoted by the index of the operator U_μ, whereas the vector $\left|\begin{matrix}\square\square\\ w\end{matrix}\right\rangle$ can be an arbitrary vector belonging to $\square\square$ (the choice of w determines, in Wigner terminology, the little group of the representation in question). Taking an inverse of g in (A.2) insures the correct transformation properties.

A natural discrete orthonormal basis in the $\mathcal{L}^2(\mathrm{Spin}(n))$ space is given by properly normalized Wigner D-functions:

$$\left\{ \left| \begin{matrix} J \\ k\,m \end{matrix} \right\rangle \equiv \int \sqrt{\dim(J)} D^J_{km}(g^{-1}) dg\, |g\rangle \right\}, \quad \left\langle \begin{matrix} J \\ k\,m \end{matrix} \middle| \begin{matrix} J' \\ k'\,m' \end{matrix} \right\rangle = \delta_{JJ'}\delta_{kk'}\delta_{mm'}, \tag{A.3}$$

where dg is an (normalized) invariant Haar measure. Here, J stands for a set of Spin(n) irreducible representation labels, while the k and m labels numerate the representation basis vectors.

An action of the $so(n)$ operators in this basis is well-known, and it can be written in terms of the Clebsch–Gordan coefficients of the Spin(n) group as follows,

$$\left\langle \begin{matrix} J' \\ k'\,m' \end{matrix} \right| M_{ab} \left| \begin{matrix} J \\ k\,m \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\, C^{J\;\begin{smallmatrix}\square\\\square\end{smallmatrix}\;J'}_{m\,(ab)\,m'}\,. \tag{A.4}$$

The matrix elements of the U_μ operators in this basis are readily found to read:

$$\begin{aligned} &\left\langle \begin{matrix} J' \\ k'\,m' \end{matrix} \right| U^{(w)}_\mu \left| \begin{matrix} J \\ k\,m \end{matrix} \right\rangle \\ &\quad = |u| \left\langle \begin{matrix} J' \\ k'\,m' \end{matrix} \right| D^{-1\,\square\square}_{w\mu} \left| \begin{matrix} J \\ k\,m \end{matrix} \right\rangle = |u| \sqrt{\frac{\dim(J)}{\dim(J')}}\, C^{J\,\square\square\,J'}_{k\;\,w\;\,k'}\, C^{J\,\square\square\,J'}_{m\;\,\mu\;\,m'}\,. \end{aligned} \tag{A.5}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ (a semidirect sum of a $\frac{n(n+1)}{2} - 1$-dimensional Abelian algebra and $so(n)$) representations is thus explicitly given in this space by (A.4) and (A.5).

Moreover, we introduce the so-called, left action generators K as:

$$K_\mu \equiv g^{\nu\lambda} D^{\square}_{\mu\nu} M_\lambda, \tag{A.6}$$

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$. The K_μ operators behave exactly as the rotation generators M_μ, it is only that they act on the lower left-hand side indices of the basis (A.3):

$$\langle K_{ab} \rangle = \left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \right| K_{ab} \left| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\; C^{\,J\;\square\;J'}_{\,k\,(ab)\,k'}\,. \tag{A.7}$$

The operators K_μ and M_μ mutually commute. However, the corresponding Casimir operators match and, in particular, we will use $\sum K_\mu^2 = \sum M_\mu^2$ in the expression for the Gell-Mann formula (3).

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Positive Energy Unitary Irreducible Representations of the Superalgebra osp$(1|8,\mathbb{R})$

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Abstract. We continue the study of positive energy (lowest weight) unitary irreducible representations of the superalgebras osp$(1|2n,\mathbb{R})$. We present the full list of these UIRs. We give a proof of the case osp$(1|8,\mathbb{R})$,

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. This makes the classification of the UIRs of these superalgebras very important. Until recently only those for $D \leqslant 6$ were studied since in these cases the relevant superconformal algebras satisfy [**1**] the Haag–Lopuszanski–Sohnius theorem [**2**]. Thus, such classification was known only for the $D = 4$ superconformal algebras su$(2,2/N)$ [**3**] (for $N = 1$), [**4, 5, 6, 7**] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1,2$) was given in [**8**] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [**9**].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras osp$(1|2n)$. Initially, the superalgebra osp$(1|32)$ was put forward for $D = 10$ [**10**]. Later it was realized that osp$(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [**11**]. In all cases we need to find first the UIRs of osp$(1|2n,\mathbb{R})$ which study was started in [**12**] and [**13**]. Later, in [**14**] we finalized the UIR classification of [**12**] as Dobrev-Zhang-Salom (DZS) Theorem. In [**14**] we proved the DZS Theorem for osp$(1|6)$.

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In the present paper, we prove the DZS Theorem for osp$(1|8)$. For the lack of space we refer for extensive literature on the subject in [**12, 14**].

2. Preliminaries on representations

Our basic references for Lie superalgebras are [**15, 16**], although in this exposition we follow [**12**].

The even subalgebra of $\mathcal{G} = \mathrm{osp}(1|2n,\mathbb{R})$ is the algebra $\mathrm{sp}(2n,\mathbb{R})$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong \mathrm{su}(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature

$$\chi \ = \ [\, d\, ;\, a_1\, , \dots , a_{n-1}\,] \tag{2.1}$$

where d is the conformal weight, and $a_1, \dots, a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $\mathrm{su}(n)$ (the simple part of $\mathcal{K}$).

In [**12**] were classified (with some omissions to be spelled out below) the positive energy (lowest weight) UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [**4, 5, 6, 7, 9**], resp. The main tool was an adaptation of the Shapovalov form [**17**] on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} = \mathrm{osp}(1|2n)$ of $\mathcal{G}$.

We recall some facts about $\mathcal{G}^{\mathbb{C}} \ = \ \mathrm{osp}(1|2n)$ (denoted $B(0,n)$ in [**15**]) as used in [**12**]. The root systems are given in terms of $\delta_1 \dots, \delta_n$, $(\delta_i, \delta_j) \ = \ \delta_{ij}$, $i, j = 1, \dots, n$. The even and odd roots systems are [**15**]

$$\Delta_{\bar{0}} = \{\pm\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ \pm 2\delta_i,\ 1 \leqslant i \leqslant n\}, \quad \Delta_{\bar{1}} = \{\pm\delta_i,\ 1 \leqslant i \leqslant n\}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [**15**] $\Pi = \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n\}$, or, introducing standard notation for the simple roots,

$$\Pi = \{\alpha_1, \dots, \alpha_n\}, \quad \alpha_j = \delta_j - \delta_{j+1},\ j = 1, \dots, n-1, \quad \alpha_n = \delta_n .$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is

$$\underset{1}{\circ} - \cdots - \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet}$$

The black dot is used to signify that the simple odd root is not nilpotent. In fact, the superalgebras $B(0,n) = \mathrm{osp}(1|2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [**15**].

The corresponding to Π positive root system is

$$\Delta^+_{\bar{0}} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ 2\delta_i,\ 1 \leqslant i \leqslant n\}, \quad \Delta^+_{\bar{1}} = \{\delta_i,\ 1 \leqslant i \leqslant n\} \tag{2.2}$$

We record how the elementary functionals are expressed through the simple roots:

$$\delta_k = \alpha_k + \cdots + \alpha_n .$$

From the point of view of representation theory, more relevant is the restricted root system, such that

$$\bar{\Delta}^+ = \bar{\Delta}^+_{\bar{0}} \cup \Delta^+_{\bar{1}}, \quad \bar{\Delta}^+_{\bar{0}} \equiv \{\alpha \in \Delta^+_{\bar{0}} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar{1}}\} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n\}$$

The superalgebra $\mathcal{G} = \mathrm{osp}(1|2n,\mathbb{R})$ is a split real form of $\mathrm{osp}(1|2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram

$$\underset{1}{\circ} - \cdots - \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ}$$

Naturally, for the B_n positive root system we drop the roots $2\delta_i$

$$\Delta^+_{\mathrm{B}_n} = \{\delta_i \pm \delta_j,\ 1 \leqslant i < j \leqslant n;\ \delta_i,\ 1 \leqslant i \leqslant n\} \cong \bar{\Delta}^+$$

This shall be used essentially below.

Besides (2.1), we shall use the Dynkin-related labelling:

$$(\Lambda, \alpha_k^\vee) = -a_k,\ 1 \leqslant k \leqslant n,$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and the minus signs are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [**16**]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [**6, 12**]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is

$$a_n = -2d - a_1 - \cdots - a_{n-1}.$$

The actual Dynkin labelling is given by $m_k = (\rho - \Lambda,\ \alpha_k^\vee)$ where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (2.2)

$$\begin{aligned} \rho \doteq \rho_{\bar{0}} - \rho_{\bar{1}} &= (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n, \\ \rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n, \\ \rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n). \end{aligned}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho,\ \alpha_i^\vee) = 1$, $i = 1, \ldots, n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$, the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ (for all k) the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$ (and of B_n).

Having in hand the values of Λ on the basis, we can recover them for any element of $\mathcal{H}^*$. We shall need only $(\Lambda, \beta^\vee)$ for all positive roots β as given in [**12**]

$$\begin{aligned} (\Lambda, (\delta_i - \delta_j)^\vee) &= (\Lambda, \delta_i - \delta_j) = -a_i - \cdots - a_{j-1} \\ (\Lambda, (\delta_i + \delta_j)^\vee) &= (\Lambda, \delta_i + \delta_j) = 2d + a_1 + \cdots + a_{i-1} - a_j - \cdots - a_{n-1} \\ (\Lambda, \delta_i^\vee) &= (\Lambda, 2\delta_i) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1} \\ (\Lambda, (2\delta_i)^\vee) &= (\Lambda, \delta_i) = d + \tfrac{1}{2}(a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1}) \end{aligned} \tag{2.3}$$

To introduce Verma modules we use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus cG^-$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$ where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that

$$Zv_0 = 0, \;\; Z \in \mathcal{G}^-; \qquad Hv_0 = \Lambda(H)v_0, \;\; H \in \mathcal{H}.$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\,v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [**16**] (which generalizes the BGG-criterion [**18**] to the super case) to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [**12**]

$$(\rho - \Lambda,\, \beta^\vee) = m_\beta, \qquad \beta \in \Delta^+, \quad m_\beta \in \mathbb{N}. \tag{2.4}$$

If a condition from (2.4) is fulfilled, then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m,β: $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the *singular vector* $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions

$$\begin{aligned} X v_s^{m,\beta} &= 0, \quad X \in \mathcal{G}^-, \\ H v_s^{m,\beta} &= \Lambda'(H) v_0, \quad H \in \mathcal{H}, \;\; \Lambda' = \Lambda + m\beta. \end{aligned}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [**19, 6**]

$$v_s^{m,\beta} = P^{m,\beta} v_0, \quad P^{m,\beta} \in U(\mathcal{G}^+).$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)P^{m,\beta}v_0$.

Note that the Casimirs of $\mathcal{G}^\mathbb{C}$ take the same values on V^Λ and $V^{\Lambda'}$.

Certainly, (2.4) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (2.4) is fulfilled, and let us denote $\tilde{I}^\Lambda \equiv \bigcup_{\beta \in \Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda / \tilde{I}^\Lambda$.

Further we shall use also the following notion. The singular vector v_1 is called *descendant* of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 = P_{12}v_2$. Clearly, in this case we have: $I^1 \subset I^2$ where I^k is the submodule generated by v_k.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [**16, 18**]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda / I^\Lambda$. (If V^Λ is irreducible, then $L_\Lambda = V^\Lambda$.)

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if $\operatorname{rank}\mathcal{G} \leqslant 2$, or when (2.4) is fulfilled for all simple roots (and, as a consequence, for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition 2.1. [**18, 20, 21**] Let V^Λ be a reducible Verma module. A vector $v_{\rm ssv} \in V^\Lambda$ is called a *subsingular vector* if $v_{\rm su} \notin \tilde{I}^\Lambda$ and $Xv_{\rm su} \in \tilde{I}^\Lambda$, for all $X \in \mathcal{G}^-$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [**12**] it was established that from (2.4) follows that the Verma module $V^{\Lambda(\chi)}$ is reducible if one of the following relations holds (following the order of (2.3

$$\mathbb{N} \ni m^-_{ij} = j - i + a_i + \cdots + a_{j-1} \tag{2.5a}$$

$$\mathbb{N} \ni m^+_{ij} = 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d \tag{2.5b}$$

$$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d \tag{2.5c}$$

$$\mathbb{N} \ni m_{ii} = n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d\ . \tag{2.5d}$$

Further we shall use the fact from [**12**] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (2.5d) is fulfilled also (2.5c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j. Naturally, $X^-_j \in \mathcal{G}^-$ corresponds to $-\alpha_j$.

Further, we notice that all reducibility conditions in (2.5a) are fulfilled. In particular, for the simple roots from those condition, (2.5a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, \ldots, n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i} v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ, we shall consider also the factor-modules $F^\Lambda_c = V^\Lambda / I^\Lambda_c \supset F^\Lambda$. We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c, i.e., $(X^+_i)^{1+a_i}|\Lambda_c\rangle = 0$, $i = 1, \ldots, n-1$.

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (2.5b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (2.5b) are obtained for $m^+_{ij} = 1$ and they are $d_{ij} \equiv n + \frac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j)$, the corresponding root being $\delta_i + \delta_j$.

The maximal d coming from the different possibilities in (2.5c,d), resp., are obtained for $m_i = 1$, $m_{ii} = 1$, resp., and they are:

$$d_i \equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}), \quad d_{ii} = d_i - \tfrac{1}{2},$$

the corresponding roots being δ_i, $2\delta_j$, resp.

There are some orderings between these maximal reduction points [**12**]:

$$\begin{aligned} d_1 &> d_2 > \cdots > d_n, \\ d_{i,i+1} &> d_{i,i+2} > \cdots > d_{in}, \\ d_{1,j} &> d_{2,j} > \cdots > d_{j-1,j}, \\ d_i &> d_{jk} > d_\ell, \qquad i \leqslant j < k \leqslant \ell. \end{aligned} \tag{2.6}$$

Obviously the first reduction point is

$$d_1 = n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}).$$

3. Unitarity

The first results on the unitarity were given in [**12**], and then improved in [**14**]. Thus, the statement below should be called Dobrev–Zhang–Salom Theorem.

THEOREM DSZ 1. *All positive energy unitary irreducible representations of the superalgebras* $\mathrm{osp}(1|2n,\mathbb{R})$ *characterized by the signature* χ *in* (2.1) *are obtained for real* d *and are given as follows:*

$$\begin{aligned}
&d \geqslant n-1+\tfrac{1}{2}(a_1+\cdots+a_{n-1}) = d_1, \quad a_1 \neq 0,\\
&d \geqslant n-\tfrac{3}{2}+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_{12}, \quad a_1 = 0,\ a_2 \neq 0,\\
&d = n-2+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_2 > d_{13}, \quad a_1 = 0,\ a_2 \neq 0,\\
&d \geqslant n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_2 = d_{13}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
&d = n-\tfrac{5}{2}+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_{23} > d_{14}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
&d = n-3+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_3 = d_{24} > d_{15}, \quad a_1 = a_2 = 0,\ a_3 \neq 0,\\
&\ \vdots\\
&d \geqslant n-1-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
&\qquad\qquad \kappa = \tfrac{1}{2}, 1, \ldots, \tfrac{1}{2}(n-1),\\
&d = n-\tfrac{3}{2}-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
&\ \vdots\\
&d = n-1-2\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}), \quad a_1 = \cdots = a_{2\kappa} = 0,\ a_{2\kappa+1} \neq 0,\\
&\ \vdots\\
&d \geqslant \tfrac{1}{2}(n-1), \quad a_1 = \cdots = a_{n-1} = 0\\
&d = \tfrac{1}{2}(n-2), \quad a_1 = \cdots = a_{n-1} = 0\\
&\ \vdots\\
&d = \tfrac{1}{2}, \quad a_1 = \cdots = a_{n-1} = 0\\
&d = 0, \quad a_1 = \cdots = a_{n-1} = 0
\end{aligned}$$

where the last case is the trivial one-dimensional irrep.

The theorem was partially proved [**12**], while in [**14**] was given a sketch of a proof, and the case $n = 3$ was proved. We are going to give a proof for $\mathrm{osp}(1|8)$.

4. The case of $\mathrm{osp}(1|8)$

For $n = 4$ formula (2.6) simplifies to

$$\begin{array}{c}
d_1 > d_{12} > d_2 > d_{23} > d_3 > d_{34} > d_4\\
\hookrightarrow > d_{13} > \nearrow \hookrightarrow > d_{24} > \nearrow\\
\hookrightarrow > d_{14} > \nearrow
\end{array}$$

In the case of $\mathrm{osp}(1|8)$ Theorem DSZ reads:

THEOREM 4.1. *All positive energy unitary irreducible representations of the superalgebras* $\mathrm{osp}(1|8,\mathbb{R})$ *characterized by the signature* χ *in* (2.1) *are obtained for real* d *and are given as follows*

$$d \geqslant 3+\tfrac{1}{2}(a_1+a_2+a_3) = d_1, \quad a_1 \neq 0,$$

$$d \geqslant \tfrac{5}{2} + \tfrac{1}{2}(a_2 + a_3) = d_{12}, \quad a_1 = 0,\ a_2 \neq 0,$$
$$d = 2 + \tfrac{1}{2}(a_2 + a_3) = d_2 > d_{13}, \quad a_1 = 0,\ a_2 \neq 0,$$
$$d \geqslant 2 + \tfrac{1}{2}a_3 = d_2 = d_{13}, \quad a_1 = a_2 = 0,\ a_3 \neq 0$$
$$d = \tfrac{3}{2} + \tfrac{1}{2}a_3 = d_{23} > d_{14}, \quad a_1 = a_2 = 0,\ a_3 \neq 0$$
$$d = 1 + \tfrac{1}{2}a_3 = d_3 > d_{24}, \quad a_1 = a_2 = 0,\ a_3 \neq 0$$
$$d \geqslant \tfrac{3}{2} = d_{23} = d_{14}, \quad a_1 = a_2 = a_3 = 0$$
$$d = 1 = d_3 = d_{24}, \quad a_1 = a_2 = a_3 = 0$$
$$d = \tfrac{1}{2} = d_{34}, \quad a_1 = a_2 = a_3 = 0,$$
$$d = 0 = d_4, \quad a_1 = a_2 = a_3 = 0$$

where the last case is the trivial one-dimensional irrep.

Proof. For $d > d_1$ there are no singular vectors and we have unitarity. At $d = d_1$ there is a singular vector of weight $\delta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ [**22, 12**]:

$$v^1_{\delta_1} = \sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\sum_{k_3=0}^{1} b_{k_1,k_2,k_3} (X_1^+)^{1-k_1}(X_2^+)^{1-k_2}(X_3^+)^{1-k_3} \times X_4^+ (X_3^+)^{k_3}(X_2^+)^{k_2}(X_1^+)^{k_1} v_0 \equiv \mathcal{P}^{1,\delta_1} v_0, \tag{4.1}$$

$$b_{k_1,k_2,k_3} = (-1)^{k_1+k_2+k_3}(a_1 + k_1)\frac{2 + a_1 + a_2}{1 + a_1 + a_2 - k_2}\,\frac{3 + a_1 + a_2 + a_3}{3 + a_1 + a_2 + a_3 - k_3}$$

where $H^s = \hat{H}_1 + \hat{H}_2 + \cdots + \hat{H}_s$, and a basis in terms of simple root vectors only is used. This singular vector is nontrivial for $a_1 \neq 0$ and must be eliminated to obtain an UIR. Below $d < d_1$ this vector is not singular but has negative norm and thus there is no unitarity for $a_1 \neq 0$. On the other hand for $a_1 = 0$ and any d the vector (4.1) is descendant of the compact root singular vector $X_1^+ v_0$ which is already factored out for $a_1 = 0$.

Thus, below we discuss only the cases with $a_1 = 0$ in which case we have unitarity for $d > d_{12} = \frac{5}{2} + \frac{1}{2}(a_2 + a_3)$. Then at the next reducibility point $d = d_{12}$, we have a singular vector corresponding to the root $\delta_1 + \delta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$ which is given by

(4.2)
$$\begin{aligned}
v^1_{\delta_1+\delta_2} = {} & \frac{1}{2 + 2a_2 + a_3} \times \\
\Big(& -\tfrac{1}{2}(Y_4 Y_3 X_3^+ (X_2^+)^2 X_1^+) - \tfrac{1}{4}(Y_4^2 (X_3^+)^2 (X_2^+)^2 X_1^+) + (Y_4^2 X_3^+ X_{23}^+ X_2^+ X_1^+) a_2 \\
& - 2(Y_4 Y_2 X_{23}^+ X_1^+) a_2 (a_2 + 1) - (Y_4^2 X_{23}^+ X_{23}^+ X_1^+) a_2 (a_2 + 1) \\
& - (Y_4 Y_3 X_{23}^+ X_2^+ X_1^+)(a_3 + 2) - 2(Y_3 Y_2 X_2^+ X_1^+)(a_2 + a_3 + 1)(a_2 + a_3 + 2) \\
& - (Y_3^2 (X_2^+)^2 X_1^+)(a_2 + a_3 + 1)(a_2 + a_3 + 2) \\
& - 4(Y_2^2 X_1^+) a_2 (a_2 + 1)(a_2 + a_3 + 1)(a_2 + a_3 + 2) \\
& + (Y_{23} X_2^+ X_1^+)(2a_2+1)(a_2+a_3+1)(a_2+a_3+2) + (Y_4 Y_2 X_3^+ X_2^+ X_1^+)(2a_2+a_3+2)
\end{aligned}$$

$$+\tfrac{1}{4}(Y_{34}X_3^+(X_2^+)^2X_1^+)2a_2+2a_3+3)+(Y_{24}X_{23}^+X_1^+)a_2(a_2+1)(2a_2+2a_3+3)$$
$$+\tfrac{1}{2}(Y_{34}X_{23}^+X_2^+X_1^+)(a_3-2a_2(a_2+a_3+1)+2)$$
$$-\tfrac{1}{2}(Y_{24}X_3^+X_2^+X_1^+)(a_3+2a_2(a_2+a_3+2)+2)$$
$$+(a_2+1)(a_2+a_3+2)$$
$$\times\big(2(Y_4Y_3X_{13}^+X_2^+)-(Y_{34}X_{13}^+X_2^+)-2(Y_4Y_3X_{23}^+X_{12}^+)+(Y_{34}X_{23}^+X_{12}^+)$$
$$+2(Y_4Y_2X_3^+X_{12}^+)-(Y_{24}X_3^+X_{12}^+)-2(Y_4Y_1X_3^+X_2^+)+(Y_{14}X_3^+X_2^+)\big)$$
$$+a_2(a_2+1)(a_2+a_3+2)\big(-4(Y_4Y_2X_{13}^+)+2(Y_{24}X_{13}^+)+4(Y_4Y_1X_{23}^+)-2(Y_{14}X_{23}^+)\big)$$
$$+(a_2+1)(a_2+a_3+1)(a_2+a_3+2)\times$$
$$\big(-4(Y_3Y_2X_{12}^+)+2(Y_{23}X_{12}^+)+4(Y_3Y_1X_2^+)-2(Y_{13}X_2^+)-8(Y_2Y_1)a_2+4a_2Y_{12}\big)\Big)v_0$$

where the root vector X_{jk}^+ corresponds to the compact root $\delta_j-\delta_{k+1}=\alpha_j+\alpha_{j+1}+\dots+\alpha_k$, Y_k corresponds to the odd (noncompact) root $\delta_k=\alpha_k+\alpha_{k+1}+\dots+\alpha_n$, (thus $Y_4\equiv X_4^+$), Y_{jk} corresponds to the even noncompact root $\delta_j+\delta_k$. In (4.2) it is more convenient to use a PBW type of basis with the compact roots $X_{\dots}^+$ to the right of the noncompact roots $Y_{\dots}$. The norm of (4.2) is

$$64a_2(a_2+1)^2(a_2+2)(a_2+a_3+1)(a_2+a_3+2)^2(a_2+a_3+3)$$
$$\times(-2d+a_2+a_3+4)(-2d+a_2+a_3+5)/(2a_2+a_3+2)^2.$$

For $d=d_{12}$, $a_1=0$, $a_2\neq0$ the singular vector (4.2) is nontrivial and gives rise to a invariant subspace which must be factored out for unitarity. For $d<\frac{5}{2}+\frac{1}{2}(a_2+a_3)$, the vector (4.2) is not singular, but has negative norm and there is no unitarity for $a_2\neq0$, except at the isolated unitary point $d=2+\frac{1}{2}(a_2+a_3)=d_2>d_{13}$ where the vector (4.2) has zero norm and can not spoil the unitarity. For that value of d there is a singular vector $v^1_{\delta_2}$ of weight $\delta_2=\alpha_2+\alpha_3+\alpha_4$ [**22, 12**]:

$$v^1_{\delta_2}=\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}b_{k_1,k_2}(X_2^+)^{1-k_1}(X_3^+)^{1-k_2}\times X_4^+(X_3^+)^{k_2}(X_2^+)^{k_1}v_0\equiv\mathcal{P}^{1,\delta_2}v_0, \tag{4.3}$$

$$b_{k_1,k_2}=(-1)^{k_1+k_2}\frac{a_2+k_1}{1+a_2+a_3-k_2}$$

which has to be factored out for unitarity for $a_2\neq0$, while for $a_2=0$ it is descendant of the compact vector $X_2^+v_0$.

Overall no further unitarity is possible for $a_2\neq0$, thus below we consider only the cases $a_1=a_2=0$. Then the singular vectors above are descendants of compact root singular vectors $X_1^+v_0$ and $X_2^+v_0$, thus, there is no obstacle for unitarity for $d>2+\frac{1}{2}a_3=d_2=d_{13}$ (for $a_1=a_2=0$). The next reducibility point is $d=d_{13}=d_2$. The singular vector for $d=d_{13}$ and $m=1$ has weight $\delta_1+\delta_3=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$:

$$v^1_{\delta_1+\delta_3}=$$
$$\Big(-4a_1(Y_4Y_3X_3^+X_{12}^+)-2a_1(Y_4^2(X_3^+)^2X_{12}^+)-2(a_1+a_2+1)(Y_4^2X_3^+X_{23}^+X_1^+)$$
$$+4a_1(a_1+a_2+1)(Y_4^2X_3^+X_{13}^+)+4(a_3+1)(Y_4Y_3X_{23}^+X_1^+)-8a_1(a_3+1)(Y_4Y_3X_{13}^+)$$

$$
\begin{aligned}
&- 4(a_1 + a_2 + a_3 + 2)(Y_4Y_2X_3^+X_1^+) + 8a_1(a_1 + a_2 + a_3 + 2)(Y_4Y_1X_3^+)\\
&+ 8a_3(a_1 + a_2 + a_3 + 2)(Y_3Y_2X_1^+) + 4a_3(a_1 + a_2 + a_3 + 2)(Y_3^2X_2^+X_1^+)\\
&- 8a_1a_3(a_1 + a_2 + a_3 + 2)(Y_3^2X_{12}^+) + 2(a_1(a_3 - 1) + a_2(a_3 - 1) - 2)(Y_{34}X_{23}^+X_1^+)\\
&- 4a_1(a_1(a_3 - 1) + a_2(a_3 - 1) - 2)(Y_{34}X_{13}^+)\\
&+ 2(a_1 + a_2 + 2)(a_1 + a_2 + a_3 + 2)(Y_{24}X_3^+X_1^+)\\
&- 4a_1(a_1 + a_2 + 2)(a_1 + a_2 + a_3 + 2)(Y_{14}X_3^+)\\
&- 4(a_1 + a_2 + 2)a_3(a_1 + a_2 + a_3 + 2)(Y_{23}X_1^+)\\
&- (a_1 + a_2 + 2a_3 + 2)(Y_{34}X_3^+X_2^+X_1^+) + 2a_1(a_1 + a_2 + 2a_3 + 2)(Y_{34}X_3^+X_{12}^+)\\
&+ 8a_1(a_1 + a_2 + 2)a_3(a_1 + a_2 + a_3 + 2)Y_{13} - 16a_1a_3(a_1 + a_2 + a_3 + 2)(Y_3Y_1)\\
&\qquad\qquad + 2(Y_4Y_3X_3^+X_2^+X_1^+) + Y_4^2(X_3^+)^2X_2^+X_1^+\Big)v_0.
\end{aligned}
$$

For $a_1 = a_2 = 0$ it is descendant of the compact root singular vector $X_1^+v_0$. However, there is a subsingular vector

$$
\begin{aligned}
&v^{ss}_{2,13} =\\
&\Big(2a_3(Y_{23}Y_1) - 2a_3(Y_{13}Y_2) + 2a_3(Y_3(Y_{12})) - 4a_3(Y_3Y_2Y_1) + 2(Y_4Y_3Y_2X_{13}^+)\\
&- Y_{34}Y_2X_{13}^+ + Y_{24}Y_3X_{13}^+ - Y_4Y_{23}X_{13}^+ - 2(Y_4Y_3Y_1X_{23}^+) + Y_{34}Y_1X_{23}^+ - Y_{14}Y_3X_{23}^+\\
&\qquad + Y_4Y_{13}X_{23}^+ + 2(Y_4Y_2Y_1X_3^+) - Y_{24}Y_1X_3^+ + Y_{14}Y_2X_3^+ - Y_4(Y_{12})X_3^+\Big)v_0
\end{aligned}
\tag{4.4}
$$

with the norm $-16a_3(a_3 + 3)(-2d + a_3 + 2)(-2d + a_3 + 3)(-2d + a_3 + 4)$. This vector must be factorized in order to obtain UR at $d = d_2 = d_{13}$. But below this value the vector (4.4) has negative norm if $a_3 \neq 0$ and there is no unitarity, except at the isolated unitary point $d = \frac{3}{2} + \frac{1}{2}a_3 = d_{23} > d_{14}$. At that value of d there is a singular vector of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3$:

$$
\begin{aligned}
v^1_{\delta_2+\delta_3} = &\Big(2(a_3 + 1)(Y_4Y_3X_{23}^+) - 2(a_3 + 1)(Y_4Y_2X_3^+) + 2a_3(a_3 + 1)(Y_3^2X_2^+)\\
&- (a_3 + 1)(Y_{34}X_{23}^+) + (a_3 + 1)(Y_{24}X_3^+) - \tfrac{1}{2}(2a_3 + 1)(Y_{34}X_3^+X_2^+)\\
&-2a_3(a_3+1)Y_{23} + 4a_3(a_3+1)(Y_3Y_2) + Y_4Y_3X_3^+X_2^+ + \tfrac{1}{2}(Y_4^2(X_3^+)^2X_2^+)\Big)v_0
\end{aligned}
\tag{4.5}
$$

with the norm $16a_3(a_3+1)^2(a_3+2)(-2d+a_3+2)(-2d+a_3+3)$. For $a_3 \neq 0$ the singular vector (4.5) should be factored for unitarity, while for $a_3 = 0$ it is descendant of the compact singular vectors.

In the same range for $a_3 \neq 0$ at $d = d_3 = 1 + \frac{1}{2}a_3$ there is a singular vector of weight $\delta_3 = \alpha_3 + \alpha_4$:

$$
v^1_{\delta_3} = \sum_{k=0}^{1} (-1)^k (a_3 + k)(X_3^+)^{1-k} X_4^+ (X_3^+) k v_0 \equiv \mathcal{P}^{1,\delta_3} v_0 \tag{4.6}
$$

which must be factored out for unitarity.

On the other hand, for $a_1 = a_2 = a_3 = 0$ all (sub)singular vectors above are descendants of the compact singular vectors $X_k^+v_0$, $k = 1, 2, 3$, and there is no obstacle for unitarity for $d > \frac{3}{2} = d_{23} = d_{14}$. For $a_3 = 0$ and $d = \frac{3}{2}$ there is also a singular vector of weight $\delta_1 + \delta_4$:

$$\begin{aligned} v^1_{\delta_1+\delta_4} = \big(- 4(Y_4Y_2X_1^+) - 2(Y_4^2X_{23}^+X_1^+) + 2(Y_4Y_3X_2^+X_1^+) \\ + Y_4^2X_3^+X_2^+X_1^+ - 3(Y_{34}X_2^+X_1^+) + 6(Y_{24}X_1^+)\big)v_0 \end{aligned}$$

but it is also descendant of compact singular vectors. Finally, for $d = \frac{3}{2}$ there is a subsingular vector of weight $\delta_1 + \delta_2 + \delta_3 + \delta_4$:

$$v^{ss}_{\delta_1+\delta_2+\delta_3+\delta_4} = \sum_{i,j,k,\ell=1}^{4} \epsilon^{ijk\ell} Y_iY_jY_kY_\ell \, v_0 \tag{4.7}$$

where $\epsilon^{ijk\ell}$ is the totally antisymmetric symbol so that $\epsilon^{1234} = 1$. The norm of the vector (4.7) is $2304(-1 + d)d(-3 + 2d)(-1 + 2d)$. Thus, for $\frac{3}{2} > d > 1$ there is no unitarity since then the vector (4.7) has negative norm. In all cases there will be no unitarity for $d \leqslant 1$, except possibly when $a_1 = a_2 = a_3 = 0$ to which we restrict below. At $d = d_3 = d_{24} = 1$ there are the singular vector (4.6) and the singular vector of weight $\delta_2 + \delta_4 = \alpha_2 + \alpha_3 + 2\alpha_4$:

$$v^1_{\delta_2+\delta_4} = \big(- (a_3 + 2)(Y_{34}X_2^+) + 2(Y_4Y_3X_2^+) + Y_4^2X_3^+X_2^+\big)v_0 \tag{4.8}$$

both of which are descendants of compact singular vectors. At $d = d_3 = d_{24} = 1$, there is also a subsingular vector

$$v^{ss}_{\delta_2+\delta_3+\delta_4} = (Y_2Y_3Y_4 - Y_4Y_3Y_2) = \frac{1}{3}\sum_{i,j,k=2}^{4} \epsilon^{ijk} Y_iY_jY_k \, v_0$$

of the norm $144d(d-1)(2d-1)$. It is not an obstacle for unitarity for $d = 1$, but for $d < 1$. Thus, there is no unitarity for $d < 1$ except at the isolated unitary point $d = \frac{1}{2} = d_{34}$. At that point all (sub)singular vectors above are descendants of compact singular vectors. Yet there is the singular vector

$$v^1_{\delta_3+\delta_4} = \big(\tfrac{1}{2}Y_4^2X_3^+ - 2Y_3Y_4 + Y_{34}\big)\, v_0$$

with the norm $8d(2d-1)$. It is not an obstacle for unitarity for $d = \frac{1}{2}$, but for $d < \frac{1}{2}$. Thus, there is no unitarity for $d < \frac{1}{2}$ except at the isolated point $d = d_4 = 0 = a_1 = a_2 = a_3$ where we have the trivial one-dimensional UIR since all possible states are descendants of factored out singular vectors. □

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Representations of Parabose Supersymmetry

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Parabose symmetry (alternative names: Generalized conformal supersymmetry with tensorial central charges, conformal M-algebra, $osp(1|2n)$ supersymmetry) has been considered as an alternative to d-dimensional conformal superalgebra. Potential relevance of the corresponding superalgebra spreads to various subfields of High Energy Physics and Astrophysics (e.g. particle classification, gauging gravity, dark matter/energy candidates, etc.). Yet, due to mathematical difficulties, even classification and analysis of its unitary irreducible representations (UIR's) have not been entirely accomplished. We complete this classification for $n = 4$ case (corresponding to four dimensional space-time) and then show how the discrete subset of these UIR's can be constructed in a less abstract manner, that allows natural physical interpretation as spaces of particular composite particle states.

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We also conjecture generalization of the obtained results to the cases relevant in the string/brane context ($n > 4$).

1 Introduction

In the standard Poincaré supersymmetry, anticommutator of two lefthanded (righthanded) supersymmetry generators either vanishes or, in the extended supersymmetry case, equals to a central charge. If this requirement is relaxed, in four space-time dimensions the following relations are obtained (in four component spinor notation):

$$\{Q_\alpha, Q_\beta\} = (C\gamma^\mu)_{\alpha\beta}P_\mu + (C\gamma^{\mu\nu})_{\alpha\beta}Z_{\mu\nu}, \{QQcovariantly\} \tag{1}$$

with C being the charge conjugation matrix, $\gamma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]$, space-time indices take values $\mu, \nu = 0, 1, 2, 3$ and spinorial indices $\alpha, \beta = 1, 2, 3, 4$. The nonstandard second term on the righthand side contains six entities $Z_{\mu\nu}$ known as "tensorial central charges".

This sort of supersymmetry generalization conveys also to the superconformal case, introducing, as we will see, a number of additional bosonic generators into the algebra. The superconformal generalization turns out to form $osp(1|8)$ superalgebra, whose enveloping algebra coincides with the, so called, $n = 4$ parabose algebra [1, 2].

Historically, first to notice interesting properties of such a construct seems to have been C. Fronsdal [3], as early as in 1985, while investigating Penrose twistors and conformal field theory. He noticed that reduction from $osp(1|8)$ symmetry to conformal symmetry of Minkowski space ($osp(1|8) \supset su(2, 2)$) can be seen as a specific type of Kaluza-Klein reduction from 10 to 4 dimensions that leads to model with infinite tower of massless fields with increasing spins. Since then the construct of generalized supersymmetry reappeared, sometimes independently, in many physical contexts. In particular, it gained lot of interest when it was realized that tensorial central charges in higher dimensions appear naturally in relation to extended objects, such as branes and that it seems to be the underlying symmetry of M-theory [4–9]. Besides, exotic BPS particles were found and studied [10–13] in this framework, and field equations corresponding to higher spin fields were obtained [14–19]. Independently, generalized conformal supersymmetry showed up as the result of a search for mathematically simple structures that could contain Poincaré symmetry and thus could be interesting as candidates for a larger space-time symmetry. The approach was based on Heisenberg [20, 21], bose and parabose [22–24] algebras.

In the first place we will be interested in the orthosymplectic generalization of supersymmetry as a candidate for a realistic symmetry of the space-time. This means that we will consider case $osp(1|8)$ that is related to four space-time dimension, but we also conjecture generalization of the results to higher dimensional cases (where $osp(1|2n)$ algebra appears in the context of branes and M-theory).

When considering a (super)group in the context of a space-time symmetry, one of the first and most natural steps to undertake is to find unitary irreducible representations (UIR's) of the group, as these give us basic information on the particle content of the free theory. In principle, only then one can know what types of fields can exist in the model, and is entitled to consider field theory, write action for the fields, attempt quantisation and/or introduce interactions. Yet, in spite of substantial interest in this type of generalized supersymmetry, no complete analysis of unitary irreducible representations, especially in this physical context, has been carried out. The probable reason is that this task is related with substantial mathematical difficulties.

The problems have been solved for low n cases: apart from the well understood case $n = 1$, even UIR's of $n = 2$ were successfully classified [25] and some families explicitly constructed [26]. We are familiar with only a few partial results pertaining to the representations of the $osp(1|2n)$ for $n > 2$ (for a brief review of the progress in the representation theory of the orthosymplectic superalgebras $osp(m|2n)$ in general, see [27]). Günayadin applied his oscillator construction to obtain some positive energy UIR's of $osp(1|2n)$ from discrete spectrum [28]. However, his approach was constructional and thus lacking in a few ways: no classification of UIR's was given, the question if there are more discrete UIR's was left open and there was no insight where is the limit of the continuous spectre. Taking parabosonic approach Lievens, Stoilova, and Van der Jeugt [29] obtained a narrow subclass of positive energy UIR's, called representations with unique vacuum (parastatistics terminology). To the best of our knowledge, the only systematic and general approach to the classification of (positive energy) $osp(1|2n)$ UIR's was attempted by Dobrev and Zhang [30], who analyzed reducibility of lowest weight Verma modules. Yet, it turned out that a complete classification of positive energy UIR's of $osp(1|8)$, at the present level of our mathematical understanding of Verma module structure, required some extremely lengthy and involved calculations that could be only performed by using computers. We thus followed the approach of Dobrev and Zhang, but developed computer algorithms to analyze Verma module structure: to search for singular and subsingular vectors and check their dependencies in each particular case. In this way we managed to make a complete list of positive energy $osp(1|8)$ UIR's, together with explicit forms of the corresponding Verma module singular and subsingular vectors. We demonstrate that there is a concrete number of discrete UIR families (precisely nine, or ten if the trivial representation is counted as a separate class), that physically should be related to elementary particles of $osp(1|8)$ models.

In addition, we also propose a method to explicitly construct discrete representations, allowing one to easily perform concrete calculations in these spaces and, in that way, give physical interpretation to the states within. The method is based on a specific generalization of the, so called, Green's ansatz (used in the context of parastatistics), but in such a way that no anticommuting operators appear when representing superalgebra elements. Curiously, it turns out that to realise all discrete families of UIR's, elementary Green's ansatz representations have to be grouped in pairs, and it takes exactly up to three such pairs to construct arbitrary discrete UIR. It is quite probably that our method for construction of representations can be connected with the one in [28], but, to our opinion, is advantageous due to lack of anticommuting operators that drastically simplifies calculations (and allows us to directly use mathematical machinery developed for non relativistic quantum mechanics).

2 Parabose algebra $n = 4$ as generalized superconformal symmetry

Parabose algebra is a generalization of the algebra of standard bose creation and annihilation operators, first suggested by H.S.Green [1]. In literature [1, 31], it is usually defined as algebra of n pairs of mutually hermitian conjugate operators $a_\alpha, a^\dagger_\alpha$, satisfying trilinear relations:

$$[\{a_\alpha, a^\dagger_\beta\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \tag{2}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma] = 0, \tag{3}$$

together with relations (additional four) that follow from these by hermitian conjugation and by use of Jacobi identities.[24]

Parabose operators, defined as above, together with all possible anticommutators $\{a_\alpha, a_\beta\}$, $\{a_\alpha, a^\dagger_\beta\}$ and $\{a^\dagger_\alpha, a^\dagger_\beta\}$ of the parabose operators, form a realization of orthosymplectic superalgebra $osp(1|2n)$. With the usual assumptions of positivity of Hilbert space metrics in the space where parabose operators act, list of unitary irreducible representation of parabose algebra reduces to, so called, "positive energy" class of $osp(1|2n)$ UIR's.

As announced in the introduction, we are primarily interested in the case of 4 physical dimensions, corresponding to $n = 4$.

Conformal ($c(1,3) \sim so(2,4)$) algebra is contained in the algebra closed by all anticommutators of parabose operators. We will demonstrate the connection by making a two-step change of basis. We first switch from operators a_α and $a^\dagger_\alpha$ to their hermitian combinations $S^\alpha \equiv (a_\alpha + a^\dagger_\alpha)$ and $Q_\alpha \equiv -i(a_\alpha - a^\dagger_\alpha)$. In the space of all anticommutators of S^α and Q_α we then introduce the following basis:

$$\begin{aligned}
&J_i \equiv \tfrac{1}{8}(\sigma_i)^\alpha{}_\beta\,\{Q_\alpha, S^\beta\}, && Y_{\underline{i}} \equiv \tfrac{1}{8}\big(\tau_{\underline{i}}\big)^\alpha{}_\beta\,\{Q_\alpha, S^\beta\}, && N_{\underline{i}j} \equiv \tfrac{1}{8}\big(\alpha_{\underline{i}j}\big)^\alpha{}_\beta\,\{Q_\alpha, S^\beta\},\\
&K_{\underline{i}j} \equiv -\tfrac{1}{8}\big(\alpha_{\underline{i}j}\big)_{\alpha\beta}\,\{S^\alpha, S^\beta\}, && K_0 \equiv \tfrac{1}{8}(\alpha_0)_{\alpha\beta}\,\{S^\alpha, S^\beta\}. && \\
&D \equiv \tfrac{1}{8}(\alpha_0)^\alpha{}_\beta\,\{Q_\alpha, S^\beta\}, && P_{\underline{i}j} \equiv \tfrac{1}{8}\big(\alpha_{\underline{i}j}\big)^{\alpha\beta}\,\{Q_\alpha, Q_\beta\}, && P_0 \equiv \tfrac{1}{8}(\alpha_0)^{\alpha\beta}\,\{Q_\alpha, Q_\beta\},
\end{aligned} \tag{4}$$

Matrices $\sigma_i, \tau_{\underline{i}}, \alpha_{\underline{i}j}$ and α_0, appearing here, represent a basis of four by four real matrices, defined as follows. Basis for antisymmetric matrices is given by six matrices σ_i and $\tau_{\underline{i}}$, $i, \underline{i} = 1, 2, 3$ that satisfy:

$$[\sigma_i, \sigma_j] = 2\varepsilon_{ijk}\sigma_k, \quad [\tau_{\underline{i}}, \tau_{\underline{j}}] = 2\varepsilon_{\underline{i}\underline{j}\underline{k}}\tau_{\underline{k}}, \quad [\sigma_i, \tau_{\underline{j}}] = 0. \tag{5}$$

Matrices $\alpha_{\underline{i}j} \equiv \tau_{\underline{i}}\sigma_j$, together with the unit matrix denoted as α_0, form a basis of symmetric matrices.

Algebra closed by parabose anticommutators, whose one particular basis is given by (4), has 36 generators and is isomorphic to $sp(8)$. Centralizer of element $Y_{\underline{i}}$ ($\underline{i}$ arbitrary) is a subalgebra isomorphic to Conformal algebra of Minkowski spacetime ($so(2,4) \subset sp(8)$) plus the element $Y_{\underline{i}}$ alone. Without loss of generality, we

[24]We note that, in a Hilbert space equipped with positive definite metrics (with respect to which one defines the adjoint $a^\dagger_\alpha$), all algebra relations actually follow from a single relation (2).

will consider centralizer of $Y_{\underline{3}}$, spanned by the operator $Y_{\underline{3}}$ itself and the operators:

$$J_k, N_i \equiv N_{\underline{3}i}, D, P_i \equiv P_{\underline{3}i}, P_0, K_i \equiv K_{\underline{3}i}, K_0, \tag{6}$$

that generate $so(2,4)$ algebra. Operators (6) play the roles of rotation generators, boost generators, dilatation generator, momenta and pure conformal generators, respectively.

We have thus demonstrated that the group generated by anticommutators of $n = 4$ parabose algebra can be seen as a particular generalization, that is, extension of the conformal symmetry group in four dimensions. If we additionally include the parabose operators Q and S themselves in the even algebra, the overall structure becomes an extension of conformal superalgebra (hence the name generalized conformal supersymmetry). Mathematically, algebra extends from $sp(8)$ to $osp(1|8)$. Operators Q and S play roles of space-time supersymmetry generators. To see this we can "invert" relations (4):

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = (\alpha_0)_{\alpha\beta}\, P_0 + (\alpha_{\underline{i}j})_{\alpha\beta}\, P_{\underline{i}j}, \{S^\alpha, S^\beta\} = (\alpha_0)^{\alpha\beta}\, K_0 - (\alpha_{\underline{i}j})^{\alpha\beta}\, K_{\underline{i}j}, \\ &\{S^\alpha, Q_\beta\} = (\alpha_0)^\alpha{}_\beta\, D + (\alpha_{\underline{i}j})^\alpha{}_\beta\, N_{\underline{i}j} + (\sigma_i)^\alpha{}_\beta\, J_i + (\tau_{\underline{i}})^\alpha{}_\beta\, Y_{\underline{i}}. \end{aligned} \tag{7}$$

Comparison of these relations with the standard conformal superalgebra relations shows appearance of extra terms on righthand sides of (7) – these are exactly the tensorial central charges from relation (1), written in a different, Lorentz noncovariant notation. In the first of the relations, apart from the expected operators $P_{\underline{3}i}$ and P_0 that we have identified with spatial momentum and energy (6), there are additional operators $P_{\underline{1}i}$ and $P_{\underline{2}i}$. These operators transform as components of a second rank antisymmetric Lorentz tensor and are linear combinations of anticommutators $\{Q_\eta, Q_\xi\}$ and $\{\overline{Q}_{\dot\eta}, \overline{Q}_{\dot\xi}\}$ (that vanish by definition in the standard supersymmetry case).

3 Unitary irreducible representations

In this section we classify unitary irreducible representations of $n = 4$ parabose algebra. We will begin with some basic observations.

As the metrics is positive definitive, an operator defined as $E \equiv \frac{1}{2}\sum_\alpha \{a_\alpha, a_\alpha^\dagger\}$ must be positive. Annihilation operators a_α reduce the eigenvalue of E, thus the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. From the parabose algebra relations follows:

$$|v\rangle \in V_0 \Rightarrow \{a_\alpha, a^\dagger_\beta\}|v\rangle \in V_0, \tag{8}$$

with α, β arbitrary. Therefore vacuum subspace carries a representation of an $U(1)\times SU(N)$ group generated by operators $\{a_\alpha, a^\dagger_\beta\}$ (with $U(1)$ part generated by E). Let $V_0^{(\mu)}$ be a subspace of V_0 carrying irreducible representation μ of $SU(N)$. For the reasons of unitarity we are interested in cases when this subspace is finite dimensional. Since generators $\{a_\alpha, a^\dagger_\beta\}$ commute with E, E acts as a multiple of unity in this subspace and its eigenvalue will be denoted as e_0. Therefore, we can uniquely label $V_0^{(\mu)}$ as $V_0^{(\mu,e_0)}$, and the parameters μ and e_0 in this way also label UIR's of parabose algebra. In the context of $osp(1|2n)$ algebra such representations are called positive energy UIR's. In analysis of this type of $osp(1|2n)$, or more concretely, of $osp(1|8)$ unitary irreducible representations we closely followed the approach from [30]: not only in method (analysis of reducibility and unitarity conditions for lowest weight Verma modules), but also in conventions, choice of root system, UIR labels, et cetera (only different letters will be sometimes used to denote quantities, in order to ensure compatibility with the rest of this paper). Thus we will run through preliminaries very briefly, referring to [30] for details.

We consider lowest weight Verma modules $V^\Lambda \cong U(\mathcal{G}^+) \otimes |v_0\rangle$. Here, $\mathcal{G}^+$ denotes subalgebra of positive roots in standard algebra decomposition $\mathcal{G}^\mathbb{C} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$ ($\mathcal{G}$ denotes superalgebra $osp(1|8)$ and $\mathcal{G}^\mathbb{C}$ its complexification; $\mathcal{H}$ is Cartan subalgebra) and $|v_0\rangle$ is a lowest weight vector of weight Λ:

$$X \in \mathcal{G}^- \Rightarrow X|v_0\rangle = 0, \quad H \in \mathcal{H} \Rightarrow H|v_0\rangle = \Lambda(H)|v_0\rangle. \tag{9}$$

Roots, expressed using elementary functionals, are:

$$\begin{aligned}\Delta &= \{\pm\delta_\alpha, 1 \le \alpha \le 4; \pm\delta_\alpha \pm \delta_\beta, 1 \le \alpha < \beta \le 4;\\ &\quad \pm 2\delta_\alpha, 1 \le \alpha \le 4\}\end{aligned} \tag{10}$$

(the two signs in $\pm\delta_\alpha\pm\delta_\beta$ not being correlated) and the corresponding root vectors we will denote as (in the same order):

$$\begin{aligned}\mathcal{G}^+ \oplus \mathcal{G}^- &= \{a^\dagger_{\pm\alpha}, 1 \le \alpha \le 4; a^\dagger_{\pm\alpha,\pm\beta}, 1 \le \alpha < \beta \le 4;\\ &\quad a^\dagger_{\pm\alpha,\pm\alpha}, 1 \le \alpha \le 4\}.\end{aligned} \tag{11}$$

Here we introduced a compact notation for superalgebra elements, that emphasises the parabose connection:

$$a^{\dagger}_{-\alpha} \equiv a_{\alpha}, \quad a^{\dagger}_{\alpha,\beta} \equiv \{a^{\dagger}_{\alpha}, a^{\dagger}_{\beta}\}. \tag{12}$$

Simple root vectors are:

$$\{a^{\dagger}_{-2,1}, a^{\dagger}_{-3,2}, a^{\dagger}_{-4,3}, a^{\dagger}_{4}\} \tag{13}$$

and the corresponding positive root vectors are:

$$\begin{aligned} \Delta^+ = \ & \Big\{a^{\dagger}_{4}, a^{\dagger}_{1,4}, a^{\dagger}_{2,4}, a^{\dagger}_{3,4}, a^{\dagger}_{3}, a^{\dagger}_{1,3}, a^{\dagger}_{2,3}, a^{\dagger}_{2}, a^{\dagger}_{1,2}, a^{\dagger}_{1}, \\ & a^{\dagger}_{-4,3}, a^{\dagger}_{-4,2}, a^{\dagger}_{-3,2}, a^{\dagger}_{-4,1}, a^{\dagger}_{-3,1}, a^{\dagger}_{-2,1}\Big\}, \end{aligned} \tag{14}$$

written in, so called, normal ordering [30] that we will use for ordering of the Poincaré-Birkhoff-Witt (PBW) basis of $U(\mathcal{G}^+)$.

We will label representations by the signature

$$\chi = \{s_1, s_2, s_3, d\}, \tag{15}$$

where parameters s_1, s_2, s_3 actually label the $su(4)$ representation μ and parameter d is related to e_0 by $e_0 = 4d + s_1 - s_3$. The connection between the signature and the lowest weight Λ is given by:

$$\begin{aligned} \Lambda = \ & (d - \frac{s_1}{2} - \frac{s_2}{2} - \frac{s_3}{2})\delta_1 + (d + \frac{s_1}{2} - \frac{s_2}{2} - \frac{s_3}{2})\delta_2 \\ & + (d + \frac{s_1}{2} + \frac{s_2}{2} - \frac{s_3}{2})\delta_3 + (d + \frac{s_1}{2} + \frac{s_2}{2} + \frac{s_3}{2})\delta_4. \end{aligned} \tag{16}$$

A corresponding shortened notation will be also used for weights: $\Lambda = (\frac{2d-s_1-s_2-s_3}{2}, \frac{2d+s_1-s_2-s_3}{2}, \frac{2d+s_1+s_2-s_3}{2}, \frac{2d+s_1+s_2+s_3}{2})$.

We introduce a (Shapovalov) norm on the Verma module via natural involutive antiautomorphism: $\omega : \omega(a_{\alpha}) = a^{\dagger}_{\alpha}$ (compatible with the assumed Hilbert space metric). Right away we note that simple unitarity considerations – calculating norms of vectors $a^{\dagger}_{-(\alpha+1),\alpha}|v_0\rangle$ and $a^{\dagger}_{1}|v_0\rangle$ – result in constraints: $s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$, $d \geq (s_1 + s_2 + s_3)/2$. Parameters s_1, s_2, s_3 must be integer, labelling an $SU(4)$ Young tableau with $s_1 + s_2 + s_3$ boxes in the first row, $s_1 + s_2$ boxes in the second and s_1 boxes in the third row.

For certain values of Λ submodules appear in the structure of the Verma module V^Λ, and the module becomes reducible. Basic case is when this happens due to existence of a singular vector $|v_s\rangle \in V^\Lambda$:

$$X|v_s\rangle = 0, \qquad \forall X \in \mathcal{G}^-. \tag{17}$$

This singular vector, in turn, generates a submodule $V^{\Lambda'} \cong U(\mathcal{G}^+)|v_s\rangle$ within V^Λ.

To ensure irreducibility, all submodules corresponding to singular vectors must be factored out. However, after factoring out these submodules, new singular vectors may appear in the remaining space – called subsingular vectors. Namely, if the union of all submodules of singular vectors is denoted by $\tilde{I}^\Lambda$ then a vector $|v_{ss}\rangle \in V^\Lambda$ is called a subsingular vector [32] if $|v_{ss}\rangle \notin \tilde{I}^\Lambda$ and:

$$X|v_{ss}\rangle \in \tilde{I}^\Lambda, \qquad \forall X \in \mathcal{G}^-. \tag{18}$$

Just as singular vectors, subsingular vectors also generate submodules that have to be factored out when looking for irreducible representations.

In the particular case of $osp(1|2n)$ there are always, irrespectively of d value, singular vectors of the form:

$$|v_s^\alpha\rangle \equiv (a^\dagger_{-(\alpha+1),\alpha})^{s_\alpha+1}|v_0\rangle, \quad \alpha = 1, 2, \ldots n-1, \tag{19}$$

(when considering cases of unitary and therefore finite dimensional $SU(n)$ representations μ, related to integer values of s_α). The union of the submodules corresponding to these singular vectors we will denote as I^Λ_{SU}. We will always consider factor modules V^Λ/I^Λ_{SU}, and due to this fact subsingular vectors will play a significant role in the the analysis.

Our analysis of the Verma module structure heavily relied on the computer analysis and was carried out in the following general manner (that we just briefly describe). First, Kac determinant of a sufficiently high level was considered as a function of parameter d (for each given class of $SU(4)$ representation μ). In this way it was possible to locate the highest value of d for which the determinant vanishes and the Verma module becomes reducible. The singular or subsingular vector responsible for the singularity of the Kac matrix was then calculated, effectively by solving an (optimized) system of linear equations. Next we would find the norm of this vector and look for possible additional discrete reduction points at (lower) values of d for which the norm also vanishes. If new reduction points with new (sub)singular vectors were found it was also necessary to check that, upon removal of the corresponding submodules, no vectors with zero or negative norm remained. For this, it was enough to check that previously found (sub)singular vectors (i.e. those occurring for higher d values) belonged to the factored-out submodules. Optimized Wolfram Mathematica code was written to perform all these calculations.

We will illustrate the procedure on a few cases, and then give the final classification. More detailed account of the (sub)singular vectors and their interrelations will be given elsewhere.

First we consider unitary irreducible representations that appear when μ is the trivial representation ($s_1 = s_2 = s_3 = 0$), i.e. cases when the lowest weight vector of Verma module is invariant w.r.t. $SU(4)$ subgroup action (space V_0 is one dimensional). The structure of the Verma module in this case is as follows.

For values $d > \frac{3}{2}$ the Verma module is irreducible, all norms are positive and the corresponding representations are unitary and irreducible.

At value $d = \frac{3}{2}$ a subsingular vector appears. In PBW basis this vector has form:

$$\begin{aligned}|v_{ss}^{(1,1,1,1)}\rangle = \ & (-2a^\dagger_{3,4}a^\dagger_2a^\dagger_1 + 2a^\dagger_{2,4}a^\dagger_3a^\dagger_1 - 2a^\dagger_4a^\dagger_{2,3}a^\dagger_1 - 2a^\dagger_{1,4}a^\dagger_3a^\dagger_2 + 2a^\dagger_4a^\dagger_{1,3}a^\dagger_2 \\ & - 2a^\dagger_4a^\dagger_3a^\dagger_{1,2} + a^\dagger_{3,4}a^\dagger_{1,2} - a^\dagger_{2,4}a^\dagger_{1,3} + a^\dagger_{1,4}a^\dagger_{2,3} + 4a^\dagger_4a^\dagger_3a^\dagger_2a^\dagger_1)|v_0\rangle.\end{aligned}$$

The notation for labeling these (sub)singular vectors is the following: ss in the lower index stands for "subsingular" whereas s means "singular" vector; in the upper index we give "relative weight" of the vector – if the (sub)singular vector generates Verma submodule of weight Λ' the the relative weight is $\Lambda' - \Lambda$ (the relative weight alone will turn out to uniquely label these vectors, in a very systematic way).

Upon removing, i.e. factoring out the submodule generated by this vector, an UIR is obtained.

The norm of the vector $|v_{ss}^{(1,1,1,1)}\rangle$ as a function of d at $s_1 = s_2 = s_3 = 0$ is $64(2d-3)(d-1)(2d-1)d$, having zeros at $d = \frac{3}{2}, 1, \frac{1}{2}$ and 0.

Between $d = \frac{3}{2}$ and $d = 1$ the norm above is negative and there are no UIR's. However, at the value $d = 1$ a new subsingular vector appears:

$$|v_{ss}^{(0,1,1,1)}\rangle = \left(a^\dagger_{3,4}a^\dagger_2 - a^\dagger_{2,4}a^\dagger_3 + a^\dagger_4a^\dagger_{2,3} - 2a^\dagger_4a^\dagger_3a^\dagger_2\right)|v_0\rangle\,. \tag{20}$$

It can be explicitly shown that the subsingular vector $|v_{ss}^{(1,1,1,1)}\rangle$ belongs to the union of submodule generated by $|v_{ss}^{(0,1,1,1)}\rangle$ and the submodule I_{SU}^{Λ}. After factoring out submodule of the vector $|v_{ss}^{(0,1,1,1)}\rangle$ no negative or zero norm vectors remain in the factor space and an UIR is obtained for $d = 1$, $s_1 = s_2 = s_3 = 0$.

Norm of the subsingular vector (20) is $16(d-1)\ (2d-1)d$. In particular, it is negative for $1 > d > \frac{1}{2}$, precluding existence of UIR's in this range.

At $d = \frac{1}{2}$ a singular vector appears:

$$|v_s^{(0,0,1,1)}\rangle = (a_4^\dagger a_4^\dagger a_{-4,3}^\dagger - a_{3,4}^\dagger + 2a_4^\dagger a_3^\dagger)|v_0\rangle, \tag{21}$$

with norm $8(2d-1)d$.

The previous subsingular vector $|v_{ss}^{(0,1,1,1)}\rangle$ belongs to the union of submodule generated by $|v_s^{(0,0,1,1)}\rangle$ and submodule I_{SU}^{Λ}. Thus, there is UIR also at $d = 1/2$, $s_1 = s_2 = s_3 = 0$ obtained upon removing the submodule of vector $|v_s^{(0,0,1,1)}\rangle$.

Norm of $|v_s^{(0,0,1,1)}\rangle$ is negative when $\frac{1}{2} > d > 0$ and, therefore, there are no UIR's in this range.

At $d = 0$ another subsingular vector, of the norm $2d$, appears:

$$|v_s^{(0,0,0,1)}\rangle = a_4^\dagger |v_0\rangle. \tag{22}$$

This reduction point corresponds to the trivial representation of $osp(1|8)$ with representation space being spanned only by vector $|v_0\rangle$.

Proceeding in the same manner, we finally obtain the following simple scheme for $n = 4$ parabose UIR classification:

- $s_1 = s_2 = s_3 = 0$:

$$\begin{aligned} & d > 3/2; \\ & d = 3/2, |v_{ss}^{(1,1,1,1)}\rangle; \\ & d = 2/2, |v_{ss}^{(0,1,1,1)}\rangle; \\ & d = 1/2, |v_s^{(0,0,1,1)}\rangle; \\ & d = 0/2, |v_s^{(0,0,0,1)}\rangle; \end{aligned} \tag{23}$$

- $s_1 = s_2 = 0, s_3 > 0$:

$$\begin{aligned} & d > s_3/2 + 4/2; \\ & d = s_3/2 + 4/2, |v_{ss}^{(1,1,1,0)}\rangle; \\ & d = s_3/2 + 3/2, |v_s^{(0,1,1,0)}\rangle; \\ & d = s_3/2 + 2/2, |v_s^{(0,0,1,0)}\rangle; \end{aligned} \tag{24}$$

- $s_1 = 0, s_2 > 0$:

$$\begin{array}{l} d > (s_2+s_3)/2+5/2; \\ d = (s_2+s_3)/2+5/2, |v_s^{(1,1,0,0)}\rangle; \\ d = (s_2+s_3)/2+4/2, |v_s^{(0,1,0,0)}\rangle; \end{array} \tag{25}$$

- $s_1 > 0$:

$$\begin{array}{l} d > (s_1+s_2+s_3)/2+6/2; \\ d = (s_1+s_2+s_3)/2+6/2, |v_s^{(1,0,0,0)}\rangle. \end{array} \tag{26}$$

The pattern of "relative weights" of (sub)singular vectors in the above scheme is obvious, and it allows us to immediately conjecture UIR classification for $n > 4$:

- $s_1 = s_2 = \cdots = s_{n-1} = 0$:

$$\begin{array}{ll} d > (n-1)/2; & \\ d = (n-1)/2, & |v_{ss}^{(1,1,1,\ldots,1,1,1,1)}\rangle; \\ d = (n-2)/2, & |v_{ss}^{(0,1,1,\ldots,1,1,1,1)}\rangle; \\ & \ldots \\ d = 2/2, & |v_{ss}^{(0,0,0,\ldots,0,1,1,1)}\rangle; \\ d = 1/2, & |v_{s}^{(0,0,0,\ldots,0,0,1,1)}\rangle; \\ d = 0/2, & |v_{s}^{(0,0,0,\ldots,0,0,0,1)}\rangle; \end{array} \tag{27}$$

- $s_1 = s_2 = \cdots = s_{n-2} = 0, s_{n-1} > 0$:

$$\begin{array}{ll} d > s_{n-1}/2+(n-1+1)/2; & \\ d = s_{n-1}/2+(n-1)/2, & |v_{ss}^{(1,1,1,\ldots,1,1,0)}\rangle; \\ d = s_{n-1}/2+(n-1-1)/2, & |v_{ss}^{(0,1,1,\ldots,1,1,0)}\rangle; \\ & \ldots \\ d = s_{n-1}/2+4/2, & |v_{ss}^{(0,0,\ldots,1,1,1,0)}\rangle; \\ d = s_{n-1}/2+3/2, & |v_{s}^{(0,0,\ldots,0,1,1,0)}\rangle; \\ d = s_{n-1}/2+2/2, & |v_{s}^{(0,0,\ldots,0,0,1,0)}\rangle; \end{array} \tag{28}$$

- $\ldots$

- $s_1 = 0, s_2 > 0$:

$$\begin{array}{ll} d > (s_2+\cdots+s_{n-1})/2+n-3/2; & \\ d = (s_2+\cdots+s_{n-1})/2+n-3/2, & |v_s^{(1,1,0,\ldots,0,0,0)}\rangle; \\ d = (s_2+\cdots+s_{n-1})/2+n-4/2, & |v_s^{(0,1,0,\ldots,0,0,0)}\rangle; \end{array} \tag{29}$$

- $s_1 > 0$:
$$\begin{aligned} &d > (s_1 + \cdots + s_{n-1})/2 + n - 1; \\ &d = (s_1 + \cdots + s_{n-1})/2 + n - 1, \quad |v_s^{(1,0,0,\ldots,0,0,0)}\rangle. \end{aligned} \tag{30}$$

4 An explicit construction of parabose UIR's

We propose a method to explicitly construct the above classified unitary irreducible representations of parabose algebra. The method cannot be applied to UIR's from the continuous spectre, i.e. those UIR's that occur for non (half)integer values of parameter d. However, from the physical viewpoint, representations from the discrete spectre (d taking discrete (half)integer values less or equal to the first reduction point) are of far greater significance since only in these cases singular or subsingular vectors appear. And it is well known that these vectors turn into important equations of motion (e.g. see [32]). In the particular case of the parabose generalization of supersymmetry, these vectors, for example, turn into Klein-Gordon, Dirac and Maxwell equations.

In the same paper where he first introduced parabose (and parafermi) algebra [1], H.S.Green has also offered a way to construct some of the unitary representations using what is nowadays known as the Green's ansatz. We demonstrate that the ansatz, originally applicable only to "unique vacuum" representations, can also accommodate other representations of the discrete type. We also combine the ansatz with, so called, Klein transformation, so that Green operators no longer satisfy strange "mixed" commutation and anticommutation relations, but instead obey usual commutation relations of bosonic algebra.

We define a Klein transformed analogue of Green's decomposition of order p (p is known as the order of the parastatistics) as the following expression for parabose operators:

$$a_\alpha = \textstyle\sum_{a=1}^{p} I_{(1)} I_{(2)} \cdots I_{(a-1)} a_\alpha^a. \tag{31}$$

In this expression operator a_α^a and its adjoint $a_\alpha^{a\dagger}$ satisfy ordinary bosonic algebra relations. There are total of $n{\cdot}p$ mutually commuting pairs of annihilation-creation operators $(a_\alpha^a, a_\alpha^{a\dagger})$:

$$[a_\alpha^a, a_\beta^{b\dagger}] = \delta_{\beta\alpha}\delta^{ab}; \quad [a_\alpha^a, a_\beta^b] = 0, \tag{32}$$

where $a, b = 1, 2, \ldots p$ and $\alpha, \beta = 1, 2, \ldots n$.

In (31) we have also introduced selfadjoint unipotent Klein "inversion" operators that act on the Green's operators in the following way:

$$I_{(a)} a_\alpha^b I_{(a)} = (-)^{\delta_{ab}} a_\alpha^b. \tag{33}$$

By their introduction we avoided appearance of anticommuting relations of original Green's operators and, by this, operators a_α^a and $a_\alpha^{a\dagger}$ become familiar mathematical objects which are easier to manipulate and interpret. The easiest way to show that such inversion operators exist is by explicit construction: $I_{(a)} = \exp(i\pi \sum_\alpha \frac{1}{2}\{a_\alpha^a, a_\alpha^{a\dagger}\})$.

The overall Green's ansatz representation space of order p can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_{(a)}$ of ordinary linear harmonic oscillator in n-dimensions: $\mathcal{H} = \mathcal{H}_{(1)} \otimes \mathcal{H}_{(2)} \otimes \cdots \otimes \mathcal{H}_{(p)}$. A single factor Hilbert space $\mathcal{H}_{(a)}$ is the space of unitary representation of n dimensional bose algebra of operators $(a_\alpha^a, a_\alpha^{a\dagger}), \alpha = 1, 2, \ldots n$, which is, at the same time, the simplest nontrivial unitary representation of parabose algebra (i.e. the simplest positive energy UIR of $osp(1|2n)$): $\mathcal{H}_{(a)} \cong U(a_\alpha^{a\dagger})|0\rangle_a$, where $|0\rangle_a$ is the usual bose vacuum of factor space $\mathcal{H}_{(a)}$. This picture is apropriate due to the fact that the action of even operators of $osp(1|2n)$ (and, in particular, of spacetime symmetry generators (4) in the $n = 4$ case) reduces simply to sum of actions in each of these factor spaces, by virtue of:

$$\{a_\alpha, a_\beta\} = \sum_{a=1}^{p} \{a_\alpha^a, a_\beta^a\}, \quad \{a_\alpha, a_\beta^\dagger\} = \sum_{a=1}^{p} \{a_\alpha^a, a_\beta^{a\dagger}\}. \tag{34}$$

As, from the mathematical point of view, the whole representation space exactly corresponds to Hilbert space of p particles in a n-dimensional non relativistic quantum mechanics, it is very clear that no negative or zero norm states appear. Therefore, if we can find, in this framework, a lowest weight vector $|v_0\rangle$ of a proper weight (corresponding to some UIR signature found in previous section) then the vectors of the form $\mathcal{P}(X)|v_0\rangle, \mathcal{P}(X) \in U(\mathcal{G}^+)$ will span that representation space. In addition, one can explicitly check that the corresponding (sub)singular vector vanishes, as it must.

The simplest nontrivial representation, with signature $s_1 = s_2 = s_3 = 0$, $d = 1/2$ corresponds to $p = 1$ space. The lowest weight vector is simply the vacuum of the $\mathcal{H}_{(1)}$: $|v_0^{(0,0,1,1)}\rangle = |0\rangle_1$. Space in $p = 1$ case is irreducible. Physical interpretation of the vectors in this space is that they correspond to tower of massless states with raising helicities. Other "unique vacuum states (i.e. $s_1 = s_2 = s_3 = 0$) are obtained for $p = 2$ and $p = 3$ with lowest weight vectors being $|0\rangle_1 \otimes |0\rangle_2$ and $|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$.

The simplest UIR class of non "unique vacuum" type has signature $s_1 = s_2 = 0, s_3 > 0$, $d = s_3/2 + 1$ and in these representations μ corresponds to single row Young tableaux. This class can be realized in $p = 2$ space, with

$$|v_0^{(0,0,1,0)}\rangle = \frac{1}{\sqrt{s_3!}}(A_4^{(1)})^{s_3}|0\rangle_1 \otimes |0\rangle_2, \tag{35}$$

where $A_\alpha^{(k)} \equiv I_{(2k)}(a_\alpha^{2k-1\dagger} + I_{(2k-1)}a_\alpha^{2k\dagger})$. We note that entire $p = 2$ space reduces w.r.t. parabose algebra action to UIR's with signatures: $s_3 = 0, 1, 2, 3, \ldots, d = s_3/2 + 1$, $s_1 = s_2 = 0$, without any additional degeneracy. From the viewpoint of physics, this is the simplest class that contains both massless and massive states with an additional charge (related to the label s_3).

There are two more classes of "single row" discrete UIR-s: those with signatures $\{0, 0, s_3, \frac{s_3}{2} + \frac{3}{2}\}$ and $\{0, 0, s_3, \frac{s_3}{2} + 2\}$. These are constructed in a similar manner as the previously considered class with signature $\{0, 0, s_3, \frac{s_3}{2} + 1\}$, only in spaces $p = 3$ and $p = 4$, respectively, with the lowest weight states given by expressions:

$$|v_0^{(0,1,1,0)}\rangle = \frac{1}{\sqrt{s_3!}}(A_4^{(1)})^{s_3}|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3, \tag{36}$$

$$|v_0^{(1,1,1,0)}\rangle = \frac{1}{\sqrt{s_3!}}(A_4^{(1)})^{s_3}|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4. \tag{37}$$

There are two "two-rows" ($s_1 = 0, s_2 > 0$) UIR classes. The class with $d = (s_2 + s_3)/2 + 2$ can be realized in $p = 4$ space, with the lowest weight state given as (up to normalization constant):

$$|v_s^{(0,1,0,0)}\rangle = (A_4^{(1)}A_3^{(2)} - A_3^{(1)}A_4^{(2)})^{s_2}(A_4^{(1)})^{s_3}|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4. \tag{38}$$

The remaining class with $d = (s_2 + s_3)/2 + 5/2$ can be realized in $p = 5$ space, with

$$|v_s^{(1,1,0,0)}\rangle = (A_4^{(1)}A_3^{(2)} - A_3^{(1)}A_4^{(2)})^{s_2}(A_4^{(1)})^{s_3}|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \otimes |0\rangle_5. \tag{39}$$

The only discrete class of representations that corresponds to three-rows Young tableaux ($s_1 > 0$, $d = (s_1 + s_2 + s_3)/2 + 3$) can be realized in $p = 6$ space, with the lowest weight state constructed as (up to normalization constant):

$$
\begin{aligned}
|v_0^{(1,0,0,0)}\rangle &= \Big(\sum_{k,l,m=1}^{3} \varepsilon_{klm} A_2^{(k)} A_3^{(l)} A_4^{(m)}\Big)^{s_1} \\
&\cdot\Big(\sum_{k,l=1}^{2} \varepsilon_{kl} A_3^{(k)} A_4^{(l)}\Big)^{s_2} (A_4^{(1)})^{s_3} \qquad (40)\\
&|0\rangle_1 \otimes \cdots \otimes |0\rangle_6,
\end{aligned}
$$

where ε denotes the Levi-Civita symbol.

Thus we demonstrated a method for realization of all discrete classes of UIR's. The presented construction method can be straightforwardly generalized both to $n > 4$ and to other (half)integer values of d that belong to continuous spectrum.

5 Conclusions

We analyzed $n = 4$ parabose supersymmetry (corresponding to $D = 4$ generalized conformal supersymmetry) using a group-theoretical approach. We gave a complete classification of unitary irreducible representations of parabose algebra. These results, although obtained in the $n = 4$ case, have proved to be readily generalizable to higher values of n, that made the analysis important also in the higher dimensional context of the string theory. Apart from classifying UIR's of the symmetry, we also proposed a method for their explicit construction.

We bring a special attention to the "pairing" of factor spaces that was observable in this setup: to obtain the simplest single box UIR ($s_1 = s_2 = 0, s_3 = 1, d = 3/2$) it takes two factor spaces $p = 2$. To form the simplest UIR with two boxes in a column ($s_1 = s_3 = 0, s_2 = 1, d = 5/2$), it turns out that $p = 4$ must be taken and the vacuum is essentially obtained by antisymmetrizing two "single-box" vacuum states. Similarly, "three-box in a column" UIR ($s_2 = s_3 = 0, s_1 = 1, d = 7/2$) is obtained by antisymmetrizing tensor product of three "single box" vacua. All discrete IR classes can be realized using tensor product of up to three "single-box" $p = 2$ spaces, in a way reminiscent of forming composite particles from simpler constituent ones.

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DECONTRACTION FORMULA FOR $sl(n, \mathbb{R})$ ALGEBRAS AND APPLICATIONS IN THEORY OF GRAVITY*

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Abstract

Special linear group $SL(n, \mathbb{R})$, as well as its covering group $\overline{SL}(n, \mathbb{R})$ in quantum domain, appear as relevant symmetry groups in many physical models based on spacetime symmetries. Applications of these symmetries and their representations in physics problems require knowledge of the $sl(n, \mathbb{R})$ algebra representations. Spinorial $sl(n, \mathbb{R})$ representations are of particular importance in various problems of quantum field theory, quantum and alternative theories of gravity, and theories of extended objects (strings, branes, etc.). Construction of the unitary and spinorial representations of the $sl(n, \mathbb{R})$ algebras is further involved by the fact that these representations are necessarily infinite dimensional. Moreover, transformation properties of physical entities, as well as their correct physical interpretation, require knowledge of the relevant $sl(n, \mathbb{R})$ algebra and $\overline{SL}(n, \mathbb{R})$ group representations in the basis of the orthogonal subgroup $Spin(n)$.

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The method used to derive expressions of the $\overline{SL}(n,\mathbb{R})$ generators is based on the so called decontraction, also known as the Gell-Mann, formula that is in the focus of this work. This formula, in our case, determines the $sl(n,\mathbb{R})$ algebra elements in terms of the algebra elements obtained by the Inönü-Wigner algebra contraction with respect to the $so(n)$ subalgebra. It is shown that this formula is valid only for some particular classes of irreducible representations that are insufficient for applications in physical models.

Next we demonstrate how the Gell-Mann formula can be generalized. The obtained generalized formula is valid for all $sl(n,\mathbb{R})$ irreducible representations: finite and infinite, unitary and non unitary, tensorial and spinorial. All expressions of the matrix elements of the $\overline{SL}(n,\mathbb{R})$, $n \geq 2$, generators are obtained explicitly by making use of the generalized decontraction formula. They are given in a closed form, in terms of the Hilbert space functions over the $Spin(n)$ subgroup, for an arbitrary irreducible representation characterized by the corresponding set of labels. This result provides, due to $sl(n,\mathbb{R})$ and $su(n)$ algebras relation, expressions of the matrix elements of the $SU(n)$ generators in the $SO(n)$ basis for all irreducible representations. An example that illustrate applications of the generalized decontraction formula in models of alternative theories of gravity based on a local affine symmetry is also presented.

1 Introduction

Special linear group over the field of real numbers $SL(n,\mathbb{R})$ is defined as a set of unit determinant $n \times n$ real matrices, equipped with usual matrix multiplication and inversion operations. In modern physics, this group appears in many different context. It appears either independently, or as the essential part of the general linear group $GL(n,\mathbb{R})$ – as it is well known that basically all mathematical problems related to the general linear group reduce to the corresponding problems for the case of the special linear subgroup: representations, topology, Clebsch-Gordan coefficients... Accordingly, special linear group also plays an important role in a number of contemporary attempts to solve existing problems in the theory of gravity.

In the first place, it is the case for the affine theories of gravity, both metric affine [1, 2] and gauge affine [3, 4]. In this context, special linear group plays the role that the Lorentz group has within the standard, Poincaré based theory of gravity. Flat space-time symmetry here is $R_n \wedge GL(n, \mathbb{R})$, i.e. semidirect product of translations in n dimensions with a subgroup of all homogenous linear transformations of these n dimensions. Already at this level understanding of the particle content of the models requires knowledge of the special linear group representations (which are the key step for finding representations of the entire affine symmetry). The gravitational interaction is introduced by localization of the affine symmetry, while the necessary symmetry breaking can be induced in various ways. Matrix elements of $SL(n, \mathbb{R})$ group and of the $SL(n, \mathbb{R})$ double covering group $\overline{SL}(n, \mathbb{R})$ appear in the interaction vertex terms. The detailed knowledge of the special linear group representations is particularly needed for the construction of concrete symmetry breaking scenarios [5].

Knowledge of representations of the special linear group, in particular of the, so called, spinorial representations is needed already in the analysis of the classical Einstein's theory of gravity. Namely, general linear group is the homogenous part of the diffeomorphism group $Diff(n, \mathbb{R})$ and the (world) tensors in the general theory of gravity transform according to the corresponding representations of exactly $GL(n, \mathbb{R})$. Therefore, the most straightforward and natural way to introduce spinorial matter in general relativity would be via fields that transform with respect to spinorial representations of the general linear group (so called *world spinors*) [6, 7]. Spinorial representations are those which, after symmetry reduction to the (pseudo)orthogonal subgroup of the general linear group (one corresponding to physical rotations), decompose into spinorial representations of the orthogonal group. The very existence of the spinorial representaions of the general linear, i.e. of the special linear group has been long unknown (up to eighties, [6]) due to the fact that all such representations are necessarily infinite dimensional. This fact certainly complicates finding and application of this type of representations, but does not diminish its physical relevance.

In last decade or two, general linear group appears in the papers also as an important subgroup in supersymmetry models with, so called, tensorial central charges [8, 9, 10, 11, 12, 13, 14, 15, 16]. In these models super-Poincaré symmetry in n dimensions is extended by adding of $n(n-1)/2$

generalized momenta ("tensorial charges") to the set of n standard generators of the space-time translations. Altogether they comprise a set of $n(n+1)/2$ operators, transforming as a second order symmetric tensor w.r.t. (with respect to) the general linear group $GL(n,\mathbb{R})$ that here extends and replaces the Lorentz group.[1] Such a generalization of the Poincaré superalgebra is additionally important as it corresponds to the symmetry of the M-theory [17, 18, 19, 20, 21, 22], so that it is also called M-algebra. Gravitational interaction ca be introduced in these models also by localization of $GL(n,\mathbb{R})$ symmetry. Thus the importance of detailed knowledge of representations of general (special) linear group in this context is also clear.

Special linear group corresponds to volume conserving transformations (i.e. area conserving in the two dimensional case). This makes the group relevant also in all cases where dynamics of the system is such that some volume (area) is conserved. Such situations occur in the context of strings and branes [23, 24].

To summarize, knowledge of $SL(n,\mathbb{R})$ group/$sl(n,\mathbb{R})$ algebra and its representations is of extreme importance in theory of gravity and theories of extended objects (in these examples it is usually enough to know representations of $sl(n,\mathbb{R})$ algebra, i.e. of the generators of $SL(n,\mathbb{R})$ group). These representations, as well as the context of their physical applications, have many specific properties.

First of all, in physics we are often interested in unitary representations, and since we are here dealing with a noncompact group, it is well known that all such representations are infinite dimensional. Next, $SL(n,\mathbb{R})$ group has its double cover group $\overline{SL}(n,\mathbb{R})$. This is most easily seen via Iwasawa decomposition $SL(n,\mathbb{R}) = NAK$: nilpotent (N) and abelian (A) subgroups are simply connected, so the covering of the entire group is determined by the covering of the maximal compact subgroup K – in this case $SO(n)$ that, as it is well known, has double cover $Spin(n) \equiv \overline{SO}(n)$. If we are interested in models that include fermionic matter, the representations of the double covering group $\overline{SL}(n,\mathbb{R})$ (that is $\overline{GL}(n,\mathbb{R})$) are of the utmost

[1] We note that this symmetry significantly differs from the affine symmetry even in the bosonic part, since the abelian subgroup generators, i.e. generalized momenta, transform as symmetric tensors of the second order in one case and as the n-dimensional vector representations of $GL(n,\mathbb{R})$ in the second case.

interest, since among these representations are those that decompose into spinorial representations of the (pseudo)orthogonal subgroups. Additional mathematical difficulty is the fact that all these spinorial representations are infinite dimensional, irrespectively of their unitarity properties [25, 6, 26].

Physical context also determines the basis of the representations space: to differentiate between fields of different spins, we usually need to know form of the symmetry generators in basis of the (pseudo)orthogonal subgroup $SO(m, n-m)$ (m depends on the signature of residual symmetry metrics in the model). This requirement reduces the number of available mathematical methods for finding of representations: for example, standard "canonical" approach (induction from maximal parabolic subgroup [27]) would provide representations in a basis of Cartan subalgebra weight vectors, and subsequent change of (infinite dimensional) basis is a difficult task. Additional problem is due to the fact that representations of the special linear group in general have nontrivial multiplicity with respect to the decomposition into (pseudo)orthogonal representations. Therefore, in this basis it is also necessary to take care of the multiplicity label.

Finally, dimension of the physical space varies from one model to another: (Kaluza-Klein theories, strings, branes) so that a generic (for arbitrary n) result in a closed form is highly preferred.

The listed technical requirements make the problem of finding the special linear group representations in this context very difficult. One possible solution to this problem, that satisfies all the above criteria, is considered in this paper. The solution is based on the generalization of the, so called, Gell-Mann (decontraction) formula.

Gell-Mann decontraction formula [28, 29, 30, 31, 32, 33] is a transformation aimed to serve as an "inverse" to the Inönü-Wigner contraction [34]. More precisely, while the Inönü-Wigner contraction is a singular transformation, more concretely a limiting procedure, that yields "contracted algebra" operators from the operators of the original algebra, the goal of the Gell-Mann formula is to provide a way to express the operators of the starting "non-contracted" algebra as functions of the contracted algebra elements. The concrete expression of the Gell-Mann formula will be written in the next section.

We are here interested in the case of $SL(n, \mathbb{R})$ group. In this context, important is its contraction contraction w.r.t. its maximal compact subgroup $SO(n)$. This procedure takes $sl(n,\mathbb{R})$ algebra into a semidirect sum of abelian subalgebra of generalized translations and a special orthogonal algebra: $r_{n(n+1)/2-1} \uplus so(n)$. Representations of this contracted group/algebra are much easier to find than the representations of the starting group/algebra (especially since the representations should be given in an $SO(n)$ adapted basis). Therefore, one approach to obtain representations of $sl(n,\mathbb{R})$ would be to convert, using the Gell-Mann formula, representations of contracted $r_{n(n+1)/2-1} \uplus so(n)$ algebra into representations of $sl(n,\mathbb{R})$.

The problem with this approach comes from the fact that the Gell-Mann formula is actually only a prescription that is not valid always, i.e. not for all algebras and all their representations. Moreover, this formula is entirely valid – i.e. as an algebraic identity – only in the case of (pseudo)orthogonal algebras, that is, for contractions $so(m+1,n) \to r_{m+n} \uplus so(m,n)$ $so(m,n+1) \to r_{m+n} \uplus so(m,n)$ [35, 36]. In other cases, including the $sl(n,\mathbb{R})$ algebra case, the Gell-Mann formula is valid only for a certain subset of representations (the validity conditions will be the subject of the third section). Thus, by using the Gell-Mann formula we can obtain only some of the $sl(n,\mathbb{R})$ representations, amongst whom, for example, there are neither spinorial nor representations with multiplicity.

On the other hand, in the $sl(n,\mathbb{R})$ case, it is possible to generalize Gell-Mann formula so to broaden its domain of applicability to all representations, including both unitary and nonunitary, both multiplicity free and with multiplicity, both tensorial and spinorial. As a direct mathematical application of the generalized formula, a closed expression for matrix elements of $SL(n,\mathbb{R})$ generators can be given – for an arbitrary (irreducible) representation, for arbitrary n, in the basis of the orthogonal subgroup. Due to the close relation of $sl(n,\mathbb{R})$ and $su(n)$ algebra, the same can be done also in the case of special unitary group/algebra.

This paper is organized as follows: the subjects of the next section are Inönü-Wigner contraction and the original form of the Gell-Mann decontraction formula; in the third section we will concentrate on the particular case of $sl(n,\mathbb{R})$ algebras and discuss the domain of validity of the formula for these algebras; the fourth section deals with the generalization of the formula; the fifth section contains discussion of the applicability of the formula in the context of affine gravity models; sixth section contains a summary of the paper; Appendix contains Clebsch-Gordan coefficients of the $SO(5)$ group necessary for finding explicit matrix values in the considered five dimensional case.

2 Inönü-Wigner contraction and the Gell-Mann decontraction formula

2.1 Contraction

Inönü and Wigner have long ago introduced the notion of algebra contraction, in order to mathematically describe transition from the relativistic Poincare symmetry to non-relativistic Galilei symmetry in the limit of infinite velocity of light [34]. At the basic of the contraction idea is the observation that change of basis A_i of algebra $\mathcal{A}$:

$$A_i \to A_i' = X_i{}^j A_j \tag{1}$$

can transform algebra $\mathcal{A}$ into a non-isomorphic algebra $\mathcal{A}'$ if the transformation coefficients $X_i{}^j$ are singular. This type of transformation is called Inönü-Wigner contraction if the singular transition coefficients can be obtained as a limit, when some parameter ϵ approaches zero, of otherwise non-singular transformation coefficients linear in ϵ: $X_i{}^j = X_i{}^j(\epsilon)$. In such a case, new structural constants of algebra $\mathcal{A}'$ have well defined limit if and only if algebra $\mathcal{A}$ contains a subalgebra $\mathcal{M}$ with respect to which the contraction is done in the following way:

$$\mathcal{M} \to \mathcal{M}' = \mathcal{M}, \qquad \mathcal{T} \to \mathcal{T}' = \epsilon\mathcal{T}$$

where $\mathcal{A} = \mathcal{M} + \mathcal{T}$ and $\mathcal{A}' = \mathcal{M}' + \mathcal{T}'$. We say that the algebra was contracted with respect to subalgebra $\mathcal{M}$ (that remained unaltered), and the elements $\mathcal{T}'$ we call "contracted". Contracted elements form an abelian ideal $\mathcal{T}'$ of algebra $\mathcal{A}'$, since in the limit $\epsilon \to 0$ it holds $[T_i', T_j'] = \epsilon^2 C_{ij}^a M_a' + \epsilon C_{ij}^k T_k' = 0$, where $M_a' \in \mathcal{M}'$, $T_i' \in \mathcal{T}'$, and C_{jk}^i are structural constants of algebra $\mathcal{A}$.

When the limit of structural constants is well defined, some other properties of the contracted algebra can be also found as a limit of the properties of the starting algebra – eg. group parameters, matrix elements of the operators, representation space basis vectors, Casimir operators. [34].

In this way, simultaneous Inönü-Wigner contraction of spatial momenta and boost generators transforms Poincare algebra into Galilei one (that is, by contraction w.r.t. subgroup generated by spatial rotations and time translation). Another example of Inönü-Wigner contraction is a transformation of three dimensional rotation algebra into Euclidean algebra in two dimension, or contraction of (anti)de Sitter algebra into Poincare algebra (by contraction of four generalized rotations into four-momenta: $P_\mu = \epsilon M_{4\mu}$, where $M_{\mu\nu} \in so(3,2)$ or $M_{\mu\nu} \in so(4,1)$).

In the case of $sl(n,\mathbb{R})$ algebras we are interested in the contraction w.r.t. the maximal compact subalgebra $so(n)$. Algebra $sl(n,\mathbb{R})$ contains $n(n-1)/2$ elements of rotational subalgebra $M_{ab} \in \mathcal{M} = so(n)$, $a,b = 1,2,...,n$ (corresponding to antisymmetric real matrices , $M_{ab} = -M_{ba}$) and $n(n+1)/2-1$ noncompact generators $T_{ab} \in \mathcal{T}$, $a,b = 1,2,...,n$ (corresponding to traceless symmetric real matrices $T_{ab} = T_{ba}$). In the context of space-time symmetries and deformations of rigid bodies, the latter are known as shear generators.

Structural relations of the special linear algebra, in Cartesian basis, are:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), && (2)\\
[M_{ab}, T_{cd}] &= i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), && (3)\\
[T_{ab}, T_{cd}] &= i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). && (4)
\end{aligned}$$

Inönü-Wigner contraction w.r.t. the maximal compact subgroup is given by the following limit:

$$U_{ab} \equiv \lim_{\epsilon\to 0}(\epsilon T_{ab}). \qquad (5)$$

Relations of the contracted algebra are:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) && (6)\\
[M_{ab}, U_{cd}] &= i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) && (7)\\
[U_{ab}, U_{cd}] &= 0. && (8)
\end{aligned}$$

(Above we used the notation U instead of T' for the contracted elements, to avoid excessive use of prime symbols.)

The connection of the two algebras, established by this contraction procedure, can be used to obtain certain classes of representations of the contracted algebra from the known representations of the starting algebra. However, more often it would be of a greater practical merit to establish the opposite type of relation, which is the subject of the next subsection.

2.2 Decontraction

As we saw, the Inönü-Wigner contraction of the $sl(n,\mathbb{R})$ algebra yields a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$, where $r_{\frac{n(n+1)}{2}-1}$ is abelian subalgebra (ideal) of "translations" in $\frac{n(n+1)}{2} - 1$ dimensions. If we knew representations of the special linear algebra, by contraction procedure we could obtain certain classes of representations of this semidirect sum algebra. However, in this case (and in most of the others, as the matter in fact) this is of not much practical use for finding representations. Namely, it is here much more easy to find, by using direct methods of representation theory, representations of the contracted algebra than of the starting special linear one. Therefore, of a great utility would be a method that would allow us the opposite: to get representations of the $sl(n,\mathbb{R})$ starting from known representations of the contracted semidirect sum. It this sense, instead of the limit (5) that expresses elements of the contracted algebra as functions of the starting one, it would be good to have expressions for the elements of the $sl(n,\mathbb{R})$ as functions of $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ operators.

An attempt to establish this type of connection resulted in the Gell-Mann formula [31, 32, 33, 37]. This formula, in its basic form, first time appeared in a paper of Dothan and Ne'eman, back in 1966 [28], and was known as the "decontraction" formula at the time [29, 30]. The formula was largely advocated by Hermann [33, 32], who, on the other hand, had learnt about it from Gell-Mann. Not knowing the details of its genesis, he referred to it as the "Gell-Mann formula". Under this latter name the formula is nowadays known in some textbooks [32] and even in a mathematical encyclopedia [31].

As it traveled a long road since its birth, this formula now appears in a few variants and forms. First we give a definition close to the one given in the encyclopedia [31].

Let $\mathcal{A}$ be a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$ with subalgebra $\mathcal{M}$, so that it holds:

$$[\mathcal{M}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T}, \mathcal{T}] \subset \mathcal{M}. \tag{9}$$

Let $\mathcal{A}'$ be its Inönü-Wigner contraction w.r.t. the subalgebra $\mathcal{M}$. Then $\mathcal{A}' = \mathcal{M}' + \mathcal{U}$ and exists an isomorphism of vector spaces $\pi : \mathcal{A} \to \mathcal{A}'$, given by the Inönü-Wigner contraction, such that $\pi(\mathcal{M}) = \mathcal{M}'$, $\pi(\mathcal{T}) = \mathcal{U}$, $[\pi(M), \pi(A)] = \pi([M, A])$ and $[\pi(T_1), \pi(T_2)] = 0$ where $M \in \mathcal{M}$, $A \in \mathcal{A}$ $T_1, T_2 \in \mathcal{T}$. Let U^2 denote quadratic element of the enveloping algebra of subalgebra $\mathcal{U}$ that is invariant w.r.t. action of subalgebra $\mathcal{M}'$. If D' is a representation of $\mathcal{A}'$ such that $D'(U^2)$ is a multiple of the unit operator, then the Gell-Mann formula for the representations D of algebra $\mathcal{A}$ is:

$$D(T) = \alpha[D'(C_2), D'(\pi(T))] + \sigma D'(\pi(T)), \qquad D(M) = D'(\pi(M)), \tag{10}$$

where $T \in \mathcal{T}, M \in \mathcal{M}$, C_2 is a second order Casimir operator of the enveloping algebra of $\mathcal{M}'$, α is a constant dependant upon $D'(U^2)$ and σ is an arbitrary parameter.

In a mathematically less rigorous way, but closer to the original formulation, the formula can be written as the following operator expression:

$$T_\mu = i\frac{\alpha'}{\sqrt{U^2}}[C_2(\mathcal{M}), U_\mu] + i\sigma U_\mu, \tag{11}$$

where $T_\mu \in \mathcal{T}$, $U_\mu \in \mathcal{U}$ and we assume, just as in the above definition, that the algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$ is Inönü-Wigner contracted into $\mathcal{A}' = \mathcal{M} + \mathcal{U}$, with $T_\mu \to U_\mu$. $C_2(\mathcal{M})$ is quadratic Casimir of the algebra $\mathcal{M}$, α' is a constant, and σ is an arbitrary parameter. By writing the square root of U^2 (U^2 is defined above) as a normalization in the denominator we cancel the dependance of α on U^2, that was present in the formulation (10). In this way we can frite the formula, at least formally, as an operator expresson, unlike the relation (10) that is given at the representations level. This form makes apparent the goal of the formula: to express the operators of the starting algebra as functions of contracted algebra elements.

It is of great importance to establish the domain of validity of the formulas (10) and (11). There is a number of papers on this subject [33, 32, 35, 36]. It is known that the formula is valid for almost all representations in the case of contractions $so(m+1,n) \to r_{m+n} \uplus so(m,n)$ and $so(m,n+1) \to r_{m+n} \uplus so(m,n)$ (problems exist only with representations where U^2 is represented as 0) [35, 36]. Namely, in these cases the relation (11) can be checked to satisfy proper commutation relations.

For example, let the operators $M_{ab}, a, b = 1, 2, ..., n$ satisfy $so(n)$ algebra commutation relation and let define contraction w.r.t. $so(n-1)$ sublagebra as a limit of transformation $P_i = \epsilon M_{ni}, i = 1, 2, ..., n-1$. Contracted algebra satisfies:

$$\begin{aligned} [M_{ij}, M_{kl}] &= i(\delta_{ik} M_{jl} + \delta_{il} M_{jk} - \delta_{jk} M_{il} - \delta_{jl} M_{ki}), & (12) \\ [M_{ij}, P_k] &= i(\delta_{ik} P_j - \delta_{jk} P_i) & (13) \\ [P_i, P_j] &= 0. & (14) \end{aligned}$$

We can explicitly check that the Gell-Mann formula will, in this case, be indeed inverse to the contraction: operators $\overline{M}_{ni}$, *defined* as in (11) as:

$$\overline{M}_{ni} = \frac{i}{2\sqrt{\sum_{j=1}^{n-1} (P_j)^2}} [\tfrac{1}{2} \sum_{j,k=1}^{n-1} (M_{jk})^2, P_i] + i\sigma P_i, \qquad i = 1, 2, ..., n-1, \quad (15)$$

together with subalgebra elements $M_{ij}, i, j = 1, 2, ..., n-1$ will again satisfy structural relations of $so(n)$ algebra (we had to fix the value of constant $\alpha' = \frac{1}{2}$). As the matter in fact, since P_i obviously transform according the the vector representation of the $so(n-1)$ subalgebra, it remains to check:

$$\begin{aligned} [\overline{M}_{ni}, \overline{M}_{nj}] &= -\tfrac{i}{16\sqrt{P^2}} [\{M^{kl}, \delta_{ik} P_l - \delta_{il} P_k\}, \{M^{k'l'}, \delta_{ik'} P_{l'} - \delta_{il'} P_{k'}\}] \\ &= \cdots = \tfrac{i}{\sqrt{P^2}} P_k P_l \delta^{kl} M_{ij} \overset{P^2 \neq 0}{=} i M_{ij}. \qquad (16) \end{aligned}$$

In the above expressions we implied summation convention and Euclidian metrics δ_{ij} with respect to the first $n-1$ coordinates and the curly brackets denote anticommutator. Therefore, we see that in the case of this algebra, the Gell-Mann formula is completely valid, that is, as an algebraic identity (apart from the special case of contracted algebra representations satisfying $P^2 = 0$, when the very formula expression is ill defined).

Unfortunately, the (pseudo)orthogonal algebras are also the only class of of algebras where the Gell-Mann formula is valid in such, algebraic sense. For example, we can try to apply the same Gell-Mann prescription in the $sl(n, \mathbb{R})$ case. In that case, the Gell-Mann formula tells us to look for the sheer generators as the following functions of the contracted algebra elements (6)-(8):

$$T_{ab} = \frac{i\alpha}{\sqrt{\sum (U_{cd})^2}} [C_2(so(n)), U_{ab}] + \sigma U_{ab}. \tag{17}$$

For the sake of later comparison, we mention that the same expression can be also written as:

$$T_{ab} = -\frac{2\alpha}{\sqrt{\sum (U_{cd})^2}} \sum_c U_{c\{a} M_{b\}c} + \sigma' U_{ab}, \tag{18}$$

where σ' differs from σ, and $\{\ \}$ denotes antisymmetrization of the indices in the bracket.

However, if we calculate commutators of so defined shear generators, it will turn out that they do not satisfy $sl(n, \mathbb{R})$ commutation relation (4), more precisely, that additional terms appear on the righthand side. These additional terms are, in general, nonzero, rendering the formula inapplicable. Only in certain representations of the contracted algebra these terms vanish, and for that subclass of representations of the contracted algebra the Gell-Mann formula is valid, resulting in the corresponding subset of representations of the special linear algebra.

The situation is similar in the case of other algebras and their contractions – although the formula is not satisfied algebraically, it can still be valid for a certain subclass of representations. A partial answer to the question of what subclasses these precisely are was given by Hermann in [33] and [32]. However, he did not even attempt to give the complete answer, concentrating, as he said, to "what seems to be the simplest case" and ignoring the cases when the little group (in Wigner's terminology) is nontrivially represented.

On the other hand, this question (for the case of $sl(n,\mathbb{R})$ algebras) is of extreme importance for us, since in the cases when the formula is applicable we have an extremely simple and convenient expression for representation of operators of the special linear algebra. Therefore, the conditions for validity of the Gell-Mann formula in the $sl(n,\mathbb{R})$ case will be discussed in the next section.

3 Domain of validity of the Gell-Mann formula for $sl(n,\mathbb{R})$

3.1 Mathematical framework

In order to make use of the Gell-Mann formula to obtain the $sl(n,\mathbb{R})$ representations, the first necessary step is to determine representation matrix elements of the contracted algebra operators. The corresponding contracted group is a semidirect product of $SO(n)$ and an Abelian group, and it is well known that the usual group induction method provides the complete set of all inequivalent irreducible representations [38]. Nevertheless, we will not pursue the induction approach here. Instead, we will rather proceed to work in the representation space of square integrable functions $\mathcal{L}^2(Spin(n))$ over the $Spin(n)$ group (in accord with the $SL(n,\mathbb{R})$ topological properties), with the standard invariant Haar measure. As for our final goal, this approach ensures certain advantages: (i) The generalized Gell-Mann formula is expressed in terms of tensor operators w.r.t. the maximal compact subgroup basis (instead w.r.t. the eigenvector basis of the Abelian subgroup), (ii) This representation space contains all inequivalent irreducible representations of the contracted group (some of the irreducible representations are multiply contained, i.e. each such representation appears as many times as is the dimension of the corresponding little group representation and all of them, irrespectively of the corresponding stabilizer, can be treated in an unified manner), and (iii) this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra representations [40]. The last feature provides the necessary requirement of a framework needed for generalization of the Gell-Mann formula, i.e. a unique framework providing for all $sl(n, \mathbb{R})$ (unitary) irreducible representations.

The generators of the contracted group are generically represented in this space as follows.

Space $\mathcal{L}^2(Spin(n))$ is the space of the vectors:

$$|\phi\rangle = \int_{Spin(n)} \phi(g)\, |g\rangle\, dg, \qquad g \in Spin(n), \tag{19}$$

where $\phi(g)$ denotes a square integrable function on the $Spin(n)$ group, $|g\rangle$ are (generalized) basis vectors of group elements, and dg is a (normalized) Haar measure.

Operators of $so(n)$ subalgebra act on these vectors in a natural way:

$$M_{ab}\, |\phi\rangle = -i \frac{d}{dt} \exp(itM_{ab}) \Big|_{t=0} |\phi\rangle\, ,$$

where the action of element g' of the $Spin(n)$ group on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(Spin(n))$ is determined by right group action on basis vectors $|g\rangle$ of this space:

$$g'\, |\phi\rangle = g' \int \phi(g)\, |g\rangle\, dg = \int \phi(g)\, |g'g\rangle\, dg, \qquad g', g \in Spin(n). \tag{20}$$

The abelian operators U_{ab} (5) of the contracted algebra in this basis act multiplicatively as Wigner's D-functions ($SO(n)$ group matrix elements as functions of the group parameters):

$$U_{ab} \to |u| D^{\square\!\square}_{w(ab)}(g^{-1}) \equiv |u| \left\langle \begin{matrix} \square\!\square \\ w \end{matrix} \right| (D^{\square\!\square}(g))^{-1} \left| \begin{matrix} \square\!\square \\ ab \end{matrix} \right\rangle, \tag{21}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and in order to simplify notation we denote by $\square\!\square$ (in a parallel to the Young tableaux) the symmetric second rank tensor representation of $SO(n)$. The vector $\left| \begin{matrix} \square\!\square \\ ab \end{matrix} \right\rangle$ from the $\square\!\square$ representation space is determined by the ab "double" index of U_{ab}, whereas the vector $\left| \begin{matrix} \square\!\square \\ v \end{matrix} \right\rangle$ can be an arbitrary vector belonging to the

$\frac{1}{2}n(n+1)-1$ dimensional $\square\!\square$ representation (the choice of v is determined, in Wigner's terminology, by the little group of the obtained representation). Taking an inverse of g in (21) ensures the correct transformation properties. The form of the representation of the Abelian operators merely reflects the fact that they transform as symmetric second rank tensor w.r.t $so(n)$ (7) and that they mutually commute.

A natural discrete orthonormal basis in the $\mathcal{L}^2(Spin(n))$ representation space is given by properly normalized Wigner D-functions:

$$\left\{\left|\begin{array}{c}\{J\}\\ \{k\}\{m\}\end{array}\right\rangle \equiv \int \sqrt{dim(\{J\})}\, D^{\{J\}}_{\{k\}\{m\}}(g^{-1})\, dg\, |g\rangle\right\}, \tag{22}$$
$$\left\langle \begin{array}{c}\{J'\}\\ \{k'\}\{m'\}\end{array}\right|\left.\begin{array}{c}\{J\}\\ \{k\}\{m\}\end{array}\right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{k'\}\{k\}}\delta_{\{m'\}\{m\}},$$

where $D^{\{J\}}_{\{k\}\{m\}}$ are matrix elements of $Spin(n)$ irreducible representations:

$$D^{\{J\}}_{\{k\}\{m\}}(g) \equiv \left\langle \begin{array}{c}\{J\}\\ \{k\}\end{array}\right| D^{\{J\}}(g) \left|\begin{array}{c}\{J\}\\ \{m\}\end{array}\right\rangle. \tag{23}$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $dim(D^{\{J\}})$ representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch-Gordan coefficients of the $Spin(n)$ group as follows:

$$\left\langle \begin{array}{c}\{J'\}\\ \{k'\}\{m'\}\end{array}\right| M_{ab} \left|\begin{array}{c}\{J\}\\ \{k\}\{m\}\end{array}\right\rangle = \delta_{\{J'\}\{J\}}\sqrt{C_2(\{J\})}\; C^{\{J\}\begin{smallmatrix}\square\\\square\end{smallmatrix}\{J'\}}_{\{m\}(ab)\{m'\}}, \tag{24}$$

where $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ denotes $Spin(n)$ representations of second order antisymmetric tensors.

The matrix elements of the U_{ab} operators in this basis are readily found to read:

$$
\begin{aligned}
&\left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| U^{w}_{ab} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle \\
&= |u| \left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| D^{-1\Box\Box}_{w(ab)} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle \\
&= |u| \sqrt{dim(\{J'\}) dim(\{J\})} \int D^{\{J'\}*}_{\{k'\}\{m'\}}(g) D^{\Box\Box}_{w(ab)}(g) D^{\{J\}}_{\{k\}\{m\}}(g) dg \qquad (25) \\
&= |u| \sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}} C^{\{J\}\Box\Box\{J'\}}_{\{k\}\ w\ \{k'\}} \ C^{\{J\}\Box\Box\{J'\}}_{\{m\}(ab)\{m'\}} \ .
\end{aligned}
$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ representations is thus explicitly given in this space by (24) and (25).

Moreover, we introduce the so called, left action generators K as:

$$
K_{ab} \equiv g^{(a''b'')(a'b')} D^{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}_{(ab)(a''b'')} M_{a'b'}, \qquad (26)
$$

where $g^{(a''b'')(a'b')}$ is the Cartan metric tensor of $SO(n)$ and $D^{\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}}$ are multiplicative operators analogous to operators $D^{\Box\Box}$, but that correspond Wigner D-functions of representation of antisymmetric second order tensors.

The K_μ operators behave exactly as the rotation generators M_μ, apart from that that they act on the lower left-hand side indices of the basis (22):

$$
\left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| K_{ab} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle = \delta_{\{J'\}\{J\}} \sqrt{C_2(\{J\})}\ C^{\{J\}\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}\{J'\}}_{\{k\}(ab)\{k'\}} \ . \qquad (27)
$$

Due to the fact that the mutually contragradient $SO(n)$ representations are equivalent, the K_{ab} operators are directly related to the "left" action of the $SO(n)$ subgroup on $\mathcal{L}^2(|g(\theta)\rangle)$: $g'\,|g\rangle = \left|gg'^{-1}\right\rangle$. For this reason we will refer to the group generated by K_{ab} simply as the left orthogonal (sub)group. The K_{ab} and M_{ab} operators mutually commute, however, the corresponding Casimir operators match, i.e.:

$$
\frac{1}{2}\sum_{a,b} K^2_{ab} = \frac{1}{2}\sum_{a,b} M^2_{ab}. \qquad (28)
$$

Commutators of K and $D^{\square\square}$ operators are:

$$[K_{ab}, D^{\square\square}_{(cd)(ef)}] = i(\delta_{ac}D^{\square\square}_{(bd)(ef)} + \delta_{ad}D^{\square\square}_{(cb)(ef)} - \delta_{bc}D^{\square\square}_{(ad)(ef)} - \delta_{bd}D^{\square\square}_{(ca)(ef)}). \quad (29)$$

We note that, in complete analogy with operators $D^{\square\square}$ and $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, it is possible to introduce also operators $D^{\{J\}}_{\{k\}\{m\}}$ that act multiplicatively in the space $\mathcal{L}^2(Spin(n))$ as corresponding Wigner D-functions of representation $\{J\}$. Due to their multiplicative action, these operators obey the same identities that are standardly fulfilled by the Wigner D-functions.

3.2 Condition for the validity of the Gell-Mann formula

The problem with validity of the Gell-Mann formula lies in the fact that commutator of two operators (from the subalgebra $\mathcal{T}$) constructed by using this formula does not always belong to the subalgebra $\mathcal{M}$ with respect to which the contraction has been performed (9), as it should. In the $sl(n, \mathbb{R})$ case that means that the commutator of two shear generators, constructed by using (17) is not equal to a linear combination of operators from $so(n)$ subalgebra. That is, the problem is in the relation (4) which is not satisfied *a priori*, i.e. without imposing additional conditions. On the other hand, relation (3) is automatically satisfied by the construction, due to obvious transformation properties of the Gell-Mann formula constructed operators T_{ab} w.r.t. the $so(n)$ subalgebra.

To investigate circumstances in which relation (4) holds, we will evaluate this commutator using relations and mathematical framework from the previous subsection. For the sake of generality of the results, we do not wish to fix the basis for algebra elements – to stress this, we will use a single letter indices (e.g. T_μ) instead of Cartesian basis double indices (T_{ab})

Using (21) and (28), the Gell-Mann formula (17) now reads:

$$T_\mu = i\alpha[C_2(so(n))_K, D^{\square\square}_{w\mu}] + i\sigma D^{\square\square}_{w\mu}, \quad (30)$$

where $C_2(so(n))_K$ is quadratic Casimir operator of the $so(n)$ subalgebra expressed using K operators (28):

$$C_2(so(n))_K = K_i K_i. \quad (31)$$

Starting from the expression (30) and using known properties of Wigner D-functions, we find:

$$
\begin{aligned}
[T_\mu, T_\nu] &= -2\alpha^2 [K_{\{i}, [K_{j\}}, D^{\yng(2)}_{w\nu}]][K_j, D^{\yng(2)}_{w\mu}]K_i - (\mu \leftrightarrow \nu) \\
&= \cdots = -\alpha^2 \textstyle\sum_J \sum_{\lambda,\lambda'} (C^{\yng(2)\,\yng(2)\,J}_{\mu\ \ \nu\ \ \lambda} - C^{\yng(2)\,\yng(2)\,J}_{\nu\ \ \mu\ \ \lambda}) \cdot \\
&\Big(2(C_2(J) - 2C_2(\yng(2))) \langle\!\langle {}^J_{\lambda'} | 1 \otimes K_i |{}^{\yng(2)}_{w}\rangle |{}^{\yng(2)}_{w}\rangle + \\
&\langle\!\langle {}^J_{\lambda'} | [1 \otimes K_i, C_{2(I+II)_K}] |{}^{\yng(2)}_{w}\rangle |{}^{\yng(2)}_{w}\rangle \Big) D^J_{\lambda'\lambda} K_i.
\end{aligned} \tag{32}
$$

The $C_{2(I+II)_K}$ operator here denotes the second order Casimir operator acting in the tensor product of two $\yng(2)$ representations, i.e. $C_{2(I+II)_K} = \sum_i (K_i \otimes 1 + 1 \otimes K_i)^2$.

The summation index J in (32) runs over all irreducible representations of the $Spin(n)$ group that appear in the tensor product $\yng(2) \otimes \yng(2)$, and λ, λ' count the vectors of these representations. Since all irreducible representations terms, apart those for which the Clebsch-Gordan coefficient $C^{\yng(2)\,\yng(2)\,J}_{\mu\ \ \nu\ \ \lambda}$ is antisymmetric w.r.t. $\mu \leftrightarrow \nu$ vanish, we are left with only two values that J takes: one corresponding to the antisymmetric second order tensor $\yng(1,1)$ and the other one corresponding to the representation that we denote as $\yng(2,2)$. The fact that in the case of $sl(n, \mathbb{R})$ algebras, there is another representation term, in addition to $\yng(1,1)$, in the antisymmetric product of two $\yng(2)$ representations (i.e. representations that correspond to abelian U operators), is in the root of the Gell-Mann formula validity problem. Note that in the case of the $so(m+1, n) \to iso(m, n)$, i.e. $so(m, n+1) \to iso(m, n)$) contractions, where the Gell-Mann formula works on the algebraic level, the contracted U operators transform as $\yng(1)$ and the antisymmetric product of two such representations certainly belongs to the $\yng(1,1)$ representation (i.e. to the representation that corresponds to $\mathcal{M} = so(m, n)$ subalgebra operators).

The $so(n)$ Casimir operator values satisfy $C^2(\yng(2,2)) = 2C^2(\yng(2)) = 4n$, implying that one of the two terms vanishes in (32) when $J = \yng(2,2)$, leaving us with:

$$
\begin{aligned}
\tfrac{1}{2\alpha^2}[T_\mu, T_\nu] = 4(n+2)\textstyle\sum_{\lambda,\lambda'} & C^{\ydiagram{2}\,\ydiagram{2}\,\ydiagram{1,1}}_{\mu\quad\nu\quad\lambda} \langle\!\langle {}^{\ydiagram{1,1}}_{\lambda'} | 1 \otimes K_i | {}^{\ydiagram{2}}_{w} \rangle | {}^{\ydiagram{2}}_{w} \rangle D^{\ydiagram{1,1}}_{\lambda'\lambda} K_i - \\
\textstyle\sum_{\lambda,\lambda'} & C^{\ydiagram{2}\,\ydiagram{2}\,\ydiagram{1,1}}_{\mu\quad\nu\quad\lambda} \langle\!\langle {}^{\ydiagram{1,1}}_{\lambda'} | [1 \otimes K_i, C_{2(I+II)_K}] | {}^{\ydiagram{2}}_{w} \rangle | {}^{\ydiagram{2}}_{w} \rangle D^{\ydiagram{1,1}}_{\lambda'\lambda} K_i - \qquad (33) \\
\textstyle\sum_{\lambda,\lambda'} & C^{\ydiagram{2}\,\ydiagram{2}\,\ydiagram{2,2}}_{\mu\quad\nu\quad\lambda} \langle\!\langle {}^{\ydiagram{2,2}}_{\lambda'} | [1 \otimes K_i, C_{2(I+II)_K}] | {}^{\ydiagram{2}}_{w} \rangle | {}^{\ydiagram{2}}_{w} \rangle D^{\ydiagram{2,2}}_{\lambda'\lambda} K_i,
\end{aligned}
$$

where we used that $C^2(\ydiagram{1,1}) = 2n - 4$.

As the coefficient α can be adjusted freely, all that is needed for the Gell-Mann formula to be valid is that (33) is proportional to the appropriate linear combination of the $Spin(n)$ generators, as determined by the Wigner-Eckart theorem, i.e.:

$$
[T_\mu, T_\nu] \sim \sum_{\lambda} C^{\ydiagram{2}\,\ydiagram{2}\,\ydiagram{1,1}}_{\mu\quad\nu\quad\lambda} M_\lambda = \sum_{\lambda,i} C^{\ydiagram{2}\,\ydiagram{2}\,\ydiagram{1,1}}_{\mu\quad\nu\quad\lambda} D^{\ydiagram{1,1}}_{i\lambda} K_i. \qquad (34)
$$

We now analyze these requirements, skipping some straightforward technical details. The third term in (33), containing D functions of the representation $\ydiagram{2,2}$, is to vanish. Since it is not possible to choose vectors w so that this term vanishes identically as an operator, the remaining possibility is to restrain the the space (22) of its domain to some subspace $V = \{|v\rangle\} \subset \mathcal{L}^2(Spin(n))$. More precisely, for this term to vanish, there must exist a subalgebra $\mathbf{L} \subset so(n)_K$, spanned by some $\{K_\alpha\}$, such that $K_\alpha \in \mathbf{L} \Rightarrow K_\alpha |v\rangle = 0$. Requiring additionally that this subspace V ought to close under an action of the shear generators, and that the first two terms of (33) ought to yield (34), we arrive at the following two necessary conditions:

1. The algebra $\mathbf{L}$, must be a symmetric subalgebra of $so(n)$, i.e.

$$
[\mathbf{L}, \mathbf{N}] \subset \mathbf{N}, [\mathbf{N}, \mathbf{N}] \subset \mathbf{L}; \mathbf{N} = \mathbf{L}^\perp. \qquad (35)
$$

2. The vector $\left| {}^{\ydiagram{2}}_{w} \right\rangle$ ought to be invariant under the L subgroup action (subgroup of $Spin(n)$ corresponding to $\mathbf{L}$), i.e.

$$
K_\alpha \in \mathbf{L} \Rightarrow K_\alpha \left| {}^{\ydiagram{2}}_{w} \right\rangle = 0. \qquad (36)
$$

The space V is thus $Spin(n)/L$. In Wigner's terminology, this means that L is the little group of the contracted algebra representation, and that necessarily it is to be represented trivially. Besides, the little group is to be a symmetric subgroup of the $Spin(n)$ group. This coincides with one class of the solutions found by Hermann [33]. However, now we demonstrated that there are no other solutions in the $sl(n, \mathbb{R})$ algebra cases, in particular, there are no solutions with little group represented non trivially.

As for the first requirement, an inspection of the tables of symmetric spaces, yields two possibilities: $L = Spin(m) \times Spin(n-m)$, where $Spin(1) \equiv 1$, and, for $n = 2k$, $L = U(k)$ (U is the unitary group). However, this second possibility certainly does not imply another solution, since it turns out that there is no vector satisfying the second above property.

Thus, *the only remaining possibility* is as follows,

$$L = Spin(m) \times Spin(n-m), \quad m = 1, 2, \ldots, n-1, \quad Spin(1) \equiv 1. \tag{37}$$

It is rather straightforward but somewhat lengthy to show that proportionality of (33) and (34) really holds in this case. The vector $\left| \begin{array}{c} \square\square \\ w \end{array} \right\rangle$ exists, and it is the one corresponding to traceless diagonal $n \times n$ matrix $diag(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m})$.

The analysis accomplished above can not be applied directly to the $n = 2$ case, thus the $sl(2, \mathbb{R})$ case must be treated separately. The maximal compact subgroup $SO(2)$, that is, its double cover $Spin(2)$, has only one generator M, and therefore it has only one-dimensional irreducible representations. In this case, there are two Abelian generators $U_\pm$ of the contracted group:

$$[M, U_\pm] = \pm U_\pm, \qquad [U_+, U_-] = 0. \tag{38}$$

Based on these relations, it is easy to verify that the $T_\pm$ operators obtained by the Gell-Mann construction as:

$$T_\pm = i[M^2, U_\pm] + i\sigma U_\pm \tag{39}$$

automatically satisfy the $sl(2, \mathbb{R})$ commutation relation:

$$[T_+, T_-] = -2M. \tag{40}$$

Therefore, we demonstrate that the Gell-Mann formula applies to the $sl(2,\mathbb{R})$ case as well.

The above results can be summarized into a conclusion that the formula is valid only in Hilbert spaces over $Spin(n)/(Spin(m) \times Spin(n-m))$, $m = 1, 2, \ldots, n-1$, $Spin(1) \equiv 1$.

3.3 Matrix elements

The presented approach allows us also to write down explicitly the matrix elements of the $sl(n,\mathbb{R})$ generators in the cases when the Gell-Mann formula is valid. The possible cases are determined by the numbers n and m. The corresponding representation space (not irreducible in general) is the one over the coset space $Spin(n)/Spin(m) \times Spin(n-m)$. The proportionality factor α is determined to be:

$$\alpha = \frac{1}{2}\sqrt{\frac{m(n-m)}{n}}, \tag{41}$$

and, in a matrix notation for $\square\!\square$ representation:

$$\left| \begin{matrix} \square\!\square \\ w \end{matrix} \right\rangle = \sqrt{\tfrac{m(n-m)}{n}}\, diag(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m}). \tag{42}$$

The Gell-Mann formula (30), and the matrix representation of the contracted Abelian generators U (25) yield:

$$\begin{gathered} \left\langle \begin{matrix} J' \\ m' \end{matrix} \right| T_\mu \left| \begin{matrix} J \\ m \end{matrix} \right\rangle = \\ i\sqrt{\tfrac{m(n-m)}{4n}}\sqrt{\tfrac{dim(J)}{dim(J')}}(C_2(J') - C_2(J) + \sigma) C^{J\,\square\!\square\,J'}_{0\;\;0\;\;0}\; C^{J\,\square\!\square\,J'}_{m\;\;\mu\;\;m'}\ . \end{gathered} \tag{43}$$

The zeroes in the indices of Clebsch-Gordan coefficients here denote vectors that are invariant w.r.t. $Spin(m) \times Spin(n-m)$ transformations (in that spirit $\left| \begin{matrix} \square\!\square \\ w \end{matrix} \right\rangle = \left| \begin{matrix} \square\!\square \\ 0 \end{matrix} \right\rangle$). In the formula (43), the space reduction from $\mathcal{L}^2(Spin(n))$ to $\mathcal{L}^2(Spin(n)/Spin(m) \times Spin(n-m))$ implies a reduction of the basis (22), i.e. $\left| \begin{matrix} J \\ 0\,m \end{matrix} \right\rangle \to \left| \begin{matrix} J \\ m \end{matrix} \right\rangle$, i.e. only the vectors invariant w.r.t.

left $Spin(m) \times Spin(n-m)$ action remain: $\left| \begin{matrix} J \\ 0\,m \end{matrix} \right\rangle$. By fixing value of the left index to be zero in the basis (22), we effectively lose multiplicity of the representation w.r.t. the $Spin(n)$ subgroup.

The expression (43), together with the action of the $Spin(n)$ generators (24) provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, that is labelled by a free parameter σ. Such representations are multiplicity free w.r.t. the maximal compact $Spin(n)$ subgroup, and all of them are *a priori* tensorial. One can obtain from these representations, for certain σ parameter values, the $sl(n,\mathbb{R})$ spinorial representations as well by explicitly evaluating the Clebsch-Gordan coefficient and performing an appropriate analytic continuation in terms of the $Spin(n)$ labels [39, 30].

3.4 Conclusion on the original Gell-Mann formula for $sl(n,\mathbb{R})$

In this section, we clarified the issue of the Gell-Mann formula validity for the $sl(n,\mathbb{R}) \to r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra contraction. We have shown that the only $sl(n,\mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $Spin(n)/Spin(m) \times Spin(n-m)$, $m = 1, 2, \ldots, n-1$. Moreover, by making use of the Gell-Mann formula in these spaces, we have obtained a closed form expressions of the noncompact operators (generating $SL(n,\mathbb{R})/SO(n)$ cosets) irreducible representations matrix elements. The matrix elements of both compact and noncompact operators of the $sl(n,\mathbb{R})$ algebra are given by (24) and (43), respectively.

In particular, it turns out that, due to Gell-Mann's formula validity conditions, no representations with $so(n)$ subalgebra representations multiplicity can be obtained in this way. Moreover, the matrix expressions of the noncompact operators as given by (43) do not account *a priori* for the $sl(n,\mathbb{R})$ spinorial representations.

Due to mutual connection of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, the results apply to the corresponding $su(n)$ case as well. The $SU(n)/SO(n)$ generators differ from the corresponding $sl(n,\mathbb{R})$ operators by the imaginary unit multiplicative factor, while the spinorial representations issue in the $su(n)$ case is pointless due to the fact that the $SU(n)$ is a simply connected (there exists no double cover) group.

In many physics applications one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present work, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(Spin(n), \kappa)$ of square integrable functions with a scalar product given in terms of an arbitrary kernel κ, and to impose the unitarity constraints both on the scalar products itself and on the noncompact operators matrix elements in that scalar product (cf. [41]). The simplest series of the $sl(n, \mathbb{R})$ unitary irreducible representations, the Principal series, of the representations constructed above are obtained when $\sigma = i\sigma_I$, $\sigma_I \in \mathbb{R}\backslash\{0\}$, i.e. when σ takes an arbitrary nonzero pure imaginary value.

4 Generalization of the Gell-Mann formula in the $sl(n, \mathbb{R})$ case

4.1 Low dimensional cases $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$

In the previous section we have shown that the Gell-Mann formula in $sl(n, \mathbb{R})$ case is of a very limited domain of validity. The reason why the formula (17) is not valid in entire space $\mathcal{L}^2(Spin(n))$ can be understood in the following way. While the $sl(n, \mathbb{R})$ operators M_{ab} are invariant w.r.t. the left action of the $Spin(n)$ group in this space, i.e. they commute with operators K_{ab}, it is not the case with the shear generators T_{ab}, as constructed by using the Gell-Mann formula (17). These transformation properties of shear generators T_{ab} are inherited from the corresponding operators U_{ab} of the contracted algebra. Their nontrivial transformation properties w.r.t. the left action of the $Spin(n)$ group are determined by the choice of the vector w in (21). As a consequence, a commutator of two such operators *a priori* will also have nontrivial transformation properties w.r.t. $SO(n)_K$ group (the one generated by K_μ).

Therefore, unlike M_μ, this commutator is not scalar w.r.t. $SO(n)_K$ action, so that commutation relation (4) is not satisfied. In certain cases it is possible to restrict representations space to a subspace in such a way that only $SO(n)_K$ invariant part of commutator $[T_\mu, T_\nu]$ remains. That is exactly what happens in the cases discussed in the last section, when the Gell-Mann formula is valid.

This analysis gives a motivation to attempt to modify the formula by adding some terms proportional to the generators of the left $SO(n)_K$ group, in such a way to cancel the unwanted terms in the commutator $[T_\mu, T_\nu]$. Indeed, such a generalization of the Gell-Mann formula can be effectively read out from the known form of the matrix elements for $sl(3,\mathbb{R})$ representations with multiplicity. Namely, in the form of $sl(3,\mathbb{R})$ matrix element expression from the paper [41] it is possible to recognize terms that correspond to the Gell-Mann formula, together with certain additional terms. Therefore, in the $n=3$ case, using the results of [41] we can directly write:

$$T_\mu = \sigma D^2_{0\mu} + \frac{i}{\sqrt{6}}[C_2(so(3)), D^2_{0\mu}] + i(D^2_{2\mu} - D^2_{-2\mu})K_0 + \delta(D^2_{2\mu} + D^2_{-2\mu}). \quad (44)$$

We used the standard spherical basis for $Spin(3)$ representations, with $\square\square$ representations here corresponding to $J=2$, and the vectors within this representation are labeled by $\mu = 0, \pm1, \pm2$. Complex numbers σ and δ are parameterizing the representations of the $sl(3,\mathbb{R})$ group. The first two terms we recognize as the original Gell-Mann formula (30), with vector w chosen to be invariant w.r.t. the action of the K_3 (i.e. chosen is vector $|J=2,\mu=0\rangle$. The additional terms to the "original" Gell-Mann formula secure that the T_μ operators satisfy the commutation relation (4) in the entire representation space. Note that there are two $sl(3,\mathbb{R})$ representation labels σ and δ, matching the algebra rank, contrary to the case of the original Gell-Mann formula whose single free parameter cannot account for the entire representation labeling. Notice also that additional terms in the formula (44) change value of projection K_3, that is, action of these terms on vector form the basis (22) will change the multiplicity label k (in general leading to nontrivial multiplicity of the representations).

The generalized expression (44) contains the original formula as a special case: by restricting the representation space $\mathcal{L}^2(Spin(3))$ to the subspace of $k=0$ (that is the subspace $\mathcal{L}^2(Spin(3)/Spin(2))$), and choosing $\delta = 0$ one arrives at the multiplicity free representations that were obtained by using the original formula. Moreover, the generalized Gell-Mann formula allows one to obtain some $sl(3,\mathbb{R})$ multiplicity free representations that cannot be reached by making use of the original formula. For example, with the choice $\sigma = \frac{3}{2}$, and $\delta = -\frac{1}{2}$ [42], a subspace spanned by the following vectors (linear combinations of basis vectors with different k values):

$$
\begin{gathered}
\Big\{ \left| \begin{smallmatrix} \frac{1}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{1}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{1}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right\rangle, \\
\left| \begin{smallmatrix} \frac{5}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{5}{2} & \\ \frac{5}{2} & m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{5}{2} & \\ -\frac{5}{2} & m \end{smallmatrix} \right\rangle, \\
\left| \begin{smallmatrix} \frac{9}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{9}{2} & m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{5}{2} & m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} & \\ \frac{1}{2} & m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{1}{2} & m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{5}{2} & m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} & \\ -\frac{9}{2} & m \end{smallmatrix} \right\rangle, \dots \Big\}
\end{gathered}
\tag{45}
$$

is invariant w.r.t. the action of $sl(3,\mathbb{R})$ operators. At the same time, subspaces with fixed value of are here one dimensional and the values of are half-integer. Therefore, this is an example of a spinorial multiplicity free $sl(3,\mathbb{R})$ representation. More precisely, this is the unique unitary $sl(3,\mathbb{R})$ spinorial multiplicity free representation $sl(3,\mathbb{R})$, and this representation cannot be obtained by application of the original Gell-Mann formula (without resorting to certain analytical continuation of the Clebsch-Gordan coefficient expressions [41]).

Matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity are also known ($n = 4$ is the largest dimension with known matrix elements). The Gell-Mann formula thus can similarly be generalized in the case of the $sl(4,\mathbb{R})$ algebra. Again, by extracting from the known matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity [30], we find:

$$
\begin{aligned}
T_{\mu_1\mu_2} &= i\Big(\sigma D^{11}_{00\mu_1\mu_2} + \tfrac{1}{2}[C_2(so(4)), D^{11}_{00\mu_1\mu_2}] \\
&+\delta_1(D^{11}_{11\mu_1\mu_2} + D^{11}_{-1-1\mu_1\mu_2}) + (D^{11}_{11\mu_1\mu_2} - D^{11}_{-1-1\mu_1\mu_2})(K^{10}_{00} + K^{01}_{00}) \\
&+\delta_2(D^{11}_{-11\mu_1\mu_2} + D^{11}_{1-1\mu_1\mu_2}) + (D^{11}_{-11\mu_1\mu_2} - D^{11}_{1-1\mu_1\mu_2})(K^{10}_{00} - K^{01}_{00})\Big),
\end{aligned}
\tag{46}
$$

where we used $Spin(4) \supset (Spin(2) \times Spin(2))$ basis and $\mu_1, \mu_2 = 0, \pm 1$. As the rank of the $sl(4,\mathbb{R})$ algebra is three, there are precisely three representation labels σ, δ_1, and δ_2 (if complex, only three real are independent).

As in the $sl(3,\mathbb{R})$ case, the generalized formula reduces, for certain values of the labels $\delta_1 = \delta_2 = 0$, in a representation subspace defined by $K^{10}_{00} = K^{01}_{00} = 0$ (i.e. in the subspace $\mathcal{L}^2(Spin(4)/(Spin(2) \times Spin(2))))$ to the original Gell-Mann formula.

If we express the generalized Gell-Mann formula for $sl(4,\mathbb{R})$ in a basis that corresponds to the subgroup chain $Spin(4) \supset Spin(3) \supset Spin(2)$, we obtain:

$$\begin{aligned} T_{j,\mu} = \gamma_1 D^{\;11}_{\substack{0\ j\\ 0\ \mu}} - \tfrac{i\sqrt{3}}{4}[C_2(so(4)), D^{\;11}_{\substack{0\ j\\ 0\ \mu}}]+ \\ \gamma_2 D^{\;11}_{\substack{2\ j\\ 0\ \mu}} + \tfrac{i}{\sqrt{6}}[C_2(so(3))_K, D^{\;11}_{\substack{2\ j\\ 0\ \mu}}]+ \\ \gamma_3 (D^{\;11}_{\substack{2\ j\\ 2\ \mu}} + D^{\;11}_{\substack{2\ j\\ -2\mu}}) + (D^{\;11}_{\substack{2\ j\\ 2\ \mu}} - D^{\;11}_{\substack{2\ j\\ -2\mu}})(K^{10}_{\substack{1\\0}} + K^{01}_{\substack{1\\0}}), \end{aligned} \tag{47}$$

where $j = 0, 1, 2$, $|\mu| \leq j$, and $C_2(so(3))_K$ denotes quadratic Casimir operator of the left $SO(3)$ subgroup, generated by

$$\{K^{10}_{\substack{1\\-1}} + K^{01}_{\substack{1\\-1}}, K^{10}_{\substack{1\\0}} + K^{01}_{\substack{1\\0}}, K^{10}_{\substack{1\\1}} + K^{01}_{\substack{1\\1}}\}$$

(in Cartesian basis: $\{K_{12}, K_{23}, K_{31}\}$). We note that in subspace $\mathcal{L}^2(Spin(4))$ it holds $C_2(so(4)) \equiv C_2(so(4))_M = C_2(so(4))_K$, but this connection does not exist for the Casimir operators of the subalgebras, i.e. $C_2(so(3))_M \neq C_2(so(3))_K$! Relation of the parameters used in (46) and (47) is $i\sigma = -\frac{1}{\sqrt{3}}\gamma_1 + \sqrt{\frac{2}{3}}\gamma_2 - 2i$, $\delta_1 = \gamma_3$, and $\delta_2 = \frac{1}{\sqrt{3}}\gamma_1 + \frac{1}{\sqrt{6}}\gamma_2 - 2i$. From this form of the expression can be easily seen that, if we choose $\gamma_2 = \gamma_3 = 0$, the generalized formula reduces to the original one in the subspace $\mathcal{L}^2(Spin(4)/Spin(3))$, since in that subspace the additional terms vanish (and only the first row of the expression remains).

4.2 Generalization of the Gell-Mann formula for $sl(5,\mathbb{R})$

In the previous subsection, thanks to the known matrix elements of the $sl(n,\mathbb{R})$, $n = 3, 4$ representations with multiplicity, we generalized Gell-Mann formula for the $\overline{SL}(3,\mathbb{R})$ and $\overline{SL}(4,\mathbb{R})$ groups, finding a formula that is valid in entire space of square integrable functions over the compact subgroup. In this subsection we will construct a generalization of the Gell-Mann formula for $sl(5,\mathbb{R})$ case and, as a direct application, we will derive matrix elements of $sl(5,\mathbb{R})$ generators for arbitrary irreducible representation. This approach to the problem of finding matrix elements is particularly important since the matrix elements for $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ were in [41] and [30] found in a very computationally involved way, that would be difficult to repeat for the higher dimensional $sl(5,\mathbb{R})$ case.

As a hint toward a way to generalize the formula in $n = 5$ case we note a certain recursive pattern in transition from $n = 3$ to $n = 4$ can be seen by comparing the expressions (44) and (47). Namely, in the formula (47) all additional terms coincide with the generalized formula for $sl(3,\mathbb{R})$ itself (44), where only the quadratic Casimir operator $C_2(so(3))$ is replaced by $C_2(so(3))_K$. It turns out that transition from $n = 4$ to $n = 5$ can be obtained in a similar manner.

Let us recall first some basic $so(5)$ algebra representation notions. The $so(5)$ algebra is of rang two, and its irreducible representations are labeled by a pair of labels $(\overline{J}_1, \overline{J}_2)$, resembling the $so(4)$ labeling. The complete labeling of the representation space vectors can be achieved by making use of the subalgebra chain: $so(5) \supset so(4) = so(3) \oplus so(3) \supset so(2) \oplus so(2)$. The basis of the $so(5)$ algebra representation space can be taken as in [43, 44]:

$$\left\{ \left| \begin{array}{cc} \overline{J}_1 & \overline{J}_2 \\ J_1 & J_2 \\ m_1 & m_2 \end{array} \right\rangle, \quad \overline{J}_i = 0, \frac{1}{2}, \ldots; \quad \overline{J}_1 \geq \overline{J}_2; \quad |m_i| \leq J_i, \quad i = 1, 2 \right\}. \tag{48}$$

The admissible values of J_1 and J_2, within an irreducible representation $(\overline{J}_1, \overline{J}_2)$ are given in [45]. Now, the basis of the $so(5)$ algebra, i.e. the $Spin(5)$ group, representation space vectors (22) is given as follows:

$$\left\{ \left| \begin{array}{cccc} \overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2 \\ k_1 & k_2 & m_1 & m_2 \end{array} \right\rangle \right\}. \tag{49}$$

The ten $so(5)$ algebra operators, generating the adjoint representation of $Spin(5)$, transform, in notation (48), under the representation $(\overline{1}, \overline{0})$. Their $so(4)$ subalgebra representation content is: $(\overline{1}, \overline{0}) \to (1,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,1)$. The shear operators transform under the 14-dimensional $so(5)$ irreducible representation $(\overline{1}, \overline{1})$ of $so(5)$ which contains $(1,1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0,0)$ representation upon reduction to $so(4)$:

$$\left\{ T_{\mu_1 \mu_2}^{j_1 j_2} \right\} = \left\{ T_{\mu_1 \mu_2}^{1\,1}, T_{\mu_1 \mu_2}^{\frac{1}{2}\frac{1}{2}}, T_{00}^{00} \right\}.$$

Now, by analogy to the transition from $n = 3$ to $n = 4$ (44, 47) we will make the following educated guess for the form of the $sl(5, \mathbb{R})$ generalized Gell-Menn formula:

$$
\begin{aligned}
T_{\mu_1\mu_2}^{j_1j_2} \;=\;& \sigma_1 D^{\substack{\overline{11}\\00}\,j_1\,j_2}_{00\,\mu_1\,\mu_2} + i\alpha_5[C_2(so(5)), D^{\substack{\overline{11}\\00}\,j_1\,j_2}_{00\,\mu_1\,\mu_2}] \\
&+i\Bigg(\sigma_2 D^{\substack{\overline{11}\\11}\,j_1\,j_2}_{00\,\mu_1\,\mu_2} + \tfrac{1}{2}[C_2(so(4)_K), D^{\substack{\overline{11}\\11}\,j_1\,j_2}_{00\,\mu_1\,\mu_2}] \\
&-D^{\substack{\bar{1}\,\bar{1}\\1\,1}\,j_1\,j_2}_{1-1\,\mu_1\,\mu_2}\,(\delta_1 + K^{\substack{\overline{10}\\10}}_{00} - K^{\substack{\overline{10}\\01}}_{00}) - D^{\substack{\bar{1}\,\bar{1}\\1\,1}\,j_1\,j_2}_{-11\,\mu_1\,\mu_2}\,(\delta_1 - K^{\substack{\overline{10}\\10}}_{00} + K^{\substack{\overline{10}\\01}}_{00}) \\
&+D^{\substack{\overline{11}\\11}\,j_1\,j_2}_{11\,\mu_1\,\mu_2}\,(\delta_2 + K^{\substack{\overline{10}\\10}}_{00} + K^{\substack{\overline{10}\\01}}_{00}) + D^{\substack{\bar{1}\,\bar{1}\\1\,1}\,j_1\,j_2}_{-1-1\,\mu_1\,\mu_2}\,(\delta_2 - K^{\substack{\overline{10}\\10}}_{00} - K^{\substack{\overline{10}\\01}}_{00})\Bigg),
\end{aligned}
\tag{50}
$$

where $j_i = 0, \frac{1}{2}, 1$, $|\mu_i| \leq j_i$, $i = 1, 2$, the representation labels $\sigma_1, \sigma_2, \delta_1$ and δ_2 are arbitrary (complex) parameters (four real are independent), and $C_2(so(4)_K)$ denotes the quadratic Casimir operator of the left action $so(4)_K$ algebra. Coefficient α_5 is determined from the requirement that that commutation relations of the $[T, T] \subset M$ type should be satisfied (4). And indeed, with the choice $\alpha_5 = \frac{1}{\sqrt{5}}$ this relation can be checked to hold.

Unlike the $sl(n, \mathbb{R})$, $n = 3, 4$ cases where we started from the known expressions for matrix elements of the shear oeprators, here, in the case of $sl(5, \mathbb{R})$ algebra, using the obtained generalization of the Gell-Mann formula (50) we can now derive matrix elements for arbitrary $sl(5, \mathbb{R})$ representation (given by the parameters $\sigma_1, \sigma_2, \delta_1$ and δ_2).

Matrix elements of the $sl(5, \mathbb{R})$ shear generators are:

$$
\begin{aligned}
&\left\langle \begin{matrix} \overline{J}'_1\,\overline{J}'_2 \\ K'_1K'_2J'_1\;J'_2 \\ k'_1\;k'_2\;m'_1m'_2 \end{matrix} \,\middle|\, T_{\mu_1\mu_2}^{j_1j_2} \,\middle|\, \begin{matrix} \overline{J}_1\,\overline{J}_2 \\ K_1K_2J_1\;J_2 \\ k_1\;k_2\;m_1m_2 \end{matrix} \right\rangle = \sqrt{\frac{dim(\overline{J}_1,\overline{J}_2)}{dim(\overline{J}'_1,\overline{J}'_2)}}\, C^{\overline{J}_1\overline{J}_2\;\bar{1}\,\bar{1}\;\overline{J}'_1\overline{J}'_2}_{\substack{J_1\,J_2\;\;j_1j_2\;\;J'_1\,J'_2\\ m_1m_2\;\mu_1\mu_2\;m'_1m'_2}} \\
&\times\Bigg(\Big(\sigma_1 + i\sqrt{\tfrac{4}{5}}(\overline{J}'_1(\overline{J}'_1+2) + \overline{J}'_2(\overline{J}'_2+1) - \overline{J}_1(\overline{J}_1+2) - \overline{J}_2(\overline{J}_2+1))\Big) C^{\overline{J}_1\overline{J}_2\;\overline{11}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;00\;K'_1K'_2\\ k_1\,k_2\;\;00\;\;k'_1\,k'_2}} \\
&+ i\Big(\sigma_2 + K'_1(K'_1+1) + K'_2(K'_2+1) - K_1(K_1+1) - K_2(K_2+1)\Big) C^{\overline{J}_1\overline{J}_2\;\overline{11}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;11\;K'_1K'_2\\ k_1\,k_2\;\;00\;\;k'_1\,k'_2}} \\
&- i(\delta_1 + k_1 - k_2) C^{\overline{J}_1\overline{J}_2\;\overline{11}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;11\;K'_1K'_2\\ k_1\,k_2\;1-1\;k'_1\,k'_2}} - i(\delta_1 - k_1 + k_2) C^{\overline{J}_1\overline{J}_2\;\bar{1}\;\bar{1}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;1\;1\;K'_1K'_2\\ k_1\,k_2\;-11\;k'_1\,k'_2}} \\
&+ i(\delta_2 + k_1 + k_2) C^{\overline{J}_1\overline{J}_2\;\overline{11}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;11\;K'_1K'_2\\ k_1\,k_2\;11\;k'_1\,k'_2}} + i(\delta_2 - k_1 - k_2) C^{\overline{J}_1\overline{J}_2\;\bar{1}\;\bar{1}\;\overline{J}'_1\overline{J}'_2}_{\substack{K_1K_2\;1\;1\;K'_1K'_2\\ k_1\,k_2\;-1-1\;k'_1\,k'_2}} \Bigg).
\end{aligned}
\tag{51}
$$

$dim(\overline{J}_1,\overline{J}_2) = (2\overline{J}_1 - 2\overline{J}_2 + 1)(2\overline{J}_1 + 2\overline{J}_2 + 3)(2\overline{J}_1 + 2)(2\overline{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation $(\overline{J}_1,\overline{J}_2)$ [45].

To summarize: matrix elements of (noncompact) shear generators (51), together with the known matrix elements of the (compact) $so(5)$ operators (24), give an action of the $sl(5,\mathbb{R})$ algebra on the basis vectors (49) space over the maximal compact subgroup $Spin(5)$ of the group $\overline{SL}(5,\mathbb{R})$. This result is general due to Corollary from the Harish-Chandra paper [40], that is directly applicable to the case of $sl(5,\mathbb{R})$ algebra.

4.3 Generalization of the Gell-Mann formula for arbitrary n

The generalized Gell-Mann formulas for $sl(3,\mathbb{R})$, $sl(4,\mathbb{R})$ and $sl(5,\mathbb{R})$ [25] are given by rather cumbersome expressions. However, when these formulas are expressed in the Cartesian basis (like formulas (2)-(4)) in terms of the K_{ab} operators and anti-commutators rather than commutators the resulting expressions become extremely simple. Moreover, this form allows for an immediate generalization to the case of an arbitrary n. We prove below that the generalized Gell-Mann formula for any $sl(n,\mathbb{R})$ algebra w.r.t its $so(n)$ subalgebra takes the following form::

$$T_{ab}^{\sigma_2\ldots\sigma_n} = -\sum_{c>d}^{n}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sum_{c=2}^{n}\sigma_c D^{\square\square}_{(cc)(ab)}, \tag{52}$$

where σ_c is a set of $n-1$ arbitrary parameters that essentially (up to some discrete parameters) label $sl(n,\mathbb{R})$ irreducible representations. Note that the sum in the first term goes only over pairs (c,d) where $c>d$ i.e. it is not symmetric in $c,\ d$.

Let us begin the proof that the expressions (52) satisfy the $sl(n,\mathbb{R})$ commutation relation (4) by introducing operators:

$$T_{ab}^{[c]} = -\sum_{d=1}^{c-1}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sigma_c D^{\square\square}_{(cc)(ab)}, \quad c=2,\ldots,n \tag{53}$$

and expressing the generalized expression (52) as:

$$T_{ab} = \sum_{c=2}^{n} T_{ab}^{[c]}. \tag{54}$$

Next we calculate the commutator $[T_{ab}^{[c]}, T_{a'b'}^{[d]}]$ for $c < d$:

$$\begin{aligned}
[T_{ab}^{[c]}, T_{a'b'}^{[d]}] &= \sum_{e=1}^{c-1}[-\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, -\sum_{f=1}^{d-1}\{K_{df}, D^{\square\square}_{(df)(a'b')}\}] \\
&+ \sum_{e=1}^{c-1} i\sigma_d[-\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, D^{\square\square}_{(dd)(a'b')}] \\
&+ \sum_{f=1}^{d-1} i\sigma_c[D^{\square\square}_{(cc)(ab)}, -\{K_{df}, D^{\square\square}_{(df)(a'b')}\}] \\
&= \sum_{e=1}^{c-1}[\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, \{K_{dc}, D^{\square\square}_{(dc)(a'b')}\} + \{K_{de}, D^{\square\square}_{(de)(a'b')}\}] \\
&+ 0 - i\sigma_c[D^{\square\square}_{(cc)(ab)}, \{K_{dc}, D^{\square\square}_{(dc)(a'b')}\}] \\
&= \cdots = i\sum_{e=1}^{c-1}\{K_{ce}, \{D^{\square\square}_{(ed)(ab)}, D^{\square\square}_{(cd)(a'b')}\} + \{D^{\square\square}_{(cd)(ab)}, D^{\square\square}_{(ed)(a'b')}\}\} \\
&+ 2\sigma_c\{D^{\square\square}_{(dc)(ab)}, D^{\square\square}_{(dc)(a'b')}\}.
\end{aligned} \tag{55}$$

The result is symmetric under the change of pair of indices $(ab) \leftrightarrow (a'b')$, and similarly can be concluded when $c > d$. We conclude that for $c \neq d$ it holds $[T_{ab}^{[c]}, T_{a'b'}^{[d]}] = [T_{a'b'}^{[c]}, T_{ab}^{[d]}]$, and thus we find::

$$[T_{ab}, T_{a'b'}] = \sum_{c}[T_{ab}^{[c]}, T_{a'b'}^{[c]}] = i\sum_{c,d,d'}\{K_{dd'}, \{D^{\square\square}_{(cd)(ab)}, D^{\square\square}_{(cd')(a'b')}\}\}. \tag{56}$$

By making use of the identity:

$$\begin{aligned}
&\textstyle\sum_c (D^{\square\square}_{(cd)(ab)} D^{\square\square}_{(cd')(a'b')} - D^{\square\square}_{(cd')(ab)} D^{\square\square}_{(cd)(a'b')}) \\
&= \tfrac{1}{2}(\delta_{aa'} D^{\substack{\square\\\square}}_{(dd')(bb')} + \delta_{bb'} D^{\substack{\square\\\square}}_{(dd')(aa')} + \delta_{ab'} D^{\substack{\square\\\square}}_{(dd')(ba')} + \delta_{ba'} D^{\substack{\square\\\square}}_{(dd')(ab')}),
\end{aligned} \tag{57}$$

and the fact that the M generators are given in terms of the K operators via the $D^{\substack{\square\\ \square}}$ operators (cf. (26)), one verifies the desired expression (4).

Note that the first equality in (56) implies that the overall sign of operators $T^{[c]}_{ab}$ is inessential. Moreover, any left rotation (generated by the K operators) of the generalized formula (52) will preserve the $[T, T]$ commutator (4) and thus lead to another valid expression for the generalized Gell-Mann formula. The generalized formulas related in this way form an equivalence class of formulas that yield the same set of $sl(n, \mathbb{R})$ irreducible representations. Besides this class there are a few alternative useful expressions of the generalized Gell-Mann formula. We point out explicitly two cases below.

Let us consider operators:

$$U^{(cd)}_{ab} \equiv D^{\square\square}_{(cd)(ab)}, \tag{58}$$

stressing that $D^{\square\square}_{(cd)(ab)}$ is just a particular representation of the U_{ab} operators (21), characterized by the choice of the vector v to be $v = (cd)$ and $|u|= 1$. Then, by making use of the commutation relations to shift the K operators to the right in (52) we find :

$$T_{ab} = -2 \sum_{c>d}^{n} U^{(cd)}_{ab} K_{cd} + i \sum_{c=2}^{n} \sigma'_c U^{(cc)}_{ab}. \tag{59}$$

The parameters in the two forms of the formula are connected by relation: $\sigma'_c = \sigma_c - 2(c-1)$.

The last expression for the generalized formula can now be directly compared to the original formula in the form (18). It is as simple as the original Gell-Mann formula, with a crucial advantage of being valid in the whole representation space over $\mathcal{L}^2(Spin(n))$. General validity of the new formula is reflected in the fact that there are now $n-1$ free parameters, i.e. representation labels, matching the $sl(n, \mathbb{R})$ algebra rank, compared to just one parameter of the original Gell-Mann formula.

Another notable form of the generalized formula relies on the fact that the operators $T^{[c]}$ (53) can be written as:

$$T^{[c]}_{ab} = \frac{i}{2}[C_2(so(c)_K), U^{(cc)}_{ab}] + i\sigma_c U^{(cc)}_{ab}, \quad c = 2, \dots, n \tag{60}$$

where $C_2(so(c)_K)$ is the second order Casimir of the $so(c)$ left action subalgebra, i.e. $C_2(so(c)_K) = \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$. The generalized Gell-Mann formula can now be written as:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}, \tag{61}$$

which is to be compared with the original formula in the form (17). Again, the generalized formula matches, by simplicity of the expression, the original one. Besides, the very term when $c = n$ is, essentially, the original Gell-Mann formula (since $C_2(so(n)_K) = C_2(so(n)_M)$), whereas the rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some representations yielding the original formula.

The generalized Gell-Mann formula expression for the noncompact "shear" generators T_{ab} holds for all cases of $sl(n,\mathbb{R})$ irreducible representations, irrespective of their $so(n)$ subalgebra multiplicity (multiplicity free of the original Gell-Mann formula, and nontrivial multiplicity) and whether they are tensorial or spinorial. The price paid is that the Generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators K_{ab} and $U^{(cd)_{ab}}$.

4.4 Direct application – martix elements of $SL(n,\mathbb{R})$ generators for arbitrary irreducible representation

The generalized Gell-Mann formula, as given by (61), can be directly applied to yield all matrix elements of the $\overline{SL}(n,\mathbb{R})$ generators for all irreducible representations, characterized by a complete set of labels σ_i, $i = 2,3,\ldots,n$ (the invariant Casimir operators are analytic functions of solely these labels), in the basis of the maximal compact subgroup $Spin(n)$. Taking the matrix elements of (61) we get:

$$\left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| T^{\sigma_2\ldots\sigma_n}_{ab} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle \tag{62}$$

$$= \left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| i\sum_{c=2}^{n} \frac{1}{2}[C_2(so(c)_K), U^{(cc)}_{ab}] + \sigma_c U^{(cc)}_{ab} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle$$

$$= \frac{i}{2}\sum_{c=2}^{n} (C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) \left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \right| U^{(cc)}_{ab} \left| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle$$

$$= \frac{i}{2}\sqrt{\frac{dim(\{J\})}{dim(\{J'\})}} \sum_{c=2}^{n} (C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) C^{\{J\}\Box\Box\{J'\}}_{\{k\}(cc)\{k'\}}\; C^{\{J\}\Box\Box\{J'\}}_{\{m\}(ab)\{m'\}}\,,$$

where, in the last equality, the expression (25) for the matrix elements of the U operators is used. The second Clebsch-Gordan coefficient, that is merely reflecting the Wigner-Eckart theorem, can be evaluated in any suitable basis, not necessarily the Cartesian one, due to the fact that the expression is covariant with respect to the free index (ab). Note, that this is not the case for the first Clebsch-Gordan coefficient – it is necessary in order to evaluate it to express the specific vector $\left| \begin{matrix}\Box\Box\\ (cc)\end{matrix} \right\rangle$ in some basis that spans the entire vector space over $Spin(n)$.

The final expression is simplified by choosing the indexes of the generalized Gell-Mann formula matrix elements to be given by labels of the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ group chain representation labels. In this notation, the basis vectors of the $Spin(n)$ irreducible representations are written as:

$$\left| \begin{matrix}\{J\}\\ \{m\}\end{matrix} \right\rangle = \left| \begin{matrix} J_{Spin(n),1}\ J_{Spin(n),2}\ J_{Spin(n),3}\cdots \\ J_{Spin(n-1),1}\ J_{Spin(n-1),2}\cdots \\ \cdots \\ J_{Spin(2)} \end{matrix} \right\rangle . \tag{63}$$

Likewise, the set of indices $\{k\}$ of (22) is thus given by the labels of the irreducible representations $\{J_{Spin(n-1),1}, J_{Spin(n-1),2}, \cdots; J_{Spin(n-2),1}, J_{Spin(n-2),2}, \cdots; \ldots; J_{Spin(2)}\}$ of the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ group chain.

To express the vector $\left| \begin{matrix}\Box\Box\\ (cc)\end{matrix} \right\rangle$ in such a basis we notice first that it corresponds to a diagonal traceless n by n matrix of the form $diag(-\frac{1}{n}, \ldots, -\frac{1}{n}$

$, \frac{n-1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n})$, with $\frac{n-1}{n}$ positioned at the c-th row and column. On the other hand, the diagonal traceless matrix $\sqrt{\frac{1}{c(c-1)}}\, diag(-1, \ldots, -1, c-1, 0, \ldots, 0)$, with first $c-1$ occurrences of -1, corresponds to a vector that belongs to a second order symmetric tensor ($\square\!\square$ representation) with respect to $Spin(c), Spin(c+1), \ldots, Spin(n)$ subgroups, and it is invariant under $Spin(c-1)$:

$$\left| \begin{array}{l} \{\square\!\square\}_{Spin(n)} \\ \ldots \\ \{\square\!\square\}_{Spin(c)} \\ \{0\}_{Spin(c-1)} \\ \ldots \\ 0 \end{array} \right\rangle . \tag{64}$$

This vector has $n-c+1$ double-boxes followed by $c-2$ zeros underneath – in shorthand notation: $\left| \begin{array}{l} \{\square\!\square\}^{n-c+1} \\ \{0\}^{c-2} \end{array} \right\rangle$. Somewhat peculiar is the matrix $\sqrt{\frac{1}{2}}\, diag(-1, 1, 0, 0, \ldots)$ that corresponds to:

$$\left| \begin{array}{l} \{\square\!\square\}^{n-1} \\ \{0\}^{0} \end{array} \right\rangle \equiv \frac{1}{\sqrt{2}} \left| \begin{array}{l} \{\square\!\square\}^{Spin(n)} \\ \ldots \\ \{\square\!\square\}^{Spin(4)} \\ 2 \\ 2 \end{array} \right\rangle + \frac{1}{\sqrt{2}} \left| \begin{array}{l} \{\square\!\square\}^{Spin(n)} \\ \ldots \\ \{\square\!\square\}^{Spin(4)} \\ 2 \\ -2 \end{array} \right\rangle , \tag{65}$$

where the standard labelling for $SO(n)$, $n \leq 3$ is implied, in particular the $\square\!\square$ representation corresponds to $J_{Spin(3)} = 2$).

By combining these facts we find:

$$\left| \begin{array}{c} \square\!\square \\ (cc) \end{array} \right\rangle + \frac{1}{c} \sum_{d=c+1}^{n} \left| \begin{array}{c} \square\!\square \\ (dd) \end{array} \right\rangle = \sqrt{\frac{c-1}{c}} \left| \begin{array}{l} \{\square\!\square\}^{n-c+1} \\ \{0\}^{c-2} \end{array} \right\rangle . \tag{66}$$

However, when evaluating the $U^{(cc)}$ operators of (61) in this basis, only the first term on the left-hand side is relevant due to the fact that:

$$d > c \quad \Rightarrow \quad [C_2(so(c)_K), U_{ab}^{(dd)}] = 0. \tag{67}$$

Having this in mind, we make use of (66) to recast, in the first equality of (62), the $U^{(cc)}$ operators accordingly. Taking into account arbitrariness of the σ_c coefficients and following the same steps as in (62), we finally obtain a rather simple expression for the shear generator matrix elements for an arbitrary $sl(n,\mathbb{R})$ representation (labelled now by parameters $\tilde{\sigma}_c$):

$$\left\langle \begin{matrix} \{J'\} \\ \{k'\}\{m'\} \end{matrix} \middle| T_{\{w\}} \middle| \begin{matrix} \{J\} \\ \{k\}\{m\} \end{matrix} \right\rangle = \tfrac{i}{2}\sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}\, C^{\{J\}\,\square\square\,\{J'\}}_{\{m\}\{w\}\{m'\}}$$
$$\times \sum_{c=2}^{n} \sqrt{\tfrac{c-1}{c}} \Big(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c \Big) C^{\{J\}\,(\square\square)^{n-c+1}\,\{J'\}}_{\{k\}\quad (0)^{c-2}\quad \{k'\}} \,. \tag{68}$$

The relation of the labelling of (43) and the one of (52), i.e. (61), is achieved provided $\sigma_c = \tilde{\sigma}_c + \sum_{d=2}^{c-1} \tilde{\sigma}_d/d$. The Clebsch-Gordan coefficient with indices $\{m\}, \{w\}, \{m'\}$ in (43) can be evaluated in an arbitrary basis (which is stressed by denoting the appropriate index by w instead by ab). The other Clebsch-Gordan coefficient can be evaluated in any basis labelled according to the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ subgroup chain (e.g. Gel'fand-Tsetlin basis) and can be, nowadays, rather easily evaluated, at least numerically.

4.5 A comment on the generalized formula

As already stated, the matrix elements of the $sl(n,\mathbb{R})/so(n)$ operators, as given by the Generalized Gell-Mann formula, apply to all tensorial, spinorial, unitary, nonunitary (both finite a infinite-dimensional) $sl(n,\mathbb{R})$ irreducible representations. In many physics applications one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present paper, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(Spin(n), \kappa)$ of square integrable functions with a scalar product in terms of an arbitrary kernel κ, and to impose the unitarity constraints both on the scalar products itself and on the $sl(n,\mathbb{R})/so(n)$ operators matrix elements in that scalar product (cf. [41]).

We note that the results of the previous subsection can be directly conveyed to the case of special unitary group $SU(n)$. Namely, operators of $su(n)$ algebra can be, similarly as in the case of $sl(n,\mathbb{R})$ algebra, split w.r.t. its $so(n)$ subalgebra into M_{ab} operators and $T^{su(n)}_{ab}$, $a, b = 1, 2, ..., n$ operators. Relation of $T^{su(n)}_{ab}$ and T_{ab} operators is a direct one: $T^{su(n)}_{ab} = iT_{ab}$, and the commutator $[T^{su(n)}_{ab}, T^{su(n)}_{cd}]$ differs from the commutator (4) only by an overall minus sign. Therefore, all formulas obtained for $SL(n,\mathbb{R})$ shear generators T_{ab} (44, 46, 47, 52, 59, 61, 68), are after multiplying by imaginary unit also applicable to $SU(n)$ generators $T_{ab}^{su(n)}$, with the following remark: since the $SU(n)$ group is its own covering group, space $L^2(Spin(n))$ has to be reduced to space $L^2(SO(n))$.

To sum up, the expressions (24) and (43) fully determine the action of the $sl(n,\mathbb{R})$ operators for an arbitrary irreducible representation given by the set of $n-1$ invariant Casimir operators labels $\tilde{\sigma}_c$. This action is given in the basis (22) of the representation spaces of the maximal compact subgroup $Spin(n)$ of the $\overline{SL}(n,\mathbb{R})$ group. This result is general due to a Corollary of Harish-Chandra [40] that explicitly applies to the case of the $sl(n,\mathbb{R})$ algebras.

5 Application in affine theories of gravity

5.1 Affine gravity models

In the introduction we have listed a few models of space-time symmetries and gravity where special linear group plays a substantial role. In this section we will briefly illustrate where the Gell-Mann formula can be applied in the context of affine theories of gravity.

Mostly due to the difficulties encountered in attempts to quantize Einstein's theory of gravity, a new ways to generalize and expand the basic concepts of Riemannian geometry and general relativity were sought. Affine theory of gravity is one of the possible directions to take. It is interesting to mention that even Einstein himself has considered affine generalizations of general relativity [51].

In affine gravity models the flat space-time symmetry of the theory (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $\overline{GL}(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of Affine Gravity models. On the other hand, the essential part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup, and that is where $\overline{SL}(n,\mathbb{R})$ generators matrix elements, obtained by using the generalized Gell-Mann formula, come into play (R_+ is subgroup of dilatations). We will apply expression for these matrix elements in order to obtain coefficients for some of the gauge field–matter interaction vertices.

Gravitational interaction is into these models usually introduced by gauging the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Since in the tensor product of two defining representation of $GL(n,\mathbb{R})$ group (also of $SL(n,\mathbb{R})$ group) does not appear any $GL(n,\mathbb{R})$ (or $SL(n,\mathbb{R})$ respectively) invariant tensor, there is also no equivalent of Minkovski metrics $\eta_{\mu\nu}$, and connection will not preserve length of vectors. Actually, as the transition from Riemannian to Riemann-Cartan space can be seen as a result of introduction of torsion, similarly, the transition from the Riemann-Cartan geometry to the affine geometry is related to abandoning of the requirement of metricity.

5.2 Gauge Affine action

A standard way to introduce interactions into affine gravity models is by localization of the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part L_g is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field φ (that ensures action invariance under local dilatations). In the case of the standard Metric affine gravity [1, 2], gravitational potentials are tetrads $e^a{}_\mu$, metrics g_{ab} and affine connection $\Gamma^a{}_{b\mu}$, so that we can write: $L_g = L_g(e, \partial e, g, \partial g, \Gamma, \partial\Gamma, \varphi)$. More precisely, due to action invariance under local affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e, g, T, R, N, \varphi)$, where $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a{}_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{b\nu} + \Gamma^c{}_{b\mu} \Gamma^a{}_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b{}^a$, where $Q_b{}^a$ denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of affine fields ϕ^I and their covariant derivatives, together with metrics and tetrads (affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of affine Lagrangians, in arbitrary number of dimensions n, of the form:

$$
\begin{aligned}
&L(e_\mu{}^a, \partial_\nu e_\mu{}^a, \Gamma_{b\mu}{}^a, \partial_\nu \Gamma_{b\mu}{}^a, g_{ab}, \Psi_A, \partial_\nu \Psi_A, \Phi_A, \partial_\nu \Phi_A, \varphi, \partial_\nu \varphi) \ = \\
&e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 + \\
&\bar{\Psi} i g^{ab}\gamma_a e_b{}^\mu D_\mu \Psi + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu (D_\mu \Phi)^+ (D_\nu \Phi) + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu D_\mu \varphi D_\nu \varphi + \\
&- L_g(n) + L_m(n)\Big].
\end{aligned}
\tag{69}
$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. affine transformations (dilatational invariance is obtained with the aid of field φ, of mass dimension $n/2 - 1$). Here T^2 and N^2 stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields (a discussion of available possibilities can be found in Appendix B of [2]). For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n,\mathbb{R})$, $n \geq 3$, transformations, are written in the second row. The field Ψ denotes a spinorial $\overline{GL}(n,\mathbb{R})$ field – components of that field transform under some appropriate spinorial $\overline{GL}(n,\mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n,\mathbb{R})$ representations are necessarily infinite dimensional [6], and thus the field Ψ will have infinite number of components. The concrete spinorial irreducible representation of field Ψ is given by a set of $n-1$ $\overline{SL}(n,\mathbb{R})$ labels $\{\sigma_c^\Psi\}$ together with the dilatation charge d_Ψ. The field Φ is a representative of a tensorial $\overline{GL}(n,\mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n,\mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^\Phi\}$ and d_Φ. Since, as it is argued in the following section, the noncompact $\overline{SL}(n-1,\mathbb{R})$ affine subgroup is to be represented unitarily, the tensorial field Φ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field φ is scalar with respect to $\overline{SL}(n,\mathbb{R})$ subgroup, and thus has only one component.

Finally, the third row contains possible additional gravitational and matter terms, denoted respectively by $L_g(n)$ and $L_m(n)$, that, due to restrictions imposed by the dilatational invariance requirement, can appear only for some concrete values of n. (E.g., in [5] dealing with the four dimensional case, authors take $L_g(4) = \alpha_1 R_{[abcd]}R^{[abcd]} + \alpha_2 R_{[ab[c]d]}R^{[ab[c]d]} + \alpha_3 R_{[a(b][c)d]}R^{[a(b][c)d]} + \alpha_4 R_{(a[b)cd]}R^{(a[b)cd]} + \alpha_5 R_{(ab[c)d]}R^{(ab[c)d]}$, and $L_m(4) = \mu\bar{\Psi}\Phi\Psi - \lambda_\Phi(\Phi^+\Phi)^2 - \lambda(\Phi^+\Phi)\varphi^2 - \lambda_\varphi\varphi^4$.)

Interaction of affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1,n-1)$ subgroup of $\overline{GL}(n,\mathbb{R})$. Such a labelling allows, in principle, to identify affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1,n-1) \subset T^n \wedge \overline{GL}(n,\mathbb{R})$. Therefore, we consider the field Ψ (and similarly for Φ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\ \{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} \Big|{}^{\{J\}}_{\{k\}\{m\}}\Big\rangle.$$

Ket vectors in this decomposition are basis vectors of the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n,\mathbb{R})$ group [26]. Sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the Lorentz $Spin(1,n-1)$ subgroup: $\{J\}$ label irreducible representation of $Spin(1,n-1)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ enumerate $Spin(1,n-1)$ multiplicity of representation $\{J\}$ within the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n,\mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathcal{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ group (for more details c.f. [26]).

The interaction term connecting fields g^{cd}, $e_d^{\ \mu}$, Γ_μ^{ab}, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd}e_d^{\ \mu}\Gamma_\mu^{ab}\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}\Psi^{\{J'\}}_{\{k'\}\{m'\}}\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}}\langle^{\{J\}}_{\{k\}\{m\}}|\gamma_c|^{\{J''\}}_{\{k''\}\{m''\}}\rangle\langle^{\{J''\}}_{\{k''\}\{m''\}}|Q_{ab}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle, \tag{70}$$

while the interaction of tensorial field with connection is given by:

$$-\tfrac{i}{2}g^{cd}e_c^{\ \mu}e_d^{\ \nu}\Gamma_\nu^{ab}\partial_\mu\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\Phi^{\{J'\}}_{\{k'\}\{m'\}}\langle^{\{J\}}_{\{k\}\{m\}}|Q_{ab}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle+ \tag{71}$$

$$\tfrac{i}{2}g^{cd}e_c^{\ \mu}e_d^{\ \nu}\Gamma_\nu^{ab}\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\partial_\mu\Phi^{\{J'\}}_{\{k'\}\{m'\}}\langle^{\{J'\}}_{\{k'\}\{m'\}}|Q_{ab}|^{\{J\}}_{\{k\}\{m\}}\rangle^*+ \tag{72}$$

$$\tfrac{1}{2}g^{cd}e_c^{\ \mu}e_d^{\ \nu}\Gamma_\mu^{ab}\Gamma_\nu^{a'b'}\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\partial_\mu\Phi^{\{J'\}}_{\{k'\}\{m'\}}\cdot$$

$$\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}}\langle^{\{J\}}_{\{k\}\{m\}}|Q_{ab}|^{\{J''\}}_{\{k''\}\{m''\}}\rangle\langle^{\{J''\}}_{\{k''\}\{m''\}}|Q_{a'b'}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle. \tag{73}$$

The scalar dilaton field interact only with the trace of affine connection:

$$\tfrac{1}{2}g^{ab}e_a^{\ \mu}e_b^{\ \nu}(\partial_\mu - i\Gamma_{a\ \mu}^{\ a}d_\varphi)\varphi(\partial_\nu - i\Gamma_{a\ \nu}^{\ a}d_\varphi)\varphi, \tag{74}$$

where d_φ denotes dilatation charge of φ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n,\mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1,n-1)$. The dilatation generator (that is, the trace Q^a_a) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n,\mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (70); more on these matrices can be found in papers of Šijački [53].) However, before we illustrate how to evaluate these matrix elements, and thus how to calculate vertex coefficients, we must make some additional general remarks on $\overline{GL}(n,\mathbb{R})$ representations that correspond to physical fields.

5.3 Deunitarizing automorphism

We will briefly discuss the matter of unitarity of the representations corresponding to fields in affine models. In standard, Poincaré symmetric models, gauge and matter fields have finite number of components and this fits well the experimental data. However, since the Lorenz group is a non compact one, this is made possible by the fact that the fields transform under the non-unitary representations of the Lorenz group. Note that it is only the compact $\overline{SO}(n-1)$ part of the Lorentz group that is represented unitary. If the unitary, so called Gelfand-Naimark, representations of the Lorenz group were used [52], the boosts would mix infinitely many field components, in contrary to observations.

For the same physical reasons, the Lorenz subgroup of $\overline{GL}(n,\mathbb{R})$ should act in an analogous way on $\overline{GL}(n,\mathbb{R})$ fields: boosts should be represented non unitarily and the Lorenz subgroup should reduce in finite dimensional subspaces of field components. On the other hand, much in the same way as spatial rotation part of the Lorenz group acts unitarily on Poincaré fields, it is physically favorable that the spatial "little group" $\overline{GL}(n-1,\mathbb{R})$, a subgroup of $\overline{GL}(n,\mathbb{R})$, acts unitarily on field components.

This can be elegantly accomplished by using a so called deunitarizing automorphism. Namely, there exists an inner automorphism [6], which leaves the $R_+ \otimes \overline{SL}(n-1,\mathbb{R})$ subgroup intact, and which maps the $Q_{(0k)}$, $Q_{[0k]}$ generators into $iQ_{[0k]}$, $iQ_{(0k)}$ respectively ($k = 1, 2, \ldots n-1$). Here $Q_{[ab]} = \frac{1}{2}(Q_{ab} - Q_{ba})$ denote the antisymmetric operators that generate the Lorentz subgroup $Spin(1, n-1)$, whereas $Q_{(ab)} = \frac{1}{2}(Q_{ab} + Q_{ba}) - \frac{1}{n} g_{ab} Q_c{}^c$ are the symmetric traceless operators that generate the proper n-volume-preserving deformations (shears).

The deunitarizing automorphism thus allows us to start with the unitary representations of the $\overline{SL}(n,\mathbb{R})$ subgroup, and upon its application, to identify the finite (unitary) representations of the abstract $\overline{SO}(n)$ compact subgroup with nonunitary representations of the physical Lorentz group, while the infinite (unitary) representations of the abstract $\overline{SO}(1,n-1)$ group now represent (non-unitarily) the compact $\overline{SO}(n)/\overline{SO}(n-1)$ generators.

5.4 Gauge affine symmetry vertex coefficients evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part of the problem is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (70)-(73). We will do that by using formula (68).

However, formula (68) is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1, n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (68) and replacing the remaining Clebsch-Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group $Spin(1, n-1)$.

Now, as a concrete example, we will consider tensorial affine field Φ in $n = 5$ dimensions. For example, let the field Φ correspond to an unitary multiplicity free $\overline{SL}(5, \mathbb{R})$ representation, defined by labels $\sigma_2 = -4, \delta_1 = \delta_2 = 0$, with σ_1 arbitrary real. The representation space is spanned by vectors (49) satisfying $\overline{J}_1 = \overline{J}_2 = \overline{J} \in N_0 + \frac{1}{2}; K_1 = K_2 = 0; J_1 = J_2 = J \leq \overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a = 1 \ldots 5$ the five Φ components with $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ (in this sense Φ^a corresponds to a Lorenz 5-vector) then the interaction vertex (71) connecting fields $\Phi^{a\dagger}$, $\partial_\mu \Phi^d$ and affine shear connection Γ_ν^{bc} is:

$$\frac{i}{2} g^{ef} e_e{}^\mu e_f{}^\nu \Phi^{a\dagger} \Gamma_\nu^{bc} \partial_\mu \Phi^d \frac{\sqrt{5}}{14}\, \sigma_1 (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}). \tag{75}$$

To obtain this result we used an easily derivable formula for Clebsch-Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$C^{L\square\,\square\square\,\square}_{\;\;a\;\;(bc)\;d} = \sqrt{\frac{n}{2(n+2)(n-1)}} (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}), \tag{76}$$

where we labelled $Spin(1, n-1)$ irreducible representations by Young diagrams, as in [26]. More importantly, we also used value of the reduced matrix element:

$$\left\langle \begin{matrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{matrix} \middle\| Q \middle\| \begin{matrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{matrix} \right\rangle = \sqrt{\frac{2}{7}}\sigma_1, \tag{77}$$

that we obtained by using formula (51) (based on this formula, a Mathematica program was generated that directly calculates $sl(5, \mathbb{R})$ matrix elements, taking into account relevant $Spin(5)$ Clebsch-Gordan coefficients given in the Appendix).

It is no more difficult to obtain coefficients of the vertices of the form (73). Lagrangian term (73) connecting Lorenz 5-vector Φ components Φ_5, $\Phi_5^\dagger$ and affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15}\left(\sigma_1^2 - 25\right) g^{cd} e_c{}^\mu e_d{}^\nu \Gamma^{55}_\mu \Gamma^{55}_\nu \Phi_5^\dagger \partial_\mu \Phi_5. \tag{78}$$

In a similar fashion, we can find vertex coefficients for more complex representations with nontrivial multiplicity.

6 Conclusion

Gell-Mann decontraction formula is, at the algebraic level, applicable only in the case of the (pseudo)orthogonal algebras. In the case of other algebras this formula is not applicable for all representations. As for the case of $sl(n,\mathbb{R})$ algebras, contracted w.r.t. the $so(n)$ subalgebra, we saw that it can be applied only to certain classes of tensorial representations without multiplicity. More specifically, we have shown that the formula is valid only in Hilbert spaces over $Spin(n)/(Spin(m)\times Spin(n-m))$, $m = 1, 2, \ldots, n-1$. When the formula is applicable, it directly yields matrix element expressions of the $sl(n,\mathbb{R})$ operators: (24) and (43).

Starting from the known expression for generator matrix elements of $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ representations with multiplicity, it was possible to easily obtain expressions for the generalized Gell-Mann formulas in the corrsponding cases, and then to follow a similar pattern and obtain generalized formula in the $sl(5,\mathbb{R})$ case. By expressing the obtained formulas in Cartesian basis, the Gell-Mann formula was generalized for arbitrary dimension n. The generalized formula is given by the expression (52). As the most direct and important application of the formula, we obtained closed form expressions for matrix elements of $sl(n,\mathbb{R})$ operators in arbitrary irreducible representation (finite, infinite, tensorial, spinorial, multiplicity free or not). The form of the generalized formula is quite elegant and comparable by simplicity to the form of the original formula.

We have also considered an application of the Gell-Mann formula in the context of affine models of gravity.

7 Appendix: Clebsch-Gordan coefficients for the 14 dimensional uinitary irreducible representation of $Spin(5)$

Analytical expressions for the $Spin(5)$ Clebsch-Gordan coefficients involving the 14-dimensional representations are a must know for obtaining and confirming all of the results pertaining to the 5 dimensional case. These coefficients were published long ago [58]. However, in attempt to use these coefficients, it turned out that some 30% of the expressions in the paper are incorrect. Therefore, a detailed analysis of the polynomial expressions had to be carried out, that led to their correction. Additionally, an algorithm for numerical evaluation of $Spin(5)$ Clebsch-Gordan coefficients was developed in order to compare their values in a vast number of points.

The obtained results are given in this appendix. More details can be found in [44].

Any $Spin(5)$ Clebsch-Gordan coefficient can be written as a multiple of two $Spin(3)$ Clebshc-Gordan coefficients and one reduced $Spin(5)$ Clebsch-Gordan coefficient:

$$\left(\begin{array}{cc|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{j}''_1 & \bar{j}''_2 \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \\ m_1 & m_2 & m'_1 & m'_2 & m''_1 & m''_2 \end{array}\right) = \left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{j}''_1 & \bar{j}''_2 \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \end{array}\right) \left(\begin{array}{c|cc} j_1 & j'_1 & j''_1 \\ m_1 & m'_1 & m''_1 \end{array}\right) \left(\begin{array}{c|cc} j_2 & j'_2 & j''_2 \\ m_2 & m'_2 & m''_2 \end{array}\right). \tag{79}$$

Since the $Spin(3)$ coefficients are well known, we will list only the reduced $Spin(5)$ Clebsch-Gordan coefficients.

The direct product of a representation $(\bar{j}_1, \bar{j}_2)$ with 14-dimensional representation $(\bar{1},\bar{1})$, decompose into the following representations:

$$\begin{aligned}(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1}) = {} & (\bar{j}_1+1,\bar{j}_2+1)\oplus(\bar{j}_1,\bar{j}_2+1)\oplus(\bar{j}_1-1,\bar{j}_2+1)\\ & \oplus(\bar{j}_1+1,\bar{j}_2)\oplus(\bar{j}_1-1,\bar{j}_2)\oplus(\bar{j}_1+1,\bar{j}_2-1)\oplus(\bar{j}_1,\bar{j}_2-1)\\ & \oplus(\bar{j}_1-1,\bar{j}_2-1)\oplus(\bar{j}_1+\frac{1}{2},\bar{j}_2+\frac{1}{2})\oplus(\bar{j}_1-\frac{1}{2},\bar{j}_2+\frac{1}{2})\\ & \oplus(\bar{j}_1+\frac{1}{2},\bar{j}_2-\frac{1}{2})\oplus(\bar{j}_1-\frac{1}{2},\bar{j}_2+\frac{1}{2})\oplus 2(\bar{j}_1,\bar{j}_2).\end{aligned} \tag{80}$$

The reduced coefficients follow:

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+1,\bar{j}_2+1)$ are:

$$\begin{aligned}N_a(\bar{j}_1,\bar{j}_2) = {} & ((2\bar{j}_1+2)\,(2\bar{j}_1+3)\,(\bar{j}_1+\bar{j}_2+2)\,(\bar{j}_1+\bar{j}_2+3)\,(2\bar{j}_2+1)\,(2\bar{j}_2\\ & +2)\,(2\bar{j}_1+2\bar{j}_2+3)\,(2\bar{j}_1+2\bar{j}_2+5))^{-\frac{1}{2}},\end{aligned} \tag{81}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1\\ j_1+1 & j_2+1\end{matrix}\middle\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right)\\ &= \Big(N_a(\bar{j}_1,\bar{j}_2)\ \ ((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2\\ &\quad +1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2\\ &\quad +3)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2\\ &\quad +3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+5)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2\\ &\quad +6))^{1/2}\Big)/\Big(4\,((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3))^{1/2}\Big),\end{aligned} \tag{82}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1\\ j_1-1 & j_2-1\end{matrix}\middle\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1\\ &\quad +j_2-\bar{j}_1+\bar{j}_2-1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ \ (-j_1-j_2+\bar{j}_1\\ &\quad +\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2\\ &\quad +3)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1\\ &\quad -j_2+\bar{j}_1+\bar{j}_2+3)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1\\ &\quad +\bar{j}_2+3))^{1/2}\Big)/\Big(4\,(j_1\,(2j_1-1)\,j_2\,(2\,j_2-1))^{1/2}\Big),\end{aligned} \tag{83}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 - \bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2 + 2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 5)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\big)^{1/2}\Big)/\Big(4\left(j_1\,(2\,j_1-1)\,(2j_2^2+5j_2+3)\right)^{1/2}\Big), \tag{84}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2 + 2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 5)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\big)^{1/2}\Big)/\Big(4\left((2j_1^2+5\,j_1+3)\,j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{85}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 5)\big)^{1/2}\Big)/\Big(4\left((j_1+1)\,(2j_1+3)\,j_2\,(j_2+1)\right)^{1/2}\Big), \tag{86}$$

$$\begin{aligned}\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1-1 & j_2\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = &-\Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 \\ &+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2 \\ &+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 \\ &+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1 \\ &+j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2 \\ &+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(4\,(j_1\,(2\,j_1-1)\,j_2\,(j_2+1))^{1/2}\Big)\,,\end{aligned} \tag{87}$$

$$\begin{aligned}\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2+1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = &\Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 \\ &+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 \\ &+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\ &+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &+5))^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big)\,,\end{aligned} \tag{88}$$

$$\begin{aligned}\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2-1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = &-\Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 \\ &-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2 \\ &+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 \\ &+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 \\ &-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2 \\ &+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big)\,,\end{aligned} \tag{89}$$

$$\left(\begin{array}{cc||cccc}\bar{j}_1+1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & j_1 & j_2 & 1 & 1\end{array}\right) = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ ((-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\ (j_1^2+j_1+j_2^2-\bar{j}_1^2-\bar{j}_2^2+j_2-\bar{j}_1+2\,\bar{j}_1\bar{j}_2+\bar{j}_2)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\ j_2\,(j_2+1))^{1/2}\Big), \tag{90}$$

$$\left(\begin{array}{cc||cccc}\bar{j}_1+1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2}\end{array}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+5))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{91}$$

$$\left(\begin{array}{cc||cccc}\bar{j}_1+1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2}\end{array}\right) = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\ (j_1j_2)^{1/2}\Big), \tag{92}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big), \tag{93}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 + 1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4))^{1/2}\Big)\Big/\Big(2\,(j_1\,(j_2+1))^{1/2}\Big), \tag{94}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right) = \frac{1}{2}\sqrt{5}N_a(\bar{j}_1,\bar{j}_2)\ ((-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1 +\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}, \tag{95}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+1,\bar{j}_2)$ are:

$$N_b(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+2)\,(2\bar{j}_1+3)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2+1)\,\bar{j}_2\,(\bar{j}_1+\bar{j}_2 + 2)\,(2\bar{j}_2+2)\,(2\bar{j}_1+2\bar{j}_2+3))^{-\frac{1}{2}} \tag{96}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1+1 & j_2+1\end{pmatrix}\!\!\left\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1\end{matrix}\right)$$
$$= -\Big(N_b(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1 -\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +5))^{1/2}\Big)/\Big(4\,((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3))^{1/2}\Big)\,, \tag{97}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1-1 & j_2-1\end{pmatrix}\!\!\left\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1\end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1 +j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+2))^{1/2}\Big)/\Big(4\,(j_1\,(2\,j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big)\,, \tag{98}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1-1 & j_2+1\end{pmatrix}\!\!\left\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1\end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+3))^{1/2}\Big)/\Big(4\,\big(j_1\,(2\,j_1-1)\,\big(2j_2^2+5j_2+3\big)\big)^{1/2}\Big)\,, \tag{99}$$

$$
\left(\begin{array}{cc||cc} \bar{j}_1+1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1\ 1 \\ j_1+1 & j_2-1 & j_1\ j_2 & 1\ 1 \end{array}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(4\left((2j_1^2+5\,j_1+3)\,j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{100}
$$

$$
\left(\begin{array}{cc||cc} \bar{j}_1+1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1\ 1 \\ j_1+1 & j_2 & j_1\ j_2 & 1\ 1 \end{array}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\,(-j_1^2+(2\,\bar{j}_1+1)\ j_1+j_2^2-\bar{j}_1^2+\bar{j}_2^2+j_2-\bar{j}_1+\bar{j}_2)\Big)\Big/\Big(4\ \left((2j_1^2+5j_1+3)\,j_2\,(j_2+1)\right)^{1/2}\Big), \tag{101}
$$

$$
\left(\begin{array}{cc||cc} \bar{j}_1+1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1\ 1 \\ j_1-1 & j_2 & j_1\ j_2 & 1\ 1 \end{array}\right) = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(2j_2\ \bar{j}_2+(j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))\Big)\Big/\Big(4\,(j_1\ (2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{102}
$$

$$
\left(\begin{array}{cc||cc} \bar{j}_1+1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1\ 1 \\ j_1 & j_2+1 & j_1\ j_2 & 1\ 1 \end{array}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\ (j_1^2+j_1-j_2^2-\bar{j}_1^2+\bar{j}_2^2-\bar{j}_1+j_2\,(2\,\bar{j}_1+1)+\bar{j}_2)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\ (2j_2^2+5j_2+3))^{1/2}\Big), \tag{103}
$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2-1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1\,\bar{j}_2 & 1\;1 \\ j_1\,j_2 & 1\;1\end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2+j_1-j_2^2-\bar{j}_1^2+\bar{j}_2^2-3\bar{j}_1-j_2\,(2\,\bar{j}_1+3)+\bar{j}_2-2\big)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\ j_2\,(2j_2-1))^{1/2}\Big), \tag{104}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1\,\bar{j}_2 & 1\;1 \\ j_1\,j_2 & 1\;1\end{matrix}\right) = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2+j_1+j_2^2-\bar{j}_1^2+\bar{j}_2^2+j_2-3\,\bar{j}_1+\bar{j}_2-2\big)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,j_2\ (j_2+1))^{1/2}\Big), \tag{105}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2}\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1\,\bar{j}_2 & 1 & 1 \\ j_1\,j_2 & \frac{1}{2} & \frac{1}{2}\end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1+j_2-\bar{j}_1)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{106}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2}\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1\,\bar{j}_2 & 1 & 1 \\ j_1\,j_2 & \frac{1}{2} & \frac{1}{2}\end{matrix}\right) = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+\bar{j}_1+2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\big)^{1/2}\Big)\Big/\Big(2\,(j_1j_2)^{1/2}\Big), \tag{107}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+1)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2 \\ +1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\ +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1 \\ -j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2 \\ +\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big), \tag{108}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1-j_2+\bar{j}_1+1)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2 \\ +1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\ +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1 \\ +j_2+\bar{j}_1+\ \bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 \\ +j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,(j_1\,(j_2+1))^{1/2}\Big), \tag{109}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 0\,0 \end{matrix}\right) = \frac{1}{2}\sqrt{5}N_b(\bar{j}_1,\bar{j}_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ \ (-j_1+j_2+\bar{j}_1 \\ -\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1 \\ -j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 \\ +\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}. \tag{110}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\rightarrow(\bar{j}_1,\bar{j}_2+1)$ are:

$$N_c(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+1)\,(2\bar{j}_1+3)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1+\bar{j}_2+2)\,(2\bar{j}_2 \\ +1)\,(2\bar{j}_2+2)\,(2\bar{j}_1+2\bar{j}_2+3))^{-\frac{1}{2}} \tag{111}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2+1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix}$$
$$= \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +5))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ \ (2j_2+3))^{1/2}\Big), \tag{112}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2-1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = -\Big(N_c(\bar{j}_1,\bar{j}_2)\quad ((j_1-j_2+\bar{j}_1-\bar{j}_2)\quad (-j_1+j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2 +\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(-j_1-j_2 +\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{113}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2+1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix}$$
$$= \Big(N_c(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2-2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 -\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +2)\,(-j_1+j_2+\bar{j}_1+\ \bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (2j_1-1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{114}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right)$$
$$= \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-2)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 +3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(2j_1+3)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{115}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2-\bar{j}_1+\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 +3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +4))^{1/2}\,(-j_1^2+2\bar{j}_2\ j_1+j_2^2+\bar{j}_1^2-\bar{j}_2^2+j_2+2\bar{j}_1 +1)\Big)/\Big(2\ \ (2)^{1/2}\,((2j_1^2+5j_1+3)\,j_2\,(j_2+1))^{1/2}\Big), \tag{116}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = -\Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ \ (j_1-j_2+\bar{j}_1 -\bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 +2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\,(j_1^2+2\ \ (\bar{j}_2+1)\,j_1-j_2^2-\bar{j}_1^2+\bar{j}_2^2-j_2-2\,\bar{j}_1 +2\bar{j}_2)\Big)/\Big(2\,(2)^{1/2}\,(j_1\,(2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{117}$$

$$
\begin{aligned}
\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2+1 & j_1 & j_2 & 1 & 1 \end{array}\right) = \Big(& N_c(\bar{j}_1, \bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2)\ (j_1 \\
&+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1 \\
&+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 \\
&+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\ (j_1^2+j_1-j_2^2+\bar{j}_1^2-\bar{j}_2^2+2\bar{j}_1+2j_2\,\bar{j}_2 \\
&+1)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,(2\,j_2^2+5j_2+3))^{1/2}\Big),
\end{aligned}
\tag{118}
$$

$$
\begin{aligned}
\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2-1 & j_1 & j_2 & 1 & 1 \end{array}\right) = \Big(& N_c(\bar{j}_1, \bar{j}_2)\ \big((-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 \\
&+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 \\
&+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 \\
&-j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ (j_1^2+j_1-j_2^2+\bar{j}_1^2-\bar{j}_2^2+2\bar{j}_1-2\,\bar{j}_2 \\
&-2j_2\,(\bar{j}_2+1))\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big),
\end{aligned}
\tag{119}
$$

$$
\begin{aligned}
\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) = \Big(& N_c(\bar{j}_1, \bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 \\
&+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1 \\
&+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 \\
&+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ (j_1^2+j_1+j_2^2+\bar{j}_1^2-\bar{j}_2^2+j_2 \\
&+2\bar{j}_1-2\,\bar{j}_2)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,j_2\ (j_2+1))^{1/2}\Big),
\end{aligned}
\tag{120}
$$

$$
\begin{aligned}
\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2+1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(& N_c(\bar{j}_1, \bar{j}_2)\,(2\ j_1+2j_2-2\bar{j}_2+1)\ \big((j_1-j_2+\bar{j}_1 \\
&-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1 \\
&+j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 \\
&+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1 \\
&+\bar{j}_2+4)\big)^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(j_2+1))^{1/2}\Big),
\end{aligned}
\tag{121}
$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(j_1-j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\,(2j_1 + 2j_2+2\ \bar{j}_2+3)\Big)/\Big(2\,(2)^{1/2}\,(j_1j_2)^{1/2}\Big), \tag{122}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ \ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1 - j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(-2j_1 + 2j_2+2\ \bar{j}_2+1)\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\,j_2)^{1/2}\Big), \tag{123}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ \ (j_1-j_2+\bar{j}_1 - \bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\ \bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3))^{1/2}\,(2j_1-2j_2+2\ \bar{j}_2 + 1)\Big)/\Big(2\,(2)^{1/2}\,(j_1\,(j_2+1))^{1/2}\Big), \tag{124}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 0\,0 \end{pmatrix} = -\sqrt{\frac{5}{2}}N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}. \tag{125}$$

Clebsch-Gordan coefficients $(\bar{j}_1, \bar{j}_2) \otimes (\bar{1}, \bar{1}) \to (\bar{j}_1 + 1, \bar{j}_2 - 1)$ are:

$$N_d(\bar{j}_1, \bar{j}_2) = (2\,(2\bar{j}_1 + 2)\,(2\bar{j}_1 + 3)\,(2\bar{j}_1 - 2\bar{j}_2 + 1)\,(2\bar{j}_1 - 2\bar{j}_2 + 3)\,(\bar{j}_1 - \bar{j}_2 + 1)\,(\bar{j}_1 - \bar{j}_2 + 2)\,\bar{j}_2\,(2\bar{j}_2 + 1))^{-\frac{1}{2}} \tag{126}$$

$$\left(\begin{matrix}\bar{j}_1 + 1 & \bar{j}_2 - 1\\ j_1 + 1 & j_2 + 1\end{matrix}\middle\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_d(\bar{j}_1, \bar{j}_2)\ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ (j_1 - j_2 + \bar{j}_1 - \bar{j}_2 + 2)\ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\,(j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 3)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 4)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 5)\ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 - 1)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2)\,(j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 4))^{1/2}\Big)\Big/\Big(4\,((j_1 + 1)\,(2j_1 + 3)\,(j_2 + 1)\,(2j_2 + 3))^{1/2}\Big), \tag{127}$$

$$\left(\begin{matrix}\bar{j}_1 + 1 & \bar{j}_2 - 1\\ j_1 - 1 & j_2 - 1\end{matrix}\middle\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_d(\bar{j}_1, \bar{j}_2)\ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ (j_1 - j_2 + \bar{j}_1 - \bar{j}_2 + 2)\ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\,(j_1 + j_2 - \bar{j}_1 + \bar{j}_2 - 3)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 - 2)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 - 1)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2)\ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 1)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(2\,j_1 - 1)\,j_2\,(2j_2 - 1))^{1/2}\Big), \tag{128}$$

$$\left(\begin{matrix}\bar{j}_1 + 1 & \bar{j}_2 - 1\\ j_1 - 1 & j_2 + 1\end{matrix}\middle\|\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_d(\bar{j}_1, \bar{j}_2)\ ((-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 3)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 4)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 3)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 - 1)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3))^{1/2}\Big)\Big/\Big(4\,\big(j_1\,(2j_1 - 1)\,\big(2j_2^2 + 5\,j_2 + 3\big)\big)^{1/2}\Big), \tag{129}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1 +\bar{j}_2+1))^{1/2}\Big)/\Big(4\left(\left(2j_1^2+5\,j_1+3\right)j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{130}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1 -\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+3))^{1/2}\Big)/\Big(4\left(\left(2j_1^2+5\,j_1+3\right)j_2\,(j_2+1)\right)^{1/2}\Big), \tag{131}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2 +\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)/\Big(4\left(j_1\,(2\,j_1-1)\,j_2\,(j_2+1)\right)^{1/2}\Big), \tag{132}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1 -\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+3))^{1/2}\Big)/\Big(4\left(j_1\,(j_1+1)\left(2j_2^2+5j_2+3\right)\right)^{1/2}\Big), \tag{133}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1 & j_2-1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{134}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1 & j_2 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}\,(2j_1\ j_2+(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))\Big)\Big/\Big(4\,(j_1\ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{135}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{136}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(j_1 j_2)^{1/2}\Big), \tag{137}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 - \bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +1))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big), \tag{138}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)\Big/\Big(2\,(j_1\,(j_2+1\,))^{1/2}\Big), \tag{139}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{pmatrix} = \frac{1}{2}\sqrt{5}N_d(\bar{j}_1,\bar{j}_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1 - \bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}. \tag{140}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+\frac{1}{2},\bar{j}_2+\frac{1}{2})$ are:

$$N_e(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+2)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,(\bar{j}_1 +\bar{j}_2+3)\,(2\bar{j}_2+1)\,(2\bar{j}_1+2\bar{j}_2+3))^{-\frac{1}{2}} \tag{141}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix}$$
$$= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+5))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ (2j_2+3))^{1/2}\Big), \tag{142}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix} = \Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1+j_2-\bar{j}_1+\bar{j}_2-1)\,(j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{143}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix}$$
$$= \Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,\big(j_1\ (2j_1-1)\,\big(2j_2^2+5j_2+3\big)\big)^{1/2}\Big), \tag{144}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix}$$
$$= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,\big((2\,j_1^2+5j_1+3)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{145}$$

$$
\begin{aligned}
&\left(\begin{array}{cc||cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) \\
&= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\
&\quad +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 \\
&\quad +j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\,(j_2\,(j_2+1) \\
&\quad +(j_1+1)\,(-j_1+\bar{j}_1+\bar{j}_2+1))\Big)\Big/\Big(2\,(2)^{1/2}\,((j_1+1)\,(2j_1+3)\,j_2\ (j_2+1))^{1/2}\Big),
\end{aligned}
\tag{146}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) \\
&= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\
&\quad +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ (j_1^2 \\
&\quad +(\bar{j}_1+\bar{j}_2+2)\,j_1-j_2\,(j_2+1))\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(2j_1-1)\,j_2\ (j_2+1))^{1/2}\Big),
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2+1 & j_1 & j_2 & 1 & 1 \end{array}\right) \\
&= \Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\
&\quad +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\,(j_1\,(j_1+1) \\
&\quad +(j_2+1)\,(-j_2+\bar{j}_1+\bar{j}_2+1))\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,(j_2+1)\ (2j_2+3))^{1/2}\Big),
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2-1 & j_1 & j_2 & 1 & 1 \end{array}\right) \\
&= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\
&\quad +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1 \\
&\quad -j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\,(j_1^2+j_1 \\
&\quad -j_2\ (j_2+\bar{j}_1+\bar{j}_2+2))\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big),
\end{aligned}
\tag{149}
$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{pmatrix} = \Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,(j_1+j_2+1)\ \ (-j_1^2-j_1-j_2^2 - j_2+(\bar{j}_1-\bar{j}_2)\ \ (\bar{j}_1-\bar{j}_2+1))\ \ ((-j_1-j_2+\bar{j}_1 + \bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1 + \bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ \ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{150}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,(2j_1+2j_2-\bar{j}_1-\bar{j}_2 + 1)\ \ ((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{151}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1 - j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(j_1-j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\,(2j_1 + 2\ j_2+\bar{j}_1+\bar{j}_2+3)\Big)/\Big(2\,(2)^{1/2}\,(j_1j_2)^{1/2}\Big), \tag{152}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1+j_2 + 1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1-j_2 + \bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(-2j_1 + 2\ j_2+\bar{j}_1+\bar{j}_2+1)\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\ \ j_2)^{1/2}\Big), \tag{153}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right) = \Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,(2j_1-2j_2+\bar{j}_1+\bar{j}_2 +1)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 +2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ (j_2+1))^{1/2}\Big), \tag{154}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 0\,0 \end{matrix} \right) = \sqrt{\frac{5}{2}}N_e(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(j_1+j_2+1)\ ((-j_1-j_2 +\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}. \tag{155}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+\frac{1}{2},\bar{j}_2-\frac{1}{2})$ are:

$$N_f(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+2)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2 +2)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,(2\bar{j}_2+1))^{-\frac{1}{2}} \tag{156}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\,)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2 +\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2-1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ (2j_2+3))^{1/2}\Big), \tag{157}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix} = \Big(N_f(\bar{j}_1,\bar{j}_2)\,(j_1-j_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+1)\;(j_1+j_2-\bar{j}_1+\bar{j}_2-2)\;(j_1+j_2-\bar{j}_1+\bar{j}_2-1)\;(j_1+j_2-\bar{j}_1+\bar{j}_2)\;(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\;(j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\;(2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{158}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix} = \Big(N_f(\bar{j}_1,\bar{j}_2)\;(j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2-\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2+1)\;(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,\big(j_1\,(2j_1-1)\;\;(2j_2^2+5j_2+3)\big)^{1/2}\Big), \tag{159}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix} = -\Big(N_f(\bar{j}_1,\bar{j}_2)\;(j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1-j_2+\bar{j}_1-\bar{j}_2+3)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2-\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2+3)\;(-j_1+j_2+\bar{j}_1+\bar{j}_2)\;(-j_1+j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)/\Big(2\,(2)^{1/2}\;\big((2j_1^2+5j_1+3)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{160}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{Vmatrix} = \Big(N_f(\bar{j}_1,\bar{j}_2)\;((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\;(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+3)\;(-j_1-j_2+\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\;(-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\;(j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(j_2\,(j_2+1)-(j_1+1)\,(j_1-\bar{j}_1+\bar{j}_2))\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\,(2j_1+3)\,j_2\;(j_2+1))^{1/2}\Big), \tag{161}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-1 & j_2 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{pmatrix} = -\Big(N_f(\bar{j}_1,\bar{j}_2)\,(j_1^2+\ (\bar{j}_1-\bar{j}_2+1)\,j_1 - j_2\,(j_2+1))\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{162}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1 & j_2+1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{pmatrix} = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\ (j_1^2+j_1-(j_2+1)\,(j_2-\bar{j}_1+\bar{j}_2))\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{163}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1 & j_2-1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{pmatrix} = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1^2+j_1-j_2\,(j_2+\bar{j}_1-\bar{j}_2+1))\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{164}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1 & j_2 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{pmatrix} = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1^2+j_1-j_2\,(j_2+1))\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}\ (-j_1^2-j_1-j_2^2+(\bar{j}_1+\bar{j}_2)^2-j_2+3\ (\bar{j}_1+\bar{j}_2)+2)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{165}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(2j_1+2j_2-\bar{j}_1+\bar{j}_2+2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{166}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(2j_1+2j_2+\bar{j}_1-\bar{j}_2+2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ j_2)^{1/2}\Big), \tag{167}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,(2j_1-2j_2-\bar{j}_1+\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ ((j_1+1)\,j_2)^{1/2}\Big), \tag{168}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,(2j_1-2j_2+\bar{j}_1-\bar{j}_2)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ (j_2+1))^{1/2}\Big), \tag{169}$$

$$\left(\begin{matrix}\bar j_1+\frac12 & \bar j_2-\frac12\\ j_1 & j_2\end{matrix}\right\Vert\left.\begin{matrix}\bar j_1 & \bar j_2 & 1\,1\\ j_1 & j_2 & 0\,0\end{matrix}\right) = \left(\frac{5}{2}\right)^{1/2} N_f(\bar j_1,\bar j_2)\ (j_1-j_2)\,(j_1+j_2 \tag{170}$$
$$+1)\ ((j_1-j_2+\bar j_1-\bar j_2+1)\ (-j_1+j_2+\bar j_1-\bar j_2+1)\ (j_1+j_2+\bar j_1-\bar j_2+2)\ (j_1+j_2-\bar j_1+\bar j_2))^{1/2}.$$

Representation $(\bar j_1,\bar j_2)$ appears twice in the decomposition of the product $(\bar j_1,\bar j_2)\otimes(\bar 1,\bar 1)$. To distinguish between the two, we will use the convention described in [58]. Clebsch-Gordan coefficients $(\bar j_1,\bar j_2)\otimes(\bar 1,\bar 1)\to(\bar j_1,\bar j_2)_1$ are:

$$\begin{aligned}N_g(\bar j_1,\bar j_2) = 2\,(5)^{1/2}\,\big(4\bar j_2^2\,(\bar j_2+1)^2 + 11\,(8\bar j_1^2+16\bar j_1+5)\,\bar j_2\,(\bar j_2+1)\\ +\bar j_1\,(\bar j_1+2)\,(2\bar j_1-1)\,(2\bar j_1+5)\big)^{-\frac12}\end{aligned} \tag{171}$$

$$\begin{aligned}&\left(\begin{matrix}\bar j_1 & \bar j_2\\ j_1+1 & j_2+1\end{matrix}\right\Vert\left.\begin{matrix}\bar j_1\,\bar j_2 & 1\,1\\ j_1\,j_2 & 1\,1\end{matrix}\right)_1\\ &= -\Big(N_g(\bar j_1,\bar j_2)\ ((j_1+j_2-\bar j_1-\bar j_2)\ (j_1+j_2-\bar j_1-\bar j_2+1)\ (j_1+j_2+\bar j_1-\bar j_2+2)\ (j_1\\ &\quad +j_2+\bar j_1-\bar j_2+3)\ (j_1+j_2-\bar j_1+\bar j_2+1)\ (j_1+j_2-\bar j_1+\bar j_2+2)\ (j_1+j_2+\bar j_1+\bar j_2\\ &\quad +3)\ (j_1+j_2+\bar j_1+\bar j_2+4))^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3))^{1/2}\Big),\end{aligned} \tag{172}$$

$$\begin{aligned}\left(\begin{matrix}\bar j_1 & \bar j_2\\ j_1-1 & j_2-1\end{matrix}\right\Vert\left.\begin{matrix}\bar j_1\,\bar j_2 & 1\,1\\ j_1\,j_2 & 1\,1\end{matrix}\right)_1 &= -\Big(N_g(\bar j_1,\bar j_2)\ ((j_1+j_2+\bar j_1-\bar j_2)\ (j_1+j_2+\bar j_1-\bar j_2\\ &+1)\ (j_1+j_2-\bar j_1+\bar j_2-1)\ (j_1+j_2-\bar j_1\\ &+\bar j_2)\ (-j_1-j_2+\bar j_1+\bar j_2+1)\,(-j_1-j_2+\bar j_1+\bar j_2\\ &+2)\,(j_1+j_2+\bar j_1+\bar j_2+1)\ (j_1+j_2+\bar j_1+\bar j_2\\ &+2))^{1/2}\Big)\Big/\Big(8\,(j_1\,(2\,j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big),\end{aligned} \tag{173}$$

$$\begin{aligned}\left(\begin{matrix}\bar j_1 & \bar j_2\\ j_1-1 & j_2+1\end{matrix}\right\Vert\left.\begin{matrix}\bar j_1\,\bar j_2 & 1\,1\\ j_1\,j_2 & 1\,1\end{matrix}\right)_1 &= -\Big(N_g(\bar j_1,\bar j_2)\ ((j_1-j_2+\bar j_1-\bar j_2-1)\ (j_1-j_2+\bar j_1\\ &-\bar j_2)\ (-j_1+j_2+\bar j_1-\bar j_2+1)\,(-j_1+j_2+\bar j_1-\bar j_2\\ &+2)\,(j_1-j_2+\bar j_1+\bar j_2)\ (j_1-j_2+\bar j_1+\bar j_2\\ &+1)\ (-j_1+j_2+\bar j_1+\bar j_2+2)\,(-j_1+j_2+\bar j_1+\bar j_2\\ &+3))^{1/2}\Big)\Big/\Big(8\,\big(j_1\,(2j_1-1)\,(2j_2^2+5\,j_2+3)\big)^{1/2}\Big),\end{aligned} \tag{174}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 -\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 +3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +1)\big)^{1/2}\Big)\Big/\Big(8\,\big((2j_1^2+5\ j_1+3)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{175}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\big)^{1/2}\Big)\Big/\Big(8\,\big((j_1+1)\,(2j_1+3)\,j_2\,(j_2+1)\big)^{1/2}\Big), \tag{176}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+2)\big)^{1/2}\Big)\Big/\Big(8\,\big(j_1\,(2\ j_1-1)\,j_2\,(j_2+1)\big)^{1/2}\Big), \tag{177}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\big)^{1/2}\Big)\Big/\Big(8\,\big(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3)\big)^{1/2}\Big), \tag{178}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2-1 & j_1 & j_2 & 1 & 1 \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 + j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(8\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{179}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \ (j_1^4+2j_1^3 - \left(10j_2^2+10j_2+2\bar{j}_1^2+2\ \bar{j}_2^2+4\bar{j}_1+2\bar{j}_2+1\right) j_1^2 - 2\left(5j_2^2+5\ j_2+\bar{j}_1^2+\bar{j}_2^2+2\bar{j}_1+\bar{j}_2+1\right) j_1+j_2^4+\ \bar{j}_1^4+\bar{j}_2^4 + 2j_2^3+4\bar{j}_1^3+2\bar{j}_2^3+5\ \bar{j}_1^2-2\bar{j}_1^2\bar{j}_2^2-4\bar{j}_1\ \bar{j}_2^2-\bar{j}_2^2+2\bar{j}_1-2\bar{j}_1^2\bar{j}_2 - 4\ \bar{j}_1\bar{j}_2-2\bar{j}_2-2j_2\left(\bar{j}_1^2+2\ \bar{j}_1+\bar{j}_2^2+\bar{j}_2+1\right)-j_2^2\left(2\bar{j}_1^2 + 4\ \bar{j}_1+2\bar{j}_2^2+2\bar{j}_2+1\right))\Big)\Big/\Big(8\,(j_1\ \ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{180}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1+2\ j_2+3)\,((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 + j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2 +\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{181}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1+2\ \ j_2+1)\,((j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(8\,(j_1 j_2)^{1/2}\Big), \tag{182}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1-2\ \ j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2 +1)\ \ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,j_2)^{1/2}\Big), \tag{183}$$

$$\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\middle\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\,(2\;\; j_1-2j_2-1)\,((j_1-j_2+\bar{j}_1 - \bar{j}_2)\;\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\;\;(j_1-j_2+\bar{j}_1+\bar{j}_2 + 1)\;\;(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2))^{1/2}\Big)/\Big(8\,(j_1\,(j_2+1\,))^{1/2}\Big)\,, \tag{184}$$

$$\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2 \end{matrix}\middle\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\,\big(5\;\; j_1^2+5j_1+5j_2^2+5j_2 - 3\,(\bar{j}_1^2+2\,\bar{j}_1+\bar{j}_2^2+\bar{j}_2)\big)\Big)/\Big(2\,(5)^{1/2}\Big)\,. \tag{185}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1,\bar{j}_2)_2$ are:

$$\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1' & j_2' \end{matrix}\middle\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right)_2 = \left(H^2-X^2\right)^{1/2}\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1' & j_2' \end{matrix}\middle\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right) - X\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1' & j_2' \end{matrix}\middle\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right)_1, \tag{186}$$

where:

$$X = -\frac{1}{10}N_g(\bar{j}_1,\bar{j}_2)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2 + 2)\,(4\bar{j}_1\,(\bar{j}_1+2)+4\bar{j}_2\,(\bar{j}_2+1)-5)\,, \tag{187}$$

$$H^2 = \frac{1}{5}\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,\big(4\bar{j}_2^4+8\bar{j}_2^3-(8\bar{j}_1\,(\bar{j}_1+2)+9)\,\bar{j}_2^2 - (8\bar{j}_1\,(\bar{j}_1+2)+13)\,\bar{j}_2+(\bar{j}_1+1)^2\,(4\bar{j}_1\,(\bar{j}_1+2)-5)\big)\,, \tag{188}$$

and a list of additional coefficients is the following:

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right.\Big)$$
$$= \Big((j_1-j_2)^2 ((j_1+j_2-\bar{j}_1-\bar{j}_2)(j_1+j_2-\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 + j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\Big)/\Big(4((j_1+1)(2j_1+3)(j_2+1)(2j_2+3))^{1/2}\Big), \tag{189}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right.\Big) = \Big((j_1-j_2)^2 ((j_1+j_2+\bar{j}_1-\bar{j}_2)(j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)(-j_1-j_2+\bar{j}_1+\bar{j}_2 +2)(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)/\Big(4(j_1(2\,j_1-1)j_2(2j_2-1))^{1/2}\Big), \tag{190}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right.\Big) = \Big((j_1+j_2+1)^2\ ((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 -\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)(-j_1+j_2+\bar{j}_1-\bar{j}_2 +2)(j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)(-j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\Big)/\Big(4(j_1(2j_1-1)(2j_2^2+5\,j_2+3))^{1/2}\Big), \tag{191}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right.\Big) = \Big((j_1+j_2+1)^2\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 -\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)(-j_1+j_2+\bar{j}_1 -\bar{j}_2)(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 +3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +1))^{1/2}\Big)/\Big(4((2j_1^2+5\,j_1+3)j_2(2j_2-1))^{1/2}\Big), \tag{192}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Bigg(\sqrt{j_2\,(j_2+1)}\left(\frac{(j_1+1)^2}{j_2\,(j_2+1)} - 1\right)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2-\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Bigg)\Big/\Big(4\,((j_1+1)\,(2j_1+3))^{1/2}\Big), \tag{193}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(\,(-j_1^2+j_2^2+j_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1+j_2-\bar{j}_1+\bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(2\;j_1-1))^{1/2}\,(j_2\,(j_2+1))^{1/2}\Big), \tag{194}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2+1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Bigg(\sqrt{j_1\,(j_1+1)}\left(\frac{(j_2+1)^2}{j_1\,(j_1+1)} - 1\right)\,((j_1-j_2+\bar{j}_1-\bar{j}_2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2-\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Bigg)\Big/\Big(4\,((j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{195}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2-1 \end{Vmatrix} \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(\,(j_1^2+j_1-j_2^2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1+j_2-\bar{j}_1+\bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_1+1))^{1/2}\,(j_2\,(2j_2-1))^{1/2}\Big), \tag{196}$$

$$
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) = -\Big(j_1^6 + 3 j_1^5 - \left(j_2^2 + j_2 + 2\bar{j}_1^2 + 2\ \bar{j}_2^2 + 4\bar{j}_1 + 2\bar{j}_2 - 1 \right) j_1^4
$$
$$
- \left(2 j_2^2 + 2\ j_2 + 4\bar{j}_1^2 + 4\bar{j}_2^2 + 8\bar{j}_1 + 4\bar{j}_2 + 3 \right)\ j_1^3
$$
$$
+ \left(-j_2^4 - 2 j_2^3 + \left(4\bar{j}_1^2 + 8\bar{j}_1 + 4\ \bar{j}_2^2 + 4\bar{j}_2 + 2 \right) j_2^2 \right.
$$
$$
+ \left(4\bar{j}_1^2 + 8\ \bar{j}_1 + 4\bar{j}_2^2 + 4\bar{j}_2 + 3 \right)\ j_2 + \bar{j}_1^4 + \bar{j}_2^4 + 4\bar{j}_1^3 + 2\bar{j}_2^3 - 3\ \bar{j}_2^2
$$
$$
\left. - 4\bar{j}_2 + \bar{j}_1^2 \left(-2\bar{j}_2^2 - 2\ \bar{j}_2 + 3 \right) - 2\bar{j}_1 \left(2\bar{j}_2^2 + 2\bar{j}_2 + 1 \right) - 2 \right) j_1^2
$$
$$
+ \left(-j_2^4 - 2 j_2^3 + \left(4\bar{j}_1^2 + 8\ \bar{j}_1 + 4\bar{j}_2^2 + 4\bar{j}_2 + 3 \right) j_2^2 \right.
$$
$$
+ 4\ \left(\bar{j}_1^2 + 2\bar{j}_1 + \bar{j}_2^2 + \bar{j}_2 + 1 \right)\ j_2 + \bar{j}_1^4 + 4\bar{j}_1^3 + \bar{j}_1 \left(-4\bar{j}_2^2 - 4\ \bar{j}_2 + 2 \right)
$$
$$
\left. + \bar{j}_1^2 \left(-2\bar{j}_2^2 - 2\bar{j}_2 + 5 \right) + \bar{j}_2 \left(\bar{j}_2^3 + 2\bar{j}_2^2 - \bar{j}_2 - 2 \right) \right) j_1
$$
$$
+ j_2 \left(j_2 + 1 \right) \left(j_2^4 + 2 j_2^3 - \left(2\ \bar{j}_1^2 + 4\bar{j}_1 + 2\bar{j}_2^2 + 2\bar{j}_2 + 1 \right) j_2^2 \right.
$$
$$
- 2\ \left(\bar{j}_1^2 + 2\bar{j}_1 + \bar{j}_2^2 + \bar{j}_2 + 1 \right)\ j_2 + \bar{j}_1^4 + 4\bar{j}_1^3
$$
$$
+ \bar{j}_1 \left(-4\bar{j}_2^2 - 4\ \bar{j}_2 + 2 \right) + \bar{j}_1^2 \left(-2\bar{j}_2^2 - 2\bar{j}_2 + 5 \right)
$$
$$
\left. + \bar{j}_2 \left(\bar{j}_2^3 + 2\bar{j}_2^2 - \bar{j}_2 - 2 \right) \right) \Big) \Big/ \Big(4 \left(j_1 \left(j_1 + 1 \right) j_2 \left(j_2 + 1 \right) \right)^{1/2} \Big), \tag{197}
$$

$$
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 + \frac{1}{2} & j_2 + \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(\left(j_1 - j_2 \right)^2 \left(2 j_1 + 2 j_2 + 3 \right)\ \big(\left(j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \right.
$$
$$
\left. + 2 \right)\ \left(j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 1 \right)\ \left(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 \right)\ \left(j_1 \right.
$$
$$
\left. + j_2 + \bar{j}_1 + \bar{j}_2 + 3 \right) \big)^{1/2} \Big) \Big/ \Big(4 \left(\left(j_1 + 1 \right) \left(j_2 + 1 \right) \right)^{1/2} \Big), \tag{198}
$$

$$
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(\left(j_1 - j_2 \right)^2 \left(2 j_1 + 2 j_2 + 1 \right)\ \big(\left(j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \right.
$$
$$
\left. + 1 \right)\ \left(j_1 + j_2 - \bar{j}_1 + \bar{j}_2 \right)\ \left(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 \right.
$$
$$
\left. + 1 \right)\ \left(j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2 \right) \big)^{1/2} \Big) \Big/ \Big(4 \left(j_1 j_2 \right)^{1/2} \Big), \tag{199}
$$

$$
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = \Big(\left(j_1 + j_2 + 1 \right)^2 \left(-2 j_1 + 2 j_2 - 1 \right)\ \big(\left(j_1 - j_2 + \bar{j}_1 - \bar{j}_2 \right.
$$
$$
\left. + 1 \right)\ \left(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \right)\ \left(j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2 \right)\ \left(-j_1 \right.
$$
$$
\left. + j_2 + \bar{j}_1 + \bar{j}_2 + 1 \right) \big)^{1/2} \Big) \Big/ \Big(4 \left(\left(j_1 + 1 \right) j_2 \right)^{1/2} \Big), \tag{200}
$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 - \frac{1}{2} & j_2 + \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = \Big((2j_1 - 2j_2 - 1)\,(j_1 + j_2 \\ + 1)^2 \;\; ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2)\;\;(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \\ + 1)\;\;(j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\;\;(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 \\ + 2))^{1/2}\Big)/\Big(4\,(j_1\,(j_2 + 1\,))^{1/2}\Big), \tag{201}$$

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 0 & 0 \end{array}\right) = -\Big(\bar{j}_1^4 + 4\bar{j}_1^3 + (-2\bar{j}_2^2 - 2\,\bar{j}_2 + 5)\,\bar{j}_1^2 + (-4\bar{j}_2^2 - 4\bar{j}_2 + 2)\,\bar{j}_1 + \bar{j}_2^4 \\ + 2\bar{j}_2^3 - 5\,(j_1^2 + j_1 - j_2\;\;(j_2 + 1))^{\,2} - \bar{j}_2^2 - 2\bar{j}_2\Big)/\Big(2\,(5)^{1/2}\Big). \tag{202}$$

The rest of the coefficients can be obtained by using the symmetries of the coefficients [43]. There are no multiplicity, so the symmetries are simply given by:

$$\left(\begin{array}{cc||cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{1} & \bar{1} \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \end{array}\right) = (-1)^{\bar{j}_1 - \bar{j}'_1 + \bar{j}'_2 - \bar{j}_2 + j_1 - j'_1 + j_2 - j'_2 + j''_1 + j''_2} \times \\ \sqrt{\frac{dim(\bar{j}_1, \bar{j}_2)(2j'_1 + 1)(2j'_2 + 1)}{dim(\bar{j}'_1, \bar{j}'_2)(2j_1 + 1)(2j_2 + 1)}} \left(\begin{array}{cc||cccc} \bar{j}'_1 & \bar{j}'_2 & \bar{j}_1 & \bar{j}_2 & \bar{1} & \bar{1} \\ j'_1 & j'_2 & j_1 & j_2 & j''_1 & j''_2 \end{array}\right),$$

where $dim(\bar{j}_1, \bar{j}_2) = (2\bar{j}_1 - 2\bar{j}_2 + 1)(2\bar{j}_1 + 2\bar{j}_2 + 3)(2\bar{j}_1 + 2)(2\bar{j}_2 + 1)/6$.

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Representations and Particles of Orthosymplectic Supersymmetry Generalization[1]

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Abstract—Orthosymplectic $osp(1|2n)$ supersymmetry (alternative names: Generalized conformal supersymmetry with tensorial central charges, conformal M-algebra, parabose algebra) has been considered as an alternative to d-dimensional conformal superalgebra. Due to mathematical difficulties, even classification of its unitary irreducible representations (UIR's) have not been entirely accomplished. We give this classification for $n = 4$ case (corresponding to four dimensional space-time) and then show how the discrete subset of these UIR's can be constructed in a Clifford algebra variation of the Green's ansatz.

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1. INTRODUCTION

Orthosymplectic type of space-time symmetry was first analyzed by C. Fronsdal [1], as early as in 1985, and since then interest for this symmetry reappeared, sometimes independently, in many contexts: M-theory [2], BPS particles [3], higher spin fields [4] and others [5].

When considering a (super)group in the context of a space-time symmetry, one of the first and most natural steps to undertake is to find unitary irreducible representations (UIR's) of the group, as these give us basic information on the particle content of the free theory. And in the case of orthosymplectic supersymmetry, physically most important class of UIR's are, so called, positive energy UIR's. The problem of finding these representations have been solved only for $n = 1$ and $n = 2$. We followed the approach of [6] and completed the task for $n = 4$ by using computer algorithms to analyze Verma module structure. In this way we managed to make a complete list of positive energy $osp(1|8)$ UIR's, together with explicit forms of the corresponding Verma module singular and subsingular vectors. In this short report we present the main features of the results, leaving the details to be published separately. In particular, we point out that there is a concrete number of discrete UIR families (precisely nine, or ten if the trivial representation is counted as a separate class), that physically should be related to elementary particles of $osp(1|8)$ models. In addition, we also point out a method to explicitly construct discrete representations, allowing one to easily perform concrete calculations in these spaces and, in that way, give physical interpretation to the states within. The method, directly generalizable to arbitrary n, is based on a Clifford algebra variation of the Green's ansatz and is mathematically related to Howe duality.

[1] The article is published in the original.

2. POSITIVE ENERGY UIR's OF $osp(1|8)$

Structural relations of $osp(1|2n)$ superalgebra can be compactly written in the form of trilinear relations of odd algebra operators a_α and $a_\alpha^\dagger$:

$$\begin{aligned} [\{a_\alpha, a_\beta^\dagger\}, a_\gamma] &= -2\delta_{\beta\gamma}a_\alpha, \\ [\{a_\alpha^\dagger, a_\beta\}, a_\gamma^\dagger] &= 2\delta_{\beta\gamma}a_\alpha^\dagger, \end{aligned} \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma], \quad [\{a_\alpha^\dagger, a_\beta^\dagger\}, a_\gamma^\dagger] = 0, \tag{2}$$

where operators $\{a_\alpha, a_\beta^\dagger\}$, $\{a_\alpha, a_\beta\}$ and $\{a_\alpha^\dagger, a_\beta^\dagger\}$ span the even part of the superalgebra and Greek indices take values 1, 2, ...n (relations obtained from these by use of Jacobi identity are also implied). If we additionally require that the dagger symbol † above denotes hermitian conjugation in the algebra representation Hilbert space (of positive definite metrics), then we have effectively constrained ourselves to the, so called, positive energy UIR's of $osp(1|2n)$.[2] Namely, in such a space, "conformal energy" operator $E \equiv 1/2\sum_\alpha \{a_\alpha, a_\alpha^\dagger\}$ must be a positive operator. Operators a_α reduce the eigenvalue ϵ of E, so the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. From the algebra relations follows: $|v\rangle \in V_0 \Rightarrow \{a_\alpha, a_\beta^\dagger\}|v\rangle \in V_0$, with α, β arbitrary.

[2] Omitting a short proof, we note that in such a Hilbert space all superalgebra relations actually follow from one single relation—the first or the second of (1).

Therefore vacuum subspace carries a representation of $U(1) \times SU(N)$ group generated by operators $\{a_\alpha, a^\dagger_\beta\}$ (with $U(1)$ part generated by E). The positive energy UIR's of $osp(1|2n)$ are entirely labelled by UIR μ of $SU(N)$ (that can be given by a Young diagram) and a positive real number ϵ_0 (energy of the vacuum subspace) that labels $U(1)$ representation. However, for a given representation μ only certain values of ϵ_0 are allowed, and this connection is highly important and nontrivial.

In this paper we are interested in the $n = 4$ case. $SU(4)$ representation μ will be explicitly parameterized by three non negative integers s_1, s_2, s_3 in a way that μ is determined by a Young diagram with $s_1 + s_2 + s_3$ boxes in the first row, $s_1 + s_2$ boxes in the second and s_1 boxes in the third row. In addition to these three numbers, we will use real parameter d given by $d = 1/4(\epsilon_0 - s_1 + s_3)$ to label $osp(1|8)$ representations.

Classification of all positive energy UIR's was done by computer analysis of the Verma module structure, carried out in the following general manner (that we just briefly describe). First, Kac determinant of a sufficiently high level was considered as a function of parameter d (for each given class of $SU(4)$ representation μ). In this way it was possible to locate the highest value of d for which the determinant vanishes and the Verma module becomes reducible. The singular or subsingular vector responsible for the singularity of the Kac matrix was then calculated, effectively by solving an (optimized) system of linear equations. Next we would find the norm of this vector and look for possible additional discrete reduction points at (lower) values of d for which the norm also vanishes. If new reduction points with new (sub)singular vectors were found it was also necessary to check that, upon removal of the corresponding submodules, no vectors with zero or negative norm remained. For this, it was enough to check that previously found (sub)singular vectors (i.e. those occurring for higher d values) belonged to the factored-out submodules. Optimized Wolfram Mathematica code was written to perform all these calculations. We now summarize the main results.

Parameter d can take the following values, depending on the labels s_1, s_2, s_3:

1. $s_1 = s_2 = s_3 = 0$: $d > 3/2$ and singular points $d = 0$, 1/2, 1, 3/2;

2. $s_1 = s_2 = 0$, $s_3 > 0$: $d > s_{3/2} + 2$ and singular points $d = s_3/2 + 1$, $s_3/2 + 3/2$, $s_3/2 + 2$;

3. $s_1 = 0$, $s_2 > 0$: $d > (s_2 + s_3)/2 + 5/2$ and singular points $d = (s_2 + s_3)/2 + 2$, $(s_2 + s_3)/2 + 5/2$;

4. $s_1 > 0$: $d > (s_1 + s_2 + s_3)/2 + 3$ and a singular point $d = (s_1 + s_2 + s_3)/2 + 3$.

Case 1 corresponds to "unique vacuum" representations, i.e. when the vacuum subspace V_0 is one dimensional and carries trivial representation of the $SU(4)$ group. Since spatial rotations are a subgroup of this $SU(4)$ group, in representations 1 the lowest conformal energy state is invariant to rotations. In this sense, representations 1 correspond to "fundamentally scalar" particles or, more precisely, multiplets (states of other spin values also belong to the multiplet of the full supersymmetry). Of particular physical interest are representations at singular points, and there are exactly three of such scalar representations. Namely, at singular points additional equations of motion appear directly related to the corresponding singular or subsingular vectors. In the case of the simplest nontrivial representation $d = 1/2$, $s_1 = s_2 = s_3 = 0$, singular vector yields the massless condition $p^2 = 0$ (this is the well known and studied *UIR* containing tower of massless particles with increasing helicities). However, relating (sub)singular vectors to physical constraints (i.e. equations of motion) is in general complicated, and this problem is effectively solved by the explicit construction of representations that is discussed in the following section.

Case 2 corresponds to representations where V subspace transforms w.r.t. $SU(4)$ subgroup as a Young diagram with s_3 boxes in a single row. The simplest representative of the kind is a single box representation—lowest energy state in these representations behaves as a Lorentz 1/2 spinor. There are again three singular points in the case 2, corresponding to three physically interesting classes of representations, including three "fundamentally" spinor multiplets.

Cases 3 and 4 correspond to more complex classes of representations. However, it turns out that states from these classes can be naturally seen as composite states built from states belonging to representations of classes 2 or of classes 1.

Overall, there turns out to be 10 singular points corresponding to 9 different classes of nontrivial multiplets ($d = 0$ point corresponds to the trivial representation).

3. CONSTRUCTION OF REPRESENTATIONS

It turns out that all representations with (half)integer d values (including all representations at (sub)singular points) can be obtained by representing the odd superalgebra operators a and $a^\dagger$ as the following sums:

$$a_\alpha = \sum_{a=1}^{p} b^a_\alpha e^a, \quad a^\dagger_\alpha = \sum_{a=1}^{p} b^{a\dagger}_\alpha e^a. \tag{3}$$

In these expressions, p is integer, e^a are elements of a real Clifford algebra:

$$\{e^a, e^b\} = 2\delta^{ab}, \quad (a, b) = 1, 2, \ldots p, \tag{4}$$

and operators b^a_α together with adjoint $b^{a\dagger}_\alpha$ satisfy ordinary bosonic algebra relations: $[b^a_\alpha, b^{b\dagger}_\beta] = \delta_{\beta\alpha}\delta^{ab}$,

$[b^a_\alpha, b^b_\beta] = 0$. The (reducible) representation space is spanned by the vectors:

$$\mathcal{H} = l.s.\{\mathcal{P}(b^\dagger)\{0\rangle \otimes \omega\}, \tag{5}$$

where $\mathcal{P}(b^\dagger)$ are monomials in mutually commutative operators $b^{a\dagger}_\alpha$, $|0\rangle$ is a bosonic vacuum and $w \in \mathcal{H}_{CL}$, where $\mathcal{H}_{CL}$ is the representation space of real Clifford algebra (4).

Representation ansatz in the form (3) possesses certain intrinsic symmetries. Operators:

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(b^{a\dagger}_\alpha b^b_\alpha - b^{b\dagger}_\alpha b^a_\alpha) + \frac{i}{4}[e^a, e^b] \tag{6}$$

commute with entire $osp(1|8)$ superalgebra. Operators G^{ab} themselves satisfy commutation relations of $so(p)$ algebra (the full symmetry is actually slightly larger, given by the orthogonal group). This symmetry we will call the gauge symmetry.

The gauge symmetry actually removes all degeneracy in decomposition of (5) to $osp(1|8)$ UIR's, i.e. the multiplicity of $osp(1|8)$ UIR's is fully taken into account by labeling transformation properties of the vector w.r.t. the gauge symmetry group. Furthermore, there is one-to-one correspondence between UIR's of $osp(1|8)$ and of the gauge group that appear in the decomposition, meaning that transformation properties under the gauge group action automatically fix the $osp(1|8)$ representation.

The vector $|v^0_{\{d, s_1, s_2, s_3\}}\rangle$ that is the lowest weight vector of $osp(1|8)$ positive energy *UIR* $\{d, s_1, s_2, s_3\}$ and the highest weight vector of the gauge group *UIR* (in a standardly defined root system) takes the following explicit form (up to multiplicative constant):

$$\begin{aligned} |v^0_{\{d,s_1,s_2,s_3\}}\rangle &= (B^{(1)\dagger}_{4+})^{s_3}(B^{(1)\dagger}_{4+}B^{(2)\dagger}_{3+} - B^{(2)\dagger}_{4+}B^{(1)\dagger}_{3+})^{s_2} \\ &\times \left(\sum_{k_1,k_2,k_3=1}^{3} \varepsilon_{k_1k_2k_3} B^{(k_1)\dagger}_{4+} B^{(k_2)\dagger}_{3+} B^{(k_3)\dagger}_{2+}\right)^{s_1} \\ &\times \left(\sum_{k_1,k_2,k_3,k_4=1}^{4} \varepsilon_{k_1k_2k_3k_4} B^{(k_1)\dagger}_{4+} B^{(k_2)\dagger}_{3+} B^{(k_3)\dagger}_{2+} B^{(k_4)\dagger}_{1+}\right)^{s_0} |0\rangle \otimes \omega_{h.w.}, \end{aligned} \tag{7}$$

with $d = s_0 + \frac{s_1 + s_2 + s_3 + p}{2}$, where $B^{(k)}_{\alpha\pm} = \frac{1}{\sqrt{2}}(b^{2k-1}_\alpha \mp i b^{2k}_\alpha)$ and $(e^{2k-1} + ie^{2k})\omega_{h.w.} = 0$, $k = 0, 1, \ldots [p/2]$. The form above assumes that p is large enough that all $B^{(k)}_{\alpha\pm}$ can be defined, i.e. $p \geq 8$: s_0 must be 0 when $p < 8$, s_1 must be 0 when $p < 6$, s_2 must be 0 when $p < 4$ and all s_0, s_1, s_2, s_3 must be 0 when $p = 1$ ($p = 0$ is trivial *UIR* of $osp(1|8)$).

In this way all positive energy *UIR*-s of $osp(1|8)$ classified in the previous section with integer or halfinteger values of d can be constructed using ansatz (3) with $p \leq 9$. Physically, corresponding states have natural interpretation as particles composed from p of the simplest $osp(1|8)$ massless particles (belonging to $d = 1/2$, $s_1 = s_2 = s_3 = 0$ *UIR*).

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Abstract Booklet

MINI-WORKSHOP 2014

Search for Classical Analysis and Quantum Integrable Systems,
15-17 November 2014, Kyoto University, Japan

1 OUTLINE

- **Title : MINI-WORKSHOP 2014 :**
Search for Classical Analysis and Quantum Integrable Systems
15-17 November 2014, Kyoto University, Japan

- **Scope and Topics : Classical and Quantum Integrable Models, Lie Theory and Symmetry in Physics**

- **Place : Graduate School of Informatics, Kyoto University, 36-1 Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan**

Nov.15	Faculty of Engineering Integrated Research Bldg. Room 213
Nov.16	Faculty of Engineering Integrated Research Bldg.Room 213
Nov.17	Faculty of Engineering Research Bldg. No.8 Room 3

- **Timetable :**

Nov.15	Talks begin at AM 10:30. Workshop dinner
Nov.16	Group photo
Nov.17	Talks end in the noon.

- **Accommodation : Guest House of Kyoto University "Seifu-Kaikan"**

- **Workshop Dinner : Japanese style bar "Momojiro-Hyakumanben"**

- **Web : http://kojima.yz.yamagata-u.ac.jp/workshop2014.html**

- **Organizing Committee :**
Takeo Kojima(Yamagata University), Satoshi Tsujimoto(Kyoto University)

2 PROGRAM

• **List of Invited Speakers :**

	Name	Affiliation and E-mail
1	**Prof. Naruhiko Aizawa**	**Osaka Prefecture University** aizawa@mi.s.osakafu-u.ac.jp
2	**Prof. Veljko Dmitrasinovic**	**Osaka University, Institute of Physics Belgrade** dmitrasin@yahoo.com
3	**Prof. Masashi Hamanaka**	**Nagoya University** hamanaka@math.nagoya-u.ac.jp
4	**Prof. Shuhei Kamioka**	**Kyoto University** kamioka.shuhei.3w@kyoto-u.ac.jp
5	**Prof. Takeo Kojima**	**Yamagata University** kojima@yz.yamagata-u.ac.jp
6	**Prof. Satoru Odake**	**Shinshu University** odake@azusa.shinshu-u.ac.jp
7	**Prof. Dimitri Polyakov**	**Sogang University, Sichuan University** twistorstring@gmail.com
8	**Prof. Yas-Hiro Quano**	**Suzuka University of Medical Science** quanoy@suzuka-u.ac.jp
9	**Prof. Igor Salom**	**University of Belgrade** isalom@ipb.ac.rs
10	**Prof. Hiroto Sekido**	**Kyoto University** sekido@amp.i.kyoto-u.ac.jp
11	**Prof. Fumihiko Sugino**	**Okayama Institute for Quantum Physics** fumihiko_sugino@pref.okayama.lg.jp
12	**Prof. Kouichi Takemura**	**Chuo University** takemura@math.chuo-u.ac.jp
13	**Prof. Satoshi Tsujimoto**	**Kyoto University** tujimoto@i.kyoto-u.ac.jp

	Name	Affiliation and E-mail
14	**Prof. Hiroshi Miki**	**Doshisha University** hmiki@mail.doshisha.ac.jp
15	**Prof. Hiroyuki Yamane**	**Toyama University** hiroyukikcoe25@yahoo.co.jp

- **Program :**

	November 15	November 16	November 17
9:30-10:30			MIKI
10:30-11:30	TSUJIMOTO	YAMANE	SUGINO
11:30-12:30	ODAKE	KAMIOKA	KOJIMA
12:30-14:00	**lunch**	**lunch**	**departure**
14:00-15:00	AIZAWA	DMITRASINOVIC	
15:00-16:00	SEKIDO	SALOM	
16:00-16:15	**coffee**	**coffee**	
16:15-17:15	QUANO	TAKEMURA	
17:15-18:15	POLYAKOV	HAMANAKA	
18:30-	**dinner**		

No. 53：Faculty of Engineering Integrated Research Bldg.

No. 59：Faculty of Engineering Research Bldg. No.8

3 ABSTRACT

• **Professor Naruhiko Aizawa (Osaka Prefecture University)**

Title : Recent developments in representation theory of nonrelativistic conformal algebras.

Abstract : Nonrelativistic conformal algebra (NRCA) is a particular class of non-semisimple Lie algebras. The member of the class is a finite or an infinite dimensional Lie algebra. The semisimple part of the finite dimensional algebras is the direct sum of sl(2) and so(d), while the Virasoro algebra is the semisimple part of the infinite dimensional algebras. This class of Lie algebra appears in various kind of problems in theoretical and mathematical physics. For instance, one can find them in connection with fluid dynamics, gravity theory, AdS/CFT correspondence and vertex operator algebras. This motivate us to study representations of NRCA. The first part of this talk is an overview on NRCA. Various members of NRCA are introduced and their structure, extensions are discussed. We also give a brief summary on what has been done on representations of NRCA. In the second part, we pick up the simplest members of finite dimensional NRCA and discuss a classification of irreducible representations of the lowest weight type. As an application of the representation theory, we construct partial differential equations with the symmetries generated by NRCA.

• **Professor Veljko Dmitrasinovic (Osaka University and Institute of Physics, University of Belgrade (Serbia))**

Title : Dynamical Symmetry of quantum and classical motion of three quarks tethered to the Torricelli point.

Abstract : Abstract: The motivation for a search for a three-body dynamical symmetry comes from the quantum dynamics of three quarks confined by the so- called Y- and Delta strings. After a brief review of the Y- and Delta strings, their spectra and the degeneracies within, we introduce a set of three-body kinematic variables that are permutation symmetric and expose the underlying dynamical O(2) symmetry of the Y-string. This symmetry exists also as an approximate symmetry of other permutation symmetric

three-body systems, such as the Newtonian gravity one. We illustrate the role of the symmetric variables by displaying the classical Newtonian periodic three-body orbits of Euler, Lagrange and Moore in these terms. Subsequently: a) we show that the above string systems also have the same classical solutions; b) we found new classical periodic orbits of three quarks in the Y-string potential; and c) we found several new periodic orbits of three bodies in the Newtonian gravity.

- **Professor Masashi Hamanaka (Nagoya University)**

Title : Noncommutative ADHM constructions and duality.
Abstract : Abstract: Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is a powerful construction method of instantons which are finite-action solutions of anti-self-dual Yang-Mills (ASDYM) equations in four-dimensional Euclidean space. This is based on a beautiful duality between moduli space of the instantons and moduli space of the ADHM data. In this talk, we discuss the ADHM construction of U(N) instantons in noncommutative (NC) space and prove the duality. This is based on collaboration with Toshio Nakatsu (Setsunan University).

- **Professor Shuhei Kamioka (Kyoto University)**

Title : Different approaches to the Aztec diamond theorem.
Abstract : Abstract: The Aztec diamond theorem by Elkies, Kuperberg and Larsen and Propp (1992) concerns a nice product formula for a combinatorial problem of domino tilings. The theorem is so beauty that many different proofs have been given in different approaches. In this talk three of those different proofs are reviewed which are based on: (i) "urban renewal" trick by Kuperberg; (ii) determinant calculation by Kamioka; and (iii) vertex operator method by Bouttier, Chapuy and Corteel.

- **Professor Takeo Kojima (Yamagata University)**

Title : Vertex operator approach to semi-infinite lattice model: recent progress.

Abstract : Vertex operator approach is a powerful method to study exactly solvable models directly in the thermodynamic limit. In this talk we review recent progresses of vertex operator approach to semi-infinite lattice models.

(1) The first progress is a generalization of the boundary condition. We study the XXZ spin chain with a triangular boundary. Bosonizations of the boundary vacuum states are realized. Integral representations of correlation functions and form factors are proposed using bosonizations. As an application, $q-$series formulae of the boundary expectation values $< \sigma_1^{\pm} >$ are derived. Exploiting the spin reversal property, relations between n-fold integrals of elliptic theta functions are conjectured.

(2) The second progress is a generalization of the symmetry. We study the elliptic $U_{q,p}(\widehat{sl_N})$ lattice model with diagonal boundary condition, which gives an elliptic deformation of the higher-rank XXZ spin chain. Bosonizations of the boundary vacuum states are realized. Integral representations of correlation functions are proposed using bosonizations. Exploiting the spin reversal property, relations between n-fold integrals of double-infinite products are conjectured. References :

[1] T.Kojima, Diagonalization of infinite transfer matrix of boundary $U_{q,p}(A_{N-1}^{(1)})$ face model, *J.Math.Phys.***52** 01351 (26pages) (2011)

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- **Professor Satoru Odake (Shinshu University)**

Title : Solvable discrete quantum mechanics: q-orthogonal polynomials with $|q| = 1$ and quantum dilogarithm.

Abstract : Several kinds of q-orthogonal polynomials with —q—=1 are constructed as the main parts of the eigenfunctions of new solvable discrete quantum mechanical systems. Their orthogonality weight functions consist of quantum dilogarithm functions, which are a natural extension of the Euler gamma functions and the q-gamma functions (q-shifted factorials). The dimensions of the orthogonal spaces are finite. These q-orthogonal polynomials are expressed in terms of the Askey-Wilson polynomials and their certain limit forms. This talk is based on the collaboration with R.Sasaki, arXiv:1406.2768.

- **Professor Yas-Hiro Quano (Suzuka University of Medical Science)**

Title : Form factors of spin 1 analogue of the eight-vertex model.

Abstract : The spin 1 analogue of the eight-vertex model (21-vertex model) is considered on the basis of free field representations of vertex operators in the 2×2-fold fusion SOS model and vertex-face transformation. Correlation functions and form factors in the 21-vertex model can be expressed in terms of type I and type II vertex operators of the corresponding fused SOS model and so-called tail operators. We need the tail operators in order to translate correlation functions and form factors in SOS model into those of elliptic vertex model. For correlation functions we use the tail operators for diagonal matrix elements with respect to the ground state sectors, and for form factors we use the ones for off diagonal matrix elements. In this talk we will construct the free field representations of the tail operators for off diagonal matrix elements with respect to the ground state sectors. As a result, integral formulae for form factors of any local operators in the 21-vertex model can be obtained, in principle.

- **Professor Igor Salom (Institute of Physics, University of Belgrade (Serbia))**

Title : Permutation-symmetric three-particle hyper-spherical harmonics.
Abstract : We consider the non-relativistic three-body body problem in quantum mechanics. Following the approach from the two-body case, the goal is to split the problem into radial and angular parts. To this end, the key element is to obtain three-body equivalent of the standard spherical harmonics (which are used for solving the two-body problem). We demonstrate the construction of the three-body permutation symmetric hyperspherical harmonics, both in the case of planar motion (2D case) and in the case of general motion (3D case).

- **Professor Hiroto Sekido (Kyoto University)**

Title : Polynomial regression with time evolution of discrete integrable systems and its applications.
Abstract : Abstract: In this talk, D-optimal designs are considered. D-optimal designs for polynomial regression models correspond to the discrete Toda equation through canonical moments and the Hankel determinant expression. We show that polynomial regression models and its D-optimal designs are generalized by using the time evolution of the discrete Toda equation. Then we introduce some applications of the generalized models.

- **Professor Fumihiko Sugino (Okayama Institute for Quantum Physics)**

Title : A SUSY double-well matrix model for 2D type IIA superstring theory.
Abstract : After a review of matrix models for bosonic string theories, we consider a double-well supersymmetric matrix model and its interpretation as a nonperturbative definition of two-dimensional type IIA superstring theory. The interpretation is confirmed by direct comparison of symmetries and amplitudes in both sides of the matrix model

and the IIA superstring theory. Next, we obtain the full nonperturbative free energy of the matrix model in terms of the Tracy-Widom distribution in random matrix theory. Its weak coupling expansion implies spontaneous supersymmetry breaking due to instantons, and strong coupling behavior suggests the existence of a well-defined S-dual theory. Furthermore, from the expression of the free energy, we see a smooth connection between a non-supersymmetric string theory and the IIA superstring theory.

- **Professor Kouichi Takemura (Chuo University)**

Title : Ultradiscrete Painlevé equations with parity variables.
Abstract : We introduce a ultradiscretization with parity variables of the q- difference Painlevé VI system of equations. We investigate solutions of ultradiscrete Painlevé equations and give a conjecture. This tale is based on a joint work with Terumitsu Tsutsui.

- **Professor Satoshi Tsujimoto (Kyoto University)**

Title : Spectral coincidence of transition operators, automata groups and box-ball systems.
Abstract : We give the automata which describe time evolution rules of the box-ball systems with a carrier. It can be shown by use of tropical geometry, such systems are ultradiscrete analogues of KdV equation. We discuss their relation with the lamplighter group generated by an automaton. We present spectral analysis of the stochastic matrices induced by these automata, and verify their spectral coincidence.

- **Professor Hiroshi Miki (Doshisha University)**

Title : Two-variable orthogonal polynomials and quantum state transfer model.
Abstract : In this talk, we show that a two-dimensional spin lattice could be solvable where an exact solution of the one excitation dynamics is provided in terms of 2-variable orthogonal polynomials. Then the perfect state transfer, the quantum state transfer with

the probability 1, is shown to take place on the lattice.

• **Professor Hiroyuki Yamane (Toyama University)**

Title : Representation theory of generalized quantum groups via Weyl groupoids.

Abstract : To any bi-homomorphism χ from a free abelian group to a field K, we can associate a generalized quantum group $U(\chi)$ in a standard way. $U(\chi)$ can be: (a) a quantum group, (b) a Lusztig's small quantum group at a root of unity, (c) a multi-parameter quantum group, (d) a quantum group associated with a basic classical Lie superalgebra, and (e) the Drinfeld quantum doubles of ta Nichols algebras of diagonal type. Let $R(\chi)$ be a generalized roo system of $U(\chi)$. Let $W(\chi)$ be the Weyl groupoid of $R(\chi)$. We apply $W(\chi)$ to obtain the classification of finite dimensional irreducible representations of $U(\chi)$, the Shapovalov determinants of $U(\chi)$, and the classification of (skew) centers of $U(\chi)$. We emphasis that especially for χ comming from the Lie superalgebra of type $B(m,n)$, there are many finite dimensional irreducible representations of $U(\chi)$ which can not be taken $q \to 1$.

Three Quarks Confined by an Area-Dependent Potential in Two Dimensions

Igor Salom and V. Dmitrašinović

Abstract We study the low-lying parts of the spectrum of three-quark states with definite permutation symmetry bound by an area-dependent three-quark potential. Such potentials generally confine three quarks in non-collinear configurations, but classically allow for free (unbound) collinear motion. We use our previous work to evaluate the low-lying parts of the spectrum in a non-adiabatic approximation. We show that the eigen-energies are positive and discrete, i.e., that the system is quantum-mechanically confined in spite of the classically allowed free collinear motion.

Keywords Potential models · Baryons · Y-junction string

1 Introduction

In a recent series of papers we have developed an algebraic theory of quantum mechanical three-body bound states in two [1–3] and in three dimensions [4–6]. This theory is based on the O(4) and O(6) symmetries, respectively, of the relativistic kinetic energy and the corresponding O(4) and O(6) hyperspherical harmonics. One expands the three-body potential and the wave functions in these hyperspherical harmonics and then uses the O(n) algebra to simplify the Schrödinger equation.

If the three-body potential is homogenous, then, under certain conditions on the expansion coefficients v of the three-body potential, allow for an energy spectrum that depends essentially only on the said coefficients. This fact leads to a well-known theorem [7–12] about energy-level ordering in the lower shells of the spectrum. Most three-body confining potentials, such as the Δ- and Y-string ones, satisfy these

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V. Dobrev (ed.), *Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 2*, Springer Proceedings in Mathematics & Statistics 255, https://doi.org/10.1007/978-981-13-2179-5_31

conditions, and consequenty their states have the "universal" ordering properties, so that their low-lying spectra look alike.

In continuation of our previous work on the quantum mechanics of three-particle bound states, here we present an example of a potential that is homogenous and generally confines classically, except under (very) special circumstances, and yet does not satisfy the aforementioned conditions. Consequently its energy spectrum is not readily calculable using our previous (adiabatic) formulae/results and does not have the "universal" properties. We present the results of a non-adiabatic calculation for low-lying parts of the spectrum in two dimensions.[1] We show that the ordering of states is significantly distorted, as compared with the conventional one, but the energy spectrum remains discrete and positive, i.e., it corresponds to a quantum-mechanically confined system.

2 An Area-Dependent Potential

We define the model potential as an harmonic oscillator perturbed by an "area term", with the coupling strength v_b,

$$V_{\rm HY} = \frac{k}{2}\left(v_a(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2) + v_b|\boldsymbol{\rho}\times\boldsymbol{\lambda}|\right). \tag{1}$$

This potential is homogenous with homogeneity coefficient $\alpha = 2$. It can be viewed as harmonic ($\alpha = 2$) generalization of the Y-string potential, which is homogenous with $\alpha = 1$, so we may call it the "harmonic Y-string".

In the limit $v_a > 0$, $v_b = 0$ this potential turns into the standard harmonic oscillator, with the well-known discrete, equidistant energy spectrum. In the limit $v_a = 0$, $v_b > 0$ the potential is still harmonic in the sense that it is proportional to the square of the hyper-radius R^2, but it depends only on the area of the triangle $|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$. Manifestly, this area vanishes for all collinear quark configurations, i.e. whenever vector $\boldsymbol{\rho}$ is parallel with the vector $\boldsymbol{\lambda}$, thus making such collinear classical motions free, i.e., unconfined.

An interesting question is what happens to this one unconfined mode of classical motion?[2] In other words, can such a "deformation" of the harmonic oscillator potential change the discrete nature of the original harmonic oscillator energy spectrum? In order to try and answer that question we shall solve the full (i.e. non-adiabatic) Schrödinger equation, Ref. [3]. The three-body potential Eq. (1) can be expanded in terms of $L = 0$ $SO(4)$ hyper-spherical harmonics $\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi)$, Ref. [3]

$$V_{3-\rm body}(\alpha, \phi) = \sqrt{\frac{\pi}{2}}\sum_{J,M}^{\infty} v_{JM}^{3-\rm body}\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi) \tag{2}$$

[1]The three-dimensional calculation will be shown elsewhere.

[2]which has measure zero as compared with the set of all three-body configurations - "shape space".

which is equivalent to an expansion in $SO(3)$ hyper-spherical harmonics,

$$V_{3-\text{body}}(\alpha, \phi) = \sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} Y_{JM}(\alpha, \phi) \tag{3}$$

that are functions of the two hyper-angles (α, ϕ). The fact that the potential does not depend on the angle ϕ implies that only $SO(3)$ hyper-spherical harmonics with $M = 0$ enter the expansion, which is equivalent to an expansion in Legendre polynomials of the variable $x = \cos\alpha$, see the Appendix.

In the following we shall keep only the first two terms in the Legendre expansion of the potential, Eq. (10) and then use it to solve the Schrodinger equation numerically for an arbitrarily large ratio of strengths of the area- and harmonic potentials $\frac{v_b}{v_a} \to \infty$, which corresponds to the $v_a \to 0$ limit, while keeping v_b finite.

In that limit, the ratio of the two "harmonic Y string" effective potential coefficients $\lim_{k\to 0}\left(v_{20}^{HY}/v_{00}^{HY}\right) = \frac{\sqrt{5}}{4}$ remains finite, however, as can be seen in the Appendix, and thus ensures that there remains an effective harmonic oscillator component in the effective potential and thus preserves confinement.

3 Low-Lying Energy Spectrum

The Hilbert space of this problem naturally separates into the even- and odd-parity parts, that are fully disconnected from each other. Moreover, other conserved quantities, ("good quantum numbers"), such as the total angular momentum L and the permutation symmetry multiplets, also provide other "super-selection rules" that further split the Hilbert space into smaller subsets. One particularly interesting Hilbert sub-space is the $L = 0$ space: this is where the deconfined ("continuum") states ought to appear, provided that they exist at all. This is because collinear motion implies vanishing angular momentum, but not *vice versa*.

Following Sect. IV.A in Ref. [3] we may use $m_1 = \frac{1}{2}\left(l_\rho + l_\lambda\right) = \frac{L}{2} = 0$ and $m_2 = G_3 = 0$ as the definition of the invariant sub-space. This condition means that these are the $[SU(6), L^P] = [56, 0^+]$ and $[SU(6), L^P] = [20, 0^+]$ states (in the spectroscopy notation), the former appears first in the $K = 0$ band and the latter in the $K = 2$ band. They re-appear at even K's, with increasing multiplicity.

We look at the strongly perturbed spectrum of the first 21 even-K states (K = 0, 2, 4, 6, 8, 10) sub-space satisfying the $m_1 = m_2 = 0$, i.e., $L = 0 = G_3$ condition. For convenience we (re)define the Hamiltonian as

$$H = H_0 + \mathrm{C}_{pot}\frac{R^2}{\sqrt{2}\pi}\left(\mathcal{Y}_{00}^{J=0} + \frac{2}{\sqrt{5}}\mathcal{Y}_{00}^{J=2}\right) \tag{4}$$

where H_0 is the harmonic oscillator Hamiltonian, with eigenvalues that are multiples of $\mathrm{C}_0 = \hbar\omega$ and C_{pot} is the coefficient multiplying the area term, i.e., $\mathrm{C}_{pot} \simeq v_b$.

Equation (4) contains the $\mathcal{Y}_{00}^{J=0}$ term which is the part of the area-term and due to the presence of this term in expansion there is no appearance of artificial negative eigenvalues.

Thus, we need the Hamiltonian matrix for the 21-dimensional even-K state (K $= 0, 2, 4, \dots, 10$) sub-space of the full Hilbert space satisfying the condition $m_1 = m_2 = 0$. We must diagonalize the corresponding 21×21 Hamiltonian matrix; below we show the upper-left-hand corner 6×6 submatrix, corresponding to $\mathrm{K} = 0, 2, 4$ states, of the full 21×21 matrix

$$\begin{pmatrix} \frac{\mathrm{C}_{pot}}{2\lambda\pi^2} + 2\mathrm{C}_0 & 0 & \frac{\sqrt{\frac{3}{2}}\mathrm{C}_{pot}}{5\lambda\pi^2} & 0 & 0 & 0 \\ 0 & \frac{9\mathrm{C}_{pot}}{10\lambda\pi^2} + 4\mathrm{C}_0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{\frac{3}{2}}\mathrm{Cpot}}{5\lambda\pi^2} & 0 & \frac{11\mathrm{C}_{pot}}{14\lambda\pi^2} + 6\mathrm{C}_0 & \frac{\sqrt{3}\mathrm{C}_{pot}}{5\lambda\pi^2} & 0 & \frac{\mathrm{C}_{pot}}{5\lambda\sqrt{2}\pi^2} \\ 0 & 0 & \frac{\sqrt{3}\mathrm{C}_{pot}}{5\lambda\pi^2} & \frac{\mathrm{C}_{pot}}{2\lambda\pi^2} + 4\mathrm{C}_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{9\mathrm{C}_{pot}}{10\lambda\pi^2} + 6\mathrm{C}_0 & 0 \\ 0 & 0 & \frac{\mathrm{C}_{pot}R^2}{5\sqrt{2}\pi^2} & 0 & 0 & \frac{\mathrm{C}_{pot}}{2\lambda\pi^2} + 6\mathrm{C}_0 \end{pmatrix} \tag{5}$$

where $\lambda = \frac{m\omega}{\hbar}$ and $\omega = \sqrt{\frac{k}{m}}$. In the vanishing "area interaction" coupling constant limit $\frac{\mathrm{C}_{pot}}{C_0} \propto \frac{v_b}{\omega} \to 0$ we recover the usual harmonic oscillator spectrum together with its characteristic degeneracy, see Fig. 1top. As we increase the ratio of the "area interaction" coupling constant and the harmonic one, first to unity, $\frac{\mathrm{C}_{pot}}{C_0} \propto \frac{v_b}{v_a} \to 1$ Fig. 1middle, and then to seven $\frac{\mathrm{C}_{pot}}{C_0} \propto \frac{v_b}{v_a} \to 7$, Fig. 1bottom, one can see that the states are shifted, at first a little, and then much more into a more-or-less smooth distribution of states, with no degeneracies, or manifest accumulation points.

Another interesting limit is $\frac{v_b}{v_a} \to \infty$, i.e., $v_a \to 0$, when this Hamiltonian does not confine all three-body configurations: the collinear classical motion is free in this potential. What this means in the quantized case is not yet clear: naively one might expect to see (at least) one continuum in the spectrum, corresponding to the unconfined ("free") collinear motion.

The lowest-lying such continuum ought to correspond to states with vanishing total angular momentum $L = 0$ and high values of K, as the collinear motion implies: 1) vanishing total angular momentum $L = 0$; 2) one (hyper)-angle in the triangle always being precisely equal to $\Phi = \pi$. The second requirement leads to the vanishing of the (hyper)-angular uncertainty $\Delta\Phi = 0$, which, in turn demands, an infinite uncertainty in the corresponding (hyper)angular momentum $\Delta K \to \infty$. That can be fulfilled only by states with vanishing total and very large/infinite/ values of the hyper-angular momentum K. In other words, one might expect the (binding) energy of some high hyper-angular momentum K states to decrease and ultimately to vanish in the infinite angular momentum limit. If there are sufficiently many such states, they may form something that resembles a continuum.

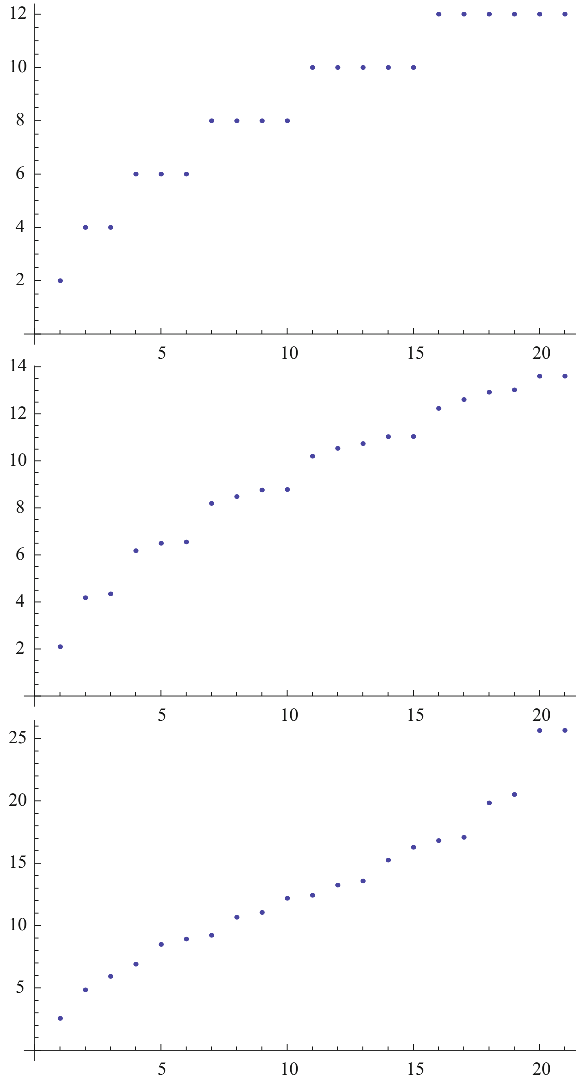

Fig. 1 The spectrum of the first six even-K bands (K = 0, 2, 4, 6, 8, 10) of the three-body harmonic oscillator perturbed by the area-dependent three-body potential with coupling constant $\frac{C_{pot}}{C_0}$ equal to 0, 1 and 7. This is a 21-dimensional sub-space of the full Hilbert space consisting of states satisfying the conditions $m_1 = m_2 = 0$, or $L = 0 = G_3$, equivalent to $[SU(6), L^P] = [20, 0^+]$ in the spectroscopy notation. Note the rearrangement of the levels until the K-shells become practically indiscernible

In order to check this limit numerically we increase the "area interaction" coupling constant ratio $\frac{v_b}{v_0}$ to e.g. $\frac{C_{pot}}{C_0} (\propto \frac{v_b}{v_a}) \to 10$, 100, and 1000, and show the results in Fig. 2.

There one sees a spectrum consisting of discrete, positive energy eigen-values. Of course, one cannot expect to find a "true" continuum with a finite number of states N, but one might see some hints thereof, if the number of states N and the off-diagonal matrix elements are large enough: our results shown in Fig. 2 do not give even a hint of such a continuum at $N = 19$ and $\frac{C_{pot}}{C_0} = 1000$.

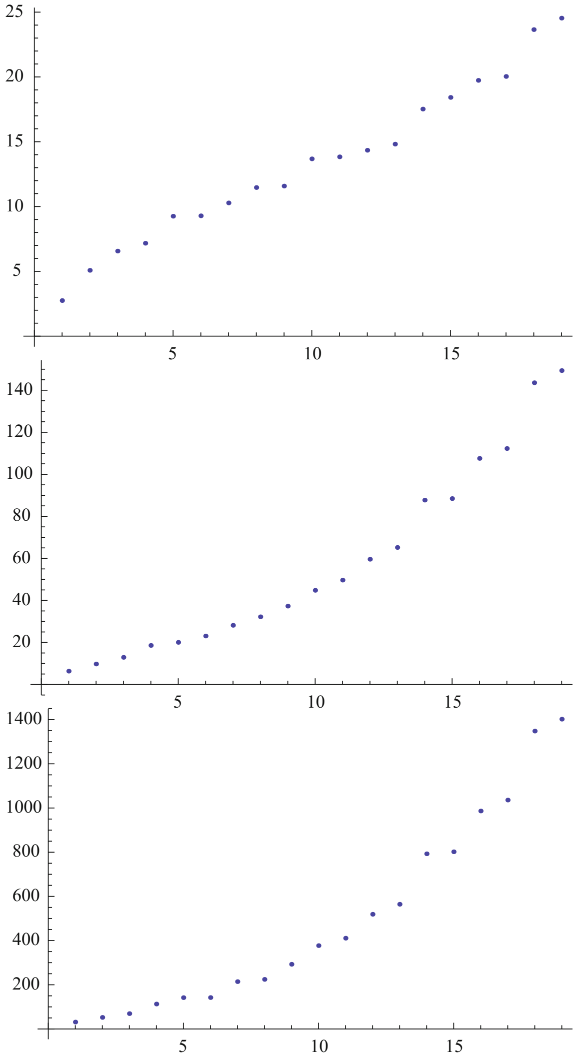

Fig. 2 The spectrum of the first six even-K bands (K = 2, 4, 6, 8, 10, 12) of the three-body harmonic oscillator perturbed by the area-dependent three-body potential with coupling constant $\frac{C_{pot}}{C_0}$ equal to 10, 100 and 1000. This is a 21-dimensional sub-space of the full Hilbert space consisting of states satisfying the conditions $m_1 = m_2 = 0$, or $L = 0 = G_3$, equivalent to $[SU(6), L^P] = [20, 0^+]$ in the spectroscopy notation. We show only $N = 19$ levels, as the last two seem to be adversely affected by the boundary. Note that the pattern of the levels is essentially unchanged, only the scale on the ordinate is different

4 Summary

In this paper we have reported our calculation of three-quark energy spectrum in a three-body potential that depends only on the area of the triangle subtended by the three quarks. The spectrum shows no signs of deconfinement in spite of classically allowed unbound one-dimensional motion.

Acknowledgements This work was financed by the Serbian Ministry of Science and Technological Development under grant numbers OI 171031, OI 171037 and III 41011.

Appendix

Equation (1) can be re-written as a function of (the absolute value of) only one O(3) (hyper-)spherical harmonic in the shape (hyper-)space: the $|Y_{10}(\alpha, \phi)|$:

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = |\cos\alpha| = \sqrt{\frac{4\pi}{3}}|Y_{10}(\alpha, \phi)|. \tag{6}$$

Now, the absolute value of $|Y_{10}(\alpha, \phi)|$ can be expressed as $\sqrt{Y^*_{10}(\alpha, \phi)Y_{10}(\alpha, \phi)}$ and the O(3) Clebsch–Gordan expansion can be applied to $Y^*_{10}(\alpha, \phi)Y_{10}(\alpha, \phi)$, which contains only the (obviously even) values of $L = 0, 2$, as in Eq. (A12) of Ref. [3].

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = \sqrt{\frac{1}{3}}\sqrt{1 + \frac{2}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)}}. \tag{7}$$

The square root can be expanded in a Taylor-like series, the first two terms of which coincide with the expansion in Legendre polynomials, or O(3) spherical harmonics, and for $L = 0$, even in O(4) hyper-spherical harmonics

$$\frac{2}{R^2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| = \sqrt{\frac{1}{3}}\left(1 + \frac{1}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)} + \cdots\right). \tag{8}$$

Manifestly the Legendre polynomial expansion, Eq. (8) is limited to even-order $J = 0, 2, 4, \ldots$ terms only,

$$\begin{aligned} V_{\mathrm{HY}}(R, \alpha, \phi) &= \frac{k}{2}\left(v_a(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2) + v_b|\boldsymbol{\rho} \times \boldsymbol{\lambda}|\right). \qquad (9) \\ &= \frac{k}{2}R^2\left(v_a + v_b\frac{1}{2}\sqrt{\frac{1}{3}}\left(1 + \frac{1}{\sqrt{5}}\frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)} + \cdots\right)\right) \\ &= \frac{k}{2}R^2\frac{v_0^{HY}}{\sqrt{4\pi}}\left(1 + \frac{v_2^{HY}}{v_0^{HY}}\sqrt{4\pi}Y_{20}(\alpha, \phi) + \cdots\right). \qquad (10) \end{aligned}$$

Note, however, that $v_b/v_a \neq v_2^{HY}/v_0^{HY}$. In particular the additive constant in the expansion Eq. (8) is important, as it ensures the (overall) positivity of this potential and leads to the change of "effective couplings"

$$v_{00}^{HY} = \sqrt{4\pi}\left(v_a + v_b\frac{1}{2}\sqrt{\frac{1}{3}}\right),$$

and

$$v_2^{HY} = v_b \frac{1}{2}\sqrt{\frac{4\pi}{15}}.$$

These two equations in turn lead to

$$\frac{v_{20}^{HY}}{v_{00}^{HY}} = \frac{v_b \frac{1}{2}\sqrt{\frac{4\pi}{15}}}{\sqrt{4\pi}\left(v_a + v_b \frac{1}{2}\sqrt{\frac{1}{3}}\right)} = \frac{v_b}{2\sqrt{15}\left(v_a + v_b \frac{1}{2}\sqrt{\frac{1}{3}}\right)},$$

and in particular in the $v_a \to 0$ limit, this ratio for the HY potential equals that of the pure area potential:

$$\lim_{v_a \to 0}\left(\frac{v_{20}^{HY}}{v_{00}^{HY}}\right) = \frac{1}{\sqrt{5}}.$$

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Quasi-classical limit of the open Jordanian XXX spin chain*

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Abstract

We study the open deformed XXX spin chain. In particular we obtain the explicit expression of the Sklyanin monodromy matrix in terms of the entries of the local Lax operator of the Jordanian chain. These results are essential in the study of the so-called quasi-classical limit of the system.

1. Introduction

A particularly interesting feature of quantum groups is the so-called twist transformation [1]. It yields new quantum groups form already known ones. More precisely, a twist of a quantum group, or more generally, of a Hopf algebra $\mathcal{A}$ is a similarity transformation of the co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by an invertible twist element

$$\Delta(a) \mapsto \Delta_t(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \qquad \forall a \in \mathcal{A}. \tag{1}$$

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In order to guarantee the co-associativity property of the twisted co-product the element $\mathcal{F}$ must satisfy the so-called twist equation [1]

$$\mathcal{F}_{12}\left(\Delta \otimes \mathrm{id}\right)(\mathcal{F}) = \mathcal{F}_{23}\left(\mathrm{id} \otimes \Delta\right)(\mathcal{F}), \tag{2}$$

where

$$(\Delta \otimes \mathrm{id})\,\mathcal{F} = (\Delta \otimes \mathrm{id}) \sum_j f_j^{(1)} \otimes f_j^{(2)} = \sum_j \Delta\left(f_j^{(1)}\right) \otimes f_j^{(2)} \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}. \tag{3}$$

Although the twist transformation generates an equivalence relation between quantum groups they produce different R-matrices. Namely, the transformation law of the co-product also determines how the corresponding universal R-matrix changes,

$$\mathcal{R} \mapsto \mathcal{R}^{(t)} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \quad \text{here} \quad \mathcal{F}_{21} = \sum_j f_j^{(2)} \otimes f_j^{(1)}. \tag{4}$$

This new R-matrix allows building and studying new integrable models [2, 3].

2. Deformed Yang R-matrix and the corresponding K-matrix

As our initial step, we briefly review the Jordanian twist element, as a particular solution of the twist equation. We consider $s\ell(2)$ generators S^α with $\alpha = +, -, 3$, with the commutation relations

$$[S^3, S^{\pm}] = \pm S^{\pm}, \quad [S^+, S^-] = 2S^3, \tag{5}$$

and Casimir operator

$$c_2 = (S^3)^2 + \frac{1}{2}(S^+S^- + S^-S^+) = (S^3)^2 + S^3 + S^-S^+ = \vec{S} \cdot \vec{S}. \tag{6}$$

The universal enveloping algebra $U(s\ell(2))$ admits the Jordanian twist element [4, 5]

$$\mathcal{F} = \exp 2\left(S^3 \otimes \sigma\right) \in U\left(s\ell(2)\right) \otimes U\left(s\ell(2)\right), \tag{7}$$

where

$$\sigma = \frac{1}{2}\log\left(1 + 2\theta S^+\right). \tag{8}$$

It straightforward to check that the Jordanian twist element satisfies the following equations [6]

$$(\Delta \otimes \mathrm{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}, \qquad (\mathrm{id} \otimes \Delta_t)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}. \tag{9}$$

In the equations above the co-product Δ is the usual co-product of the $U(s\ell(2))$ and Δ_t is the twisted co-product. Evidently the equations (9) imply the twist equation (2) [6]. Thus the the Jordanian twist element (7) satisfies the Drinfeld twist equation (2).

The XXX Heisenberg spin chain is related to the Yangian $\mathcal{Y}(s\ell(2))$ and the $SL(2)$-invariant Yang R-matrix [7]

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{10}$$

where λ is a spectral parameter, η is a quasi-classical parameter. We use $\mathbb{1}$ for the identity operator and $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The universal enveloping algebra of $s\ell(2)$ is a Hopf sub-algebra of the Yangian, $U(s\ell(2)) \subset \mathcal{Y}(s\ell(2))$. Notice that the matrix form of $\mathcal{F}$ in the spin-1/2 representation $\rho_{1/2}$ is $F_{12} \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$,

$$F_{12} = \left(\rho_{1/2} \otimes \rho_{1/2}\right)(\mathcal{F}) = \mathbb{1} + 2\theta S^3 \otimes S^+ = \begin{pmatrix} 1 & \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

in particular, in the spin-1/2 representation the generators S^α are given by the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

Therefore the transformation of the Yang R-matrix by the Jordanina twist element yields the R-matrix of the twisted Yangian $\mathcal{Y}_\theta(s\ell(2))$ [8, 9, 6]

$$R^J(\lambda) = F_{21} R_{12}(\lambda) F_{12}^{-1} = \begin{pmatrix} \lambda+\eta & -\lambda\theta & \lambda\theta & \lambda\theta^2 \\ 0 & \lambda & \eta & -\lambda\theta \\ 0 & \eta & \lambda & \lambda\theta \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{12}$$

where $F_{21} = \mathcal{P} F_{12} \mathcal{P}$. In what follows, we will use only twisted R-matrix (12) and in order to simplify the notation we will omit the symbol J in the superscript.

The R-matrix (12) satisfies the Yang-Baxter equation

$$R_{12}(\lambda-\mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda-\mu). \tag{13}$$

By setting $\theta = -\xi\eta$ we can guarantee the quasi-classical property

$$\frac{1}{\lambda+\eta} R(\lambda, \eta, \theta)|_{\theta=-\xi\eta} = \mathbb{1} + \eta r(\lambda) + O(\eta^2), \tag{14}$$

where $r(\lambda)$ is the classical r-matrix

$$r(\lambda) = \begin{pmatrix} 0 & \xi & -\xi & 0 \\ 0 & -\frac{1}{\lambda} & \frac{1}{\lambda} & \xi \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda} & -\xi \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

which satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] = 0 \tag{16}$$

and has the unitarity property $r_{21}(-\lambda) = -r_{12}(\lambda)$.

The R-matrix (12) has the so-called regularity property

$$R(0,\eta) = \eta\mathcal{P}, \tag{17}$$

and the unitarity property

$$R_{12}(\lambda)R_{21}(-\lambda) = g(\lambda)\mathbb{1}, \quad \text{with} \quad g(\lambda) = \eta^2 - \lambda^2. \tag{18}$$

The PT symmetry is broken

$$R_{21}(\lambda) \neq R_{12}^{t_1 t_2}(\lambda), \tag{19}$$

where the indices t_1 and t_2 denote the respective transpositions in the first and second space of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. The R-matrix does not have the crossing symmetry, but it satisfies the weaker condition

$$\left(\left(\left(R_{12}^{t_2}(\lambda)\right)^{-1}\right)^{t_2}\right)^{-1} = \frac{g(\lambda+\eta)}{g(\lambda+2\eta)}\, M_2 R_{12}(\lambda+2\eta) M_2^{-1}, \tag{20}$$

with

$$M = \begin{pmatrix} 1 & -2\theta \\ 0 & 1 \end{pmatrix}.$$

In [9] it was shown that the general solution to the reflection equation

$$R_{12}(\lambda-\mu)K_1^-(\lambda)R_{21}(\lambda+\mu)K_2^-(\mu) = K_2^-(\mu)R_{12}(\lambda+\mu)K_1^-(\lambda)R_{21}(\lambda-\mu) \tag{21}$$

is given by

$$K^-(\lambda) = \begin{pmatrix} \zeta + \lambda - \frac{\phi\theta}{\eta}\lambda^2 & \psi\lambda \\ \phi\lambda & \zeta - \lambda - \frac{\phi\theta}{\eta}\lambda^2 \end{pmatrix}. \tag{22}$$

Also, the dual reflection equation was derived and it was shown that its the general solution is given by [9]

$$K^+(\lambda) = K^-(-\lambda-\eta)M. \tag{23}$$

Final observation is that by setting $\theta = -\xi\eta$ we achieve that the matrix $K^-(\lambda)$ (22) does not depend on the parameter η i.e.,

$$\frac{\partial K^-(\lambda)}{\partial \eta} = 0. \tag{24}$$

This is an important step in the so-called quasi-classical limit of the corresponding chain [10, 11, 12, 13].

Using the results obtained above, in the next section, we will study the open deformed XXX spin chain following Sklyanin's approach [14] as we have successfully done in the case of XXX Heisenberg spin chain [10] and XXZ Heisenberg spin chain [12].

3. Jordanian deformation of the XXX spin chain

The Hilbert space of the system is

$$\mathcal{H} = \overset{N}{\underset{m=1}{\otimes}} V_m = (\mathbb{C}^{2s+1})^{\otimes N}, \tag{25}$$

we study the deformed inhomogeneous spin chain with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$, corresponding inhomogeneous parameter α_m and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{26}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

We introduce the Lax operator

$$\begin{aligned} \mathbb{L}_{0m}(\lambda) = & \begin{pmatrix} e^{-\sigma_m} & 2\theta S_m^3 e^{\sigma_m} \\ 0 & e^{\sigma_m} \end{pmatrix} \\ & + \frac{\eta}{\lambda} \begin{pmatrix} S_m^3 \left(\mathbb{1}_m + 2\theta S_m^+\right) e^{-\sigma_m} & \left(S_m^- - 2\theta (S_m^3)^2\right) e^{\sigma_m} \\ S_m^+ e^{-\sigma_m} & -S_m^3 e^{\sigma_m} \end{pmatrix}. \end{aligned} \tag{27}$$

In the case when the quantum space is a spin $\frac{1}{2}$ representation, the Lax operator is equal to the R-matrix, $\mathbb{L}_{0m}(\lambda) = \frac{1}{\lambda} R_{0m}(\lambda - \eta/2)$.

Due to the commutation relations (5), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m) = \mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu). \tag{28}$$

The so-called monodromy matrix

$$T_0(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N) \cdots \mathbb{L}_{01}(\lambda - \alpha_1) \tag{29}$$

is used to describe the system. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{30}$$

From RLL-relations (28) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda-\mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda-\mu). \tag{31}$$

The RTT-relations define the commutation relations for the entries of the monodromy matrix.

Also, we define the Lax operator

$$\begin{aligned}\widetilde{\mathbb{L}}_{0m}(\lambda) = &\begin{pmatrix} e^{\sigma_m} & -2\theta e^{\sigma_m} S_m^3 \\ 0 & e^{-\sigma_m} \end{pmatrix} \\ &+ \frac{\eta}{\lambda}\begin{pmatrix} e^{\sigma_m} S_m^3 & e^{\sigma_m}\left(S_m^- - 2\theta (S_m^3)^2\right) \\ e^{-\sigma_m} S_m^+ & -e^{-\sigma_m}\left(\mathbb{1}_m + 2\theta S_m^+\right) S_m^3 \end{pmatrix}.\end{aligned} \tag{32}$$

It obeys the following important identity

$$\mathbb{L}_{0m}(\lambda)\widetilde{\mathbb{L}}_{0m}(\eta-\lambda) = \left(1 + \eta^2 \frac{s_m(s_m+1)}{\lambda(\eta-\lambda)}\right)\mathbb{1}_0\,, \tag{33}$$

where s_m is the value of spin in the space V_m. Thus the monodromy matrix

$$\widetilde{T}_0(\lambda) = \begin{pmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{pmatrix} = \widetilde{\mathbb{L}}_{01}(\lambda+\alpha_1+\eta)\cdots\widetilde{\mathbb{L}}_{0N}(\lambda+\alpha_N+\eta), \tag{34}$$

obeys the following relations

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda+\mu)T_0(\lambda) = T_0(\lambda)R_{00'}(\lambda+\mu)\widetilde{T}_{0'}(\mu), \tag{35}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu-\lambda) = R_{00'}(\mu-\lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda). \tag{36}$$

By construction it follows that the entries of the Sklyanin monodromy matrix

$$\mathcal{T}_0(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} = T_0(\lambda)K_0^-(\lambda)\widetilde{T}_0(\lambda), \tag{37}$$

obey the exchange relations of the so-called reflection equation algebra [14, 10, 12]

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu) = \mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu). \tag{38}$$

4. Conclusions

The formulae (37), together with (27) and (32), yields, along the lines previously used successfully in the cases of the XXX and XXZ Heisenberg spin chains [10, 12], the quasi-classical expansion of the Sklyanin monodromy of the deformed chain. We believe that these results will help complete the study of the open deformed Gaudin model which we have initiated in [6]. Notice that the open trigonometric Gaudin was reviewed in [15]. The algebraic Bethe ansatz for the periodic deformed Gaudin model was done in [16, 17]. It is very likely that the implementation of the algebraic Bethe ansatz for the open deformed Gaudin model would require specific set of generators of the corresponding generalized Gaudin algebra, as in the $s\ell(2)$ case [18]. These considerations will be reported elsewhere.

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Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, IR)$ and Character Formulae for $n = 3$

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Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, I\!R)$ and Character Formulae for $n = 3$

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Abstract. We overview our study of the positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, I\!R)$. We give more explicitly character formulae for these representations in the case $n = 3$.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. Until recently only those for $D \leqslant 6$ were studied since in these cases the relevant superconformal algebras satisfy [1] the Haag-Lopuszanski-Sohnius theorem [2]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [3] (for $N = 1$), [4, 5, 6, 7] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [8] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [9].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras $osp(1|2n)$. Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [10]. Later it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [11]. In all cases we need to find first the UIRs of $osp(1|2n, I\!R)$ which study was started in [12] and [13]. Later, in [14] we finalized the UIR classification of [12] as Dobrev-Zhang-Salom (DZS) Theorem. There we also proved the DZS Theorem for $osp(1|6)$, while the case $osp(1|8)$ was proved in [15].

In the present paper we present more explicitly the character formulae for $osp(1|6)$. For the lack of space we refer for extensive literature on the subject in [12, 14].

2. Preliminaries on representations

Our basic references for Lie superalgebras are [16, 17], although in this exposition we follow [12].

The even subalgebra of $\mathcal{G} = osp(1|2n, I\!R)$ is the algebra $sp(2n, I\!R)$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, a_1\, , ..., a_{n-1}\,] \tag{1}$$

where d is the conformal weight, and $a_1, ..., a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

We present the classification of the positive energy (lowest weight) UIRs of $\mathcal{G}$ following [12, 14] where were used the methods used for the $D = 4, 6$ conformal superalgebras, cf. [4, 5, 6, 7, 9]. The main tool is an adaptation of the Shapovalov form [18] on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} = osp(1|2n)$ of $\mathcal{G}$.

The root system of $\mathcal{G}^{\mathbb{C}}$ are given in terms of $\delta_1 \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, ..., n$. The even and odd roots systems are [16]:

$$\begin{aligned} \Delta_{\bar{0}} &= \{\pm\delta_i \pm \delta_j\ ,\ 1 \leqslant i < j \leqslant n\ ,\quad \pm 2\delta_i\ ,\ 1 \leqslant i \leqslant n\}\ , \\ \Delta_{\bar{1}} &= \{\pm\delta_i\ ,\ 1 \leqslant i \leqslant n\} \end{aligned} \tag{2}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [16]:

$$\Pi = \{\, \delta_1 - \delta_2\, ,\, , \ldots, \delta_{n-1} - \delta_n\, ,\, \delta_n\, \}\ , \tag{3}$$

or introducing standard notation for the simple roots:

$$\begin{aligned} \Pi &= \{ \alpha_1 , ..., \alpha_n \} , \\ &\alpha_j \ = \ \delta_j - \delta_{j+1} \ , \quad j = 1, ..., n-1 \ , \quad \alpha_n \ = \ \delta_n \ . \end{aligned} \tag{4}$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is:

$$\underset{1}{\circ} \text{——} \cdots \text{——} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet} \tag{5}$$

The black dot is used to signify that the simple odd root is not nilpotent. In fact, the superalgebras $B(0,n) = osp(1|\,2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [16].

The corresponding to Π positive root system is:

$$\Delta^+_{\bar 0} \ = \{\delta_i \pm \delta_j \, , \ 1 \leqslant i < j \leqslant n, \ 2\delta_i \, , \ 1 \leqslant i \leqslant n\}, \qquad \Delta^+_{\bar 1} \ = \{\delta_i \, , \ 1 \leqslant i \leqslant n\} \tag{6}$$

Conversely, we give the elementary functionals through the simple roots:

$$\delta_k \ = \ \alpha_k + \cdots + \alpha_n \ . \tag{7}$$

From the point of view of representation theory more relevant is the restricted root system, such that:

$$\begin{aligned} \bar\Delta^+ \ &= \ \bar\Delta^+_{\bar 0} \cup \Delta^+_{\bar 1} \ , \\ \bar\Delta^+_{\bar 0} \ &\equiv \ \{\alpha \in \Delta^+_{\bar 0} \ | \ \tfrac{1}{2}\alpha \notin \Delta^+_{\bar 1}\} \ = \ \{\delta_i \pm \delta_j \ , \ 1 \leqslant i < j \leqslant n\} \end{aligned} \tag{8}$$

The superalgebra $\mathcal{G} = osp(1|\,2n, I\!R)$ is a split real form of $osp(1|\,2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram:

$$\underset{1}{\circ} \text{——} \cdots \text{——} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ} \tag{9}$$

and root system:

$$\Delta^+_{\mathrm{B_n}} \ = \ \{\delta_i \pm \delta_j \ , \ 1 \leqslant i < j \leqslant n \ , \quad \delta_i \ , \ 1 \leqslant i \leqslant n\} \ \cong \ \bar\Delta^+ \tag{10}$$

This shall be used essentially below.

We need explicitly the lowest weight $\Lambda \in \mathcal{H}^*$ (where $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$) the values of which should be related to the signature (1):

$$(\Lambda, \alpha_k^\vee) = -a_k , \quad 1 \leqslant k \leqslant n , \tag{11}$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and the minus signs anticipate the fact that we shall use lowest weight Verma modules (instead of the highest weight modules used in [17]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [6, 12]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

$$a_n = -2d - a_1 - \cdots - a_{n-1} . \tag{12}$$

The actual Dynkin labelling is given by:

$$m_k = (\rho - \Lambda, \alpha_k^\vee) \tag{13}$$

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (6):

$$\begin{aligned} \rho &\doteq \rho_{\bar{0}} - \rho_{\bar{1}} = (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n , \quad (14) \\ \rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n , \\ \rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n) . \end{aligned}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho, \alpha_i^\vee) = 1, i = 1, ..., n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ $(\forall k)$ the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$.

To introduce Verma modules we use the standard decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{15}$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned} Z\, v_0 &= 0 , \quad Z \in \mathcal{G}^- \\ H\, v_0 &= \Lambda(H)\, v_0 , \quad H \in \mathcal{H} . \end{aligned} \tag{16}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\,v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [17] to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [12]:

$$(\rho - \Lambda, \beta^\vee) = m_\beta \ , \qquad \beta \in \Delta^+ \ , \quad m_\beta \in \mathbb{N} \ . \tag{17}$$

If a condition from (17) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m, β : $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the **singular vector** $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions [19]:

$$\begin{aligned} X\ v_s^{m,\beta} &= 0\ , \quad X \in \mathcal{G}^-\ , \\ H\ v_s^{m,\beta} &= \Lambda'(H)\ v_0\ , \quad H \in \mathcal{H}\ , \quad \Lambda' \ = \ \Lambda + m\beta\ . \end{aligned} \tag{18}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [20, 21]:

$$v_s^{m,\beta} \ = \ P^{m,\beta}\, v_0 \ , \quad P^{m,\beta} \in U(\mathcal{G}^+) \ . \tag{19}$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)\, P^{m,\beta}\, v_0$.

Certainly, (17) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (17) is fulfilled, and let us denote: $\tilde{I}^\Lambda \equiv \cup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [17, 22]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$.

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if rank $\mathcal{G} \leqslant 2$, or when (17) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition: [22, 23, 24] *Let* V^Λ *be a reducible Verma module. A vector* $v_{\rm ssv} \in V^\Lambda$ *is called a* **subsingular vector** *if* $v_{\rm su} \notin \tilde{I}^\Lambda$ *and the following holds:*

$$X\ v_{\rm su} \ \in \ \tilde{I}^\Lambda\ , \quad \forall X \in \mathcal{G}^- \tag{20}$$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [12] it was established that from (17) follows that the Verma module $V^{\Lambda(\chi)}$ is

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reducible if one of the following relations holds:

$$\mathbb{N} \ni m^-_{ij} = j - i + a_i + \cdots + a_{j-1} \tag{21a}$$

$$\mathbb{N} \ni m^+_{ij} = 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d \tag{21b}$$

$$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d \tag{21c}$$

$$\mathbb{N} \ni m_{ii} = n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d \tag{21d}$$

corresponding to the roots $\delta_i - \delta_j$, $\delta_i + \delta_j$, $(i < j)$, δ_i, $2\delta_i$, resp. Further we shall use the fact from [12] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (21d) is fulfilled also (21c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j.

Further, we notice that all reducibility conditions in (21a) are fulfilled. In particular, for the simple roots from those condition (21a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, ..., n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)\, v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}\, v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ we shall consider also the factor-modules:

$$F^\Lambda_c \; = \; V^\Lambda \,/\, I^\Lambda_c \; \supset \; F^\Lambda \; . \tag{22}$$

We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c :

$$(X^+_i)^{1+a_i}\, |\Lambda_c\rangle \; = \; 0 \; , \quad i = 1, ..., n-1. \tag{23}$$

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (21b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (21b) are obtained for $m^+_{ij} = 1$ and they are:

$$d_{ij} \; \equiv \; n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j) \; , \quad i < j, \tag{24}$$

the corresponding root being $\delta_i + \delta_j$.

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The maximal d coming from the different possibilities in (21c) are obtained for $m_i = 1$ and they are:

$$d_i \equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) \ , \tag{25}$$

the corresponding roots being δ_i .

There are some orderings between these maximal reduction points [12]:

$$\begin{aligned} d_1 &> d_2 > \cdots > d_n \ , \\ d_{i,i+1} &> d_{i,i+2} > \cdots > d_{in} \ , \\ d_{1,j} &> d_{2,j} > \cdots > d_{j-1,j} \ , \\ d_i &> d_{jk} > d_\ell \ , \qquad i \leqslant j < k \leqslant \ell \ . \end{aligned} \tag{26}$$

Obviously the first reduction point is:

$$d_1 = n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \ . \tag{27}$$

Below we shall use the following notion. The singular vector v_1 is called **descendant** of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 = P_{12}\, v_2$. Clearly, in this case we have: $I^1 \subset I^2$, where I^k is the submodule generated by v_k . Thus, when we factor the submodule I_2 this means factoring also the submodule I_1.

3. Unitarity

The first results on the unitarity were given in [12], and then improved in [14]. Thus, the statement below should be called *Dobrev-Zhang-Salom Theorem*:

Theorem DZS: All positive energy unitary irreducible representations of the superalgebras $osp(1|\,2n, I\!R)$ characterized by the signature χ in (1) are obtained for real d and are given as follows:

$$\begin{aligned}
d &\geqslant n-1+\tfrac{1}{2}(a_1+\cdots+a_{n-1}) \ = \ d_1 \ , \quad a_1\neq 0 \ , && (28)\\
d &\geqslant n-\tfrac{3}{2}+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) \ = \ d_{12} \ , \quad a_1=0,\ a_2\neq 0 \ ,\\
d &= n-2+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) \ = \ d_2 > d_{13} \ , \quad a_1=0,\ a_2\neq 0 \ ,\\
d &\geqslant n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) \ = \ d_2 \ = \ d_{13} \ , \quad a_1=a_2=0,\ a_3\neq 0 \ ,\\
d &= n-\tfrac{5}{2}+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) \ = \ d_{23} \ > \ d_{14} \ , \quad a_1=a_2=0,\ a_3\neq 0 \ ,\\
d &= n-3+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) \ = \ d_3 \ = \ d_{24} \ > \ d_{15} \ , \quad a_1=a_2=0,\ a_3\neq 0 \ , && (29)\\
&...\\
d &\geqslant n-1-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) \ , \quad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 \ ,\\
&\qquad \kappa=\tfrac{1}{2},1,...,\tfrac{1}{2}(n-1) \ ,\\
d &= n-\tfrac{3}{2}-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) \ , \quad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 \ ,\\
&...\\
d &= n-1-2\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) \ , \quad a_1=...=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 \ ,\\
&...\\
d &\geqslant \tfrac{1}{2}(n-1) \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0\\
d &= \tfrac{1}{2}(n-2) \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0\\
&...\\
d &= \tfrac{1}{2} \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0\\
d &= 0 \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0
\end{aligned}$$

Parts of the *Proof* were given in [12], while in [14] was given a detailed sketch of the Proof. In [14] was given also the Proof for the case $n=3$, while the proof for $n=4$ was given in [15].

4. Character formulae

Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda\in\hat{\mathcal{H}}^*$ such that $(\lambda,\alpha_i^\vee)\in\mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, $(\alpha_i^\vee\equiv 2\alpha_i/(\alpha_i,\alpha_i))$. Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$\begin{aligned}
V &= \mathop{\oplus}_{\mu\in\Gamma_+} V_\mu \ , && (30)\\
V_\mu &= \{u\in V \mid Hu=(\Lambda+\mu)(H)u,\ \forall\ H\in\mathcal{H}\}
\end{aligned}$$

(Note that $V_0 = \mathbb{C}v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C},\ c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geqslant \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1,\ e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of V is defined by:

$$\begin{aligned} ch_0\ V &= \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\Lambda+\mu)\ = \\ &= e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\mu) \end{aligned} \tag{31}$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has $\dim\ V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim \mathcal{G}_\beta$ ($= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$\begin{aligned} ch_0\ V^\Lambda &= e(\Lambda) \sum_{\mu\in\Gamma_+} P(\mu)e(\mu)\ = \\ &= e(\Lambda) \prod_{\alpha\in\Delta^+} (1-e(\alpha))^{-1} \end{aligned} \tag{32}$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda)\ =\ \lambda - (\lambda,\alpha^\vee)\alpha\,, \quad \lambda\in\hat{\mathcal{H}}^*\,, \quad \alpha\in\hat{\Delta}. \tag{33}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch_0\ L_\Lambda\ =\ \sum_{w\in W} (-1)^{\ell(w)}\ ch_0\ V^{w\cdot\Lambda}\,, \quad \Lambda\in-\Gamma_+ \tag{34}$$

where the dot $\cdot$ action is defined by $w\cdot\lambda = w(\lambda-\rho)+\rho$. For future reference we note:

$$s_\alpha\ \cdot\ \Lambda\ =\ \Lambda\ +\ n_\alpha\alpha \tag{35}$$

where

$$n_\alpha \;=\; n_\alpha(\Lambda) \;\doteq\; (\rho - \Lambda, \alpha^\vee) \;=\; (\rho - \Lambda)(H_\alpha)\,, \qquad \alpha \in \Delta^+. \tag{36}$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac. They are more complicated than the bosonic case, except for the algebras we consider. Actually, for $osp(1/2n)$ the Verma module character formula is the same as (32):

$$ch\ V^\Lambda \;=\; e(\Lambda) \prod_{\alpha\in\bar\Delta^+} \frac{1}{1 - e(\alpha)} \tag{37}$$

using the restricted root system $\bar\Delta^+$. Naturally, the character formula for the finite-dimensional irreducible LWM L_Λ is again (34) using the Weyl group W_n of B_n.

Multiplets

A Verma module V^Λ may be reducible w.r.t. to many positive roots, and thus there maybe many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

One main ingredient of the approach of [20] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called *multiplets* [20]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

If a Verma module V^Λ is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., $m_k \in \mathbb{N}$ for all k, then the irreducible submodules are isomorphic to the finite-dimensional irreps of $\mathcal{G}^\mathbb{C}$ [20]. (Actually, this is a condition only for m_n since $m_k \in \mathbb{N}$ for $k = 1, \ldots, n-1$.) In these cases we have the *main multiplets* which are isomorphic to the Weyl group of $\mathcal{G}^\mathbb{C}$ [20].

In the cases of non-dominant weight Λ the character formula for the irreducible LWM is [25] :

$$ch\ L_\Lambda \;=\; \sum_{\substack{w\in W\\ w\leqslant w_\Lambda}} (-1)^{\ell(w_\Lambda w)}\ P_{w,w_\Lambda}(1)\ ch\ V^{w\cdot(w_\Lambda^{-1}\cdot\Lambda)}\,, \quad \Lambda\in\Gamma \tag{38}$$

where $P_{y,w}(u)$ are the Kazhdan–Lusztig polynomials $y, w \in W$ [25] (for an easier exposition see [24]), w_Λ is a unique element of W with minimal length such that the signature of $\Lambda_0 = w_\Lambda^{-1} \cdot \Lambda$ is anti-dominant or semi-anti-dominant:

$$\chi_0 = (m'_1, \ldots, m'_n), \qquad m'_k = 1 - \Lambda_0(H_k) \in \mathbb{Z}_- \ . \tag{39}$$

Note that $P_{y,w}(1) \in \mathbb{N}$ for $y \leqslant w$.

When Λ_0 is semi-anti-dominant, i.e., at least one $m'_k = 0$, then in fact W is replaced by a reduced Weyl group W_R.

Most often the value of $P_{y,w}(1)$ is equal to 1 (as in the character formula for the finite-dimensional irreps), while the cases $P_{y,w}(1) > 1$ are related to the appearance of subsingular vectors, though the situation is more subtle, see [24].

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. In the case of d_{ij} using (24) we have:

$$m_n(d_{ij}) = 1 - 2m_j - \cdots - 2m_{n-1} - m_i - \cdots - m_{j-1} \ . \tag{40}$$

In the case of d_i (25) we have:

$$m_n(d_i) = 1 - 2m_i - \cdots - 2m_{n-1} \ . \tag{41}$$

As expected the weights related to positive energy d are not dominant $(m_n(d_{ij}) \in \mathbb{Z}_-,\ m_n(d_i) \in -\mathbb{N},\ (i < n))$, since the positive energy UIRs are infinite-dimensional. (Naturally, $m_n(d_n) = 1$ falls out of the picture since $d_n < 0$.)

Thus, the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the w_Λ in the character formula (38). This we do in the next subsection in the case $n = 3$.

5. The case of osp(1|6)

For $n = 3$ formula (26) simplifies to:

$$\begin{array}{c} d_1 \ > \ d_{12} \ > \ d_2 \ > \ d_{23} \ > \ d_3 \\ > \ d_{13} \ > \end{array}$$

The Theorem now reads:

$$\begin{aligned}
d &\geqslant 2+\tfrac{1}{2}(a_1+a_2) \ = \ d_1 \ , \quad a_1\neq 0 \ , \\
d &\geqslant \tfrac{3}{2}+\tfrac{1}{2}a_2 \ = \ d_{12} \ , \quad a_1=0,\ a_2\neq 0 \ , \\
d &= 1+\tfrac{1}{2}a_2 \ = \ d_2>d_{13} \ , \quad a_1=0,\ a_2\neq 0 \ , \\
d &\geqslant 1 \ = \ d_2=d_{13} \ , \quad a_1=a_2=0 \ , \\
d &= \tfrac{1}{2} \ = \ d_{23} \ , \quad a_1=a_2=0 \ , \\
d &= 0 \ = \ d_3 \ , \quad a_1=a_2=0 \ .
\end{aligned} \tag{42}$$

The Weyl group W_n of B_n has $2^n n!$ elements, i.e., 48 for B_3. Let $S = (s_1, s_2, s_3)$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

$$s_1^2=s_2^2=s_3^2=e, \quad (s_1s_2)^3=e, \quad (s_2s_3)^4=e, \quad s_1s_3 \ = \ s_3s_1\,, \tag{43}$$

e being the identity of W_3. The 48 elements may be listed as:

$$\begin{aligned}
& e\,,\ s_1\,,\ s_2\,,\ s_3 \\
& s_1s_2\,,\ s_1s_3\,,\ s_2s_1\,,\ s_2s_3\,,\ s_3s_2\,, \\
& s_1s_2s_1\,,\ s_1s_2s_3\,,\ s_1s_3s_2\,,\ s_2s_1s_3\,,\ s_2s_3s_2\,, \\
& \ s_3s_2s_1\,,\ s_3s_2s_3\,, \\
& s_1s_2s_1s_3\,,\ s_1s_2s_3s_2\,,\ s_1s_3s_2s_1\,,\ s_1s_3s_2s_3\,, \\
& s_2s_3s_2s_1\,,\ s_2s_1s_3s_2\,,\ s_3s_2s_3s_1\,,\ s_3s_2s_3s_2\,, \\
& s_1s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3\,,\ s_1s_2s_1s_3s_2\,, \\
& s_1s_3s_2s_3s_2\,, s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_3\,, \\
& s_3s_2s_3s_1s_2\,,\ s_3s_2s_3s_2s_1\,, \\
& s_1s_3s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3s_2\,,\ s_1s_2s_1s_3s_2s_1\,, \\
& s_2s_1s_3s_2s_1s_3\,, s_2s_1s_3s_2s_3s_2\,,\ s_3s_2s_3s_1s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_3\,, s_2s_1s_3s_2s_3s_2s_1\,, \\
& s_2s_1s_3s_2s_3s_1s_2\,, s_3s_2s_1s_2s_3s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_1s_3\,,\ s_3s_2s_3s_1s_2s_3s_2\,, \\
& s_2s_3s_2s_1s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_2s_1\,, \\
& s_3s_2s_1s_3s_2s_3s_1s_2\,, s_2s_3s_2s_1s_3s_2s_3s_2s_1\ .
\end{aligned} \tag{44}$$

The character formula for the Verma modules in our case is given explicitly by:

$$
\begin{aligned}
ch\ V^\Lambda = & \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)} \ \times \\
& \times \frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\
& \times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}
\end{aligned} \tag{45}
$$

where $t_j \equiv e(\alpha_j)$.

Now we give the character formulae of the five boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight Λ_0 which determines the main multiplet by (m'_1, m'_2, m'_3), $m'_k \in \mathbb{N}$, using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some $m'_k = 0$.

- In the case of $d = d_1 = 2 + \frac{1}{2}(a_1 + a_2)$ there are twelve members of the multiplet which is a submiltiplet of a main multiplet. (Remember that that $m_1 > 1$ since $a_1 \neq 0$.) They are grouped into two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_1}}$, where $\Lambda_0^{d_1} = w \cdot \Lambda_0$, $w = w_{\Lambda_0^{d_1}} = s_2s_1s_3s_2s_3$, with signature:

$$
\begin{aligned}
\Lambda_0^{d_1} \quad : \quad & (m_1, m_2, m'_3 = 1 - 2m_{12}) \ , \\
& m_1, m_2 \in \mathbb{N} \ , \quad m_{12} \equiv m_1 + m_2 \quad .
\end{aligned} \tag{46}
$$

The other submultiplet starts from $V^{\Lambda'_0}$ with $\Lambda'_0 = \Lambda_0^{d_1} + \delta_1 = \Lambda_0^{d_1} + \alpha_1 + \alpha_2 + \alpha_3$, with signature: $\Lambda'_0 : (m_1 - 1, m_2, m'_3 = 1 - 2m_{12})$, $m_1 > 1$. The character formula is (38) with $w_\Lambda = w_{\Lambda_0^{d_1}}$:

$$
\begin{aligned}
\mathrm{ch}\, \Lambda_0^{d_1} = & \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \ \times \\
\times & \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \ \times \\
\times & \{ \ \mathrm{ch}\, \Lambda_{m_1,m_2}(t_1,t_2) \ - \ t_1t_2t_3\, \mathrm{ch}\, \Lambda_{m_1-1,m_2}(t_1,t_2) \ \} \ , \ m_1 > 1
\end{aligned} \tag{47}
$$

where $\mathrm{ch}\, \Lambda_{m_1,m_2}(t_1,t_2)$ is the normalized character of the finite-dimensional $sl(3)$ irrep with Dynkin labels (m_1, m_2) (and dimension $m_1m_2(m_1 + m_2)/2$):

$$
\mathrm{ch}\ \Lambda_{m_1,m_2}(t_1,t_2) \ = \ \frac{1 - t_1^{m_1} - t_2^{m_2} + t_1^{m_1}t_2^{m_{12}} + t_1^{m_{12}}t_2^{m_2} - t_1^{m_{12}}t_2^{m_{12}}}{(1-t_1)(1-t_2)(1-t_1t_2)} \tag{48}
$$

Naturally, the latter formula is a polynomial in t_1, t_2, e.g., $\mathrm{ch}\,\Lambda_{1,1}(t_1,t_2) = 1$, $\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2) = 1 + t_1 + t_1t_2$.

In the case $m_1 = 2, m_2 = 1$ the character formula (47) simplifies to:

$$\mathrm{ch}\,\Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ \times \left(1 + \frac{t_1(1+t_2)}{1-t_1t_2t_3} \right), \qquad m_1 = 2, m_2 = 1 \tag{49}$$

• In the case of $d = d_{12} = \frac{1}{2}(3 + a_2)$ which is relevant for unitarity, i.e., $m_1 = 1$, there are again twelve members of the multiplet. Omitting the details [14] the character f-la is:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \tag{50}$$
$$\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$
$$\times \{ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) - (t_1t_2^2t_3^2)^{m_2}\,\mathrm{ch}\,\Lambda_{1,m_2-1}(t_1,t_2) \}, \quad m_2 > 1$$

In the case $m_2 = 2$ it simplifies to:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)} \times \\ \times \{ 1 + t_1t_2^2t_3^2 + \frac{t_2(1+t_1)}{1-t_1t_2^2t_3^2} \} \tag{51}$$

• In the case $d = d_2 = d_{13} = 1$ and $a_1 = a_2 = 0$, $m_1 = m_2 = 1$, the signature is:

$$\Lambda_0^{d_2=d_{13}} \ : \ (1, 1, -1)\ . \tag{52}$$

Again there are twelve members of the multiplet which has two $sl(3)$ submultiplets. First there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}}$ with parameters $(1,1)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}} + \alpha_1 + 2\alpha_2 + 3\alpha_3$ with parameters $(1,1)$. Note that that $\alpha_1 + 2\alpha_2 + 3\alpha_3 = \delta_1 + \delta_2 + \delta_3$ is the weight of a subsingular vector [14], yet the corresponding KL polynomial $P_{y,w}(1)$ is equal to 1. Thus, the character formula is [14]:

$$\mathrm{ch}\,\Lambda_0^{d_2=d_{13}} = \tag{53}$$
$$= \frac{e(\Lambda_0^{d_2=d_{13}})\ (\ 1 - t_1t_2^2t_3^3\ )}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

Note that the above formula may be rewritten as:

$$\mathrm{ch}\,\Lambda_0^{d_2=d_{13}} = \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)} \times \times \left(\frac{1}{1-t_1t_2^2t_3^2} + \frac{t_3}{1-t_3} \right) \tag{54}$$

• In the case of $d = d_2 = 1+\frac{1}{2}a_2 > d_{13} = 1$, i.e., $m_1 = 1$, $m_2 = 1+a_2 > 1$. The multiplet has 24 members for $m_2 > 2$. Omitting the details [14] the character f-la is:

$$\begin{aligned} \mathrm{ch}\,\Lambda_0^{\prime d_2} &= \frac{e(\Lambda_0^{\prime d_2})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\ &\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) - t_2t_3\,\mathrm{ch}\,\Lambda_{2,m_2-1}(t_1,t_2) + \\ &+ t_1t_2^3t_3^3\,\mathrm{ch}\,\Lambda_{2,m_2-2}(t_1,t_2) - t_1^2t_2^4t_3^4\,\mathrm{ch}\,\Lambda_{1,m_2-2}(t_1,t_2) \} \end{aligned} \tag{55}$$

When $m_2 = 2$ $(a_2 = 1)$ the multiplet reduces to only 12 members, and the character formula simplifies to:

$$\begin{aligned} \mathrm{ch}\,\Lambda_0^{\prime d_2} = &\frac{e(\Lambda_0^{\prime d_2})}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \frac{1}{(1-t_3)(1-t_1t_2t_3)} + \frac{t_2}{(1-t_3)(1-t_2t_3)} \\ &\quad + \frac{t_1t_2}{(1-t_2t_3)(1-t_1t_2t_3)} \} \end{aligned} \tag{56}$$

• In the case of $d = d_{23} = \frac{1}{2}$, $a_1 = a_2 = 0$, i.e., $m_1 = m_2 = 1$, again we have a multiplet with 24 members. Omitting the details [14] the character formula is:

$$\begin{aligned}
\mathrm{ch}\,\Lambda_0^{d_{23}} \;&=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \qquad (57)\\
&\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\
&\times \{\; 1 \;-\; t_2t_3^2\,\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2) \;+\; t_1t_2^2t_3^4\,\mathrm{ch}\,\Lambda_{1,2}(t_1,t_2) \;-\; t_1^2t_2^4t_3^6 \;\} \;= \\
&= \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)\;(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\
&\times \{\; 1 \;-\; t_2t_3^2\,(1+t_1+t_1t_2) \;+\; t_1t_2^2t_3^4\,(1+t_2+t_1t_2) \;-\; t_1^2t_2^4t_3^6 \;\}
\end{aligned}$$

Note that the above formula may be rewritten as:

$$\mathrm{ch}\,\Lambda_0^{d_{23}} \;=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \qquad (58)$$

Note that formulae (49),(51),(54),(56),(58) are new w.r.t. [14].

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Permutation-Symmetric Three-Body O(6) Hyperspherical Harmonics in Three Spatial Dimensions

Igor Salom and V. Dmitrašinović

Abstract We have constructed the three-body permutation symmetric O(6) hyperspherical harmonics which can be used to solve the non-relativistic three-body Schrödinger equation in three spatial dimensions. We label the states with eigenvalues of the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras and we present the corresponding $K \leq 4$ harmonics. Concrete transformation properties of the harmonics are discussed in some detail.

1 Introduction

Hyperspherical harmonics are an important tool for dealing with quantum-mechanical three-body problem, being of a particular importance in the context of bound states [1–6]. However, before our recent progress [7], a systematical construction of permutation-symmetric three-body hyperspherical harmonics was, to our knowledge, lacking (with only some particular cases being worked out – e.g. those with total orbital angular momentum $L = 0$, see Refs. [5, 8]).

In this note, we report the construction of permutation-symmetric three-body O(6) hyperspherical harmonics using the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras, where $U(1)$ is the "democracy transformation", or "kinematic rotation" group for three particles, $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. This particular chain of algebras is mathematically very natural, since the $U(1)$ group of "democracy transformations" is the only nontrivial (Lie) subgroup of full hyperspherical $SO(6)$ symmetry (the symmetry of nonrelativistic kinetic energy) that commutes with spatial rotations. Historically, this chain was also suggested in the recent review of the Russian school's work, Ref. [9], and indicated

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by the previous discovery of the dynamical $O(2)$ symmetry of the Y-string potential, Ref. [10]. The name "democracy transformations" comes from the close relation of these transformations with permutations: (cyclic) particle permutations form a discrete subgroup of this $U(1)$ group.

2 Three-Body Hyperspherical Coordinates

A natural set of coordinates for parametrization of three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ (in the center-of-mass frame of reference) is given by the Euclidean relative position Jacobi vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)$. The overall six components of the two vectors can be seen as specifying a position in a six-dimensional configuration space $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$, which, in turn, can be parameterized by hyperspherical coordinates as $\Psi(R, \Omega_5)$. Here $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyper-radius, and five angles Ω_5 parametrize a hyper-sphere in the six-dimensional Euclidean space. Three $(\Phi_i;\ i = 1, 2, 3)$ of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. Due to the connection $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$, this shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere. A spherical coordinate system can be further introduced in this shape space. Among various (in principle infinitely many) ways that this can be accomplished, the one due to Iwai [6] stands out as the one that fully observes the permutation symmetry of the problem. Namely, of the two Iwai (hyper)spherical angles (α, ϕ): $(\sin\alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right)$, the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions.

Nevertheless, in the construction of hyperspherical harmonics, we will, unlike the most of the previous attempts in this context, refrain from use of any explicit set of angles, and express harmonics as functions of Cartesian Jacobi coordinates.

3 O(6) Symmetry of the Hyperspherical Approach

The motivation for hyperspherical approach to the three-body problem comes from the fact that the equal-mass three-body kinetic energy T is O(6) invariant and can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}. \tag{1}$$

Here, $K_{\mu\nu}, (\mu, \nu = 1, 2, \ldots, 6)$ denotes the $SO(6)$ "grand angular" momentum tensor

$$K_{\mu\nu} = m\left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu\right) \\ = \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu\right). \tag{2}$$

$K_{\mu\nu}$ has 15 linearly independent components, that contain, among themselves three components of the "ordinary" orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m\left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}}\right)$.

It is due to this symmetry of the kinetic energy that the decomposition of the wave function and potential energy into $SO(6)$ hyperspherical harmonics becomes a natural way to tackle the three-body quantum problem.

In this particular physical context, the six dimensional hyperspherical harmonics need to have some desirable properties. Quite generally, apart from the hyperangular momentum K, which labels the O(6) irreducible representation, all hyperspherical harmonics must carry additional labels specifying the transformation properties of the harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group. The symmetries of most three-body potentials, including the three-quark confinement ones, are: parity, rotations and permutations (spatial exchange of particles).

Therefore, the goal is to find three-body hyperspherical harmonics with well defined transformation properties with respect to thee symmetries. Parity is directly related to K value: $P = (-1)^{\mathrm{K}}$, the rotation symmetry implies that the hyperspherical harmonics must carry usual quantum numbers L and m corresponding to $SO(3)_{rot} \supset SO(2)$ subgroups and permutation properties turn out to be related with a continuous $U(1)$ subgroup of "democracy transformations", as will be discussed below.

4 Labels od Permutation-Symmetric Three-Body Hyperspherical Harmonics

We introduce the complex coordinates:

$$X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3. \tag{3}$$

Nine of 15 hermitian $SO(6)$ generators $K_{\mu\nu}$ in these new coordinates become

$$iL_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} + X_i^- \frac{\partial}{\partial X_j^-} - X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}, \tag{4}$$

$$2Q_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} - X_i^- \frac{\partial}{\partial X_j^-} + X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}. \tag{5}$$

Here L_{ij} have physical interpretation of components of angular momentum vector $\mathbf{L}$. The symmetric tensor Q_{ij} decomposes as $(5) + (1)$ w.r.t. rotations, while the trace:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+ \frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^- \frac{\partial}{\partial X_i^-} \tag{6}$$

is the only scalar under rotations, among all of the $SO(6)$ generators. Therefore, the only mathematically justified choice is to take eigenvalues of this operator for an additional label of the hyperspherical harmonics. Besides, this trace Q is the generator of the forementioned democracy transformations, a special case of which are the cyclic permutations – which in addition makes this choice particularly convenient on an route to construction of permutation-symmetric hyperspherical harmonics. The remaining five components of the symmetric tensor Q_{ij}, together with three antisymmetric tensors L_{ij} generate the $SU(3)$ Lie algebra, which together with the single scalar Q form an $U(3)$ algebra, Ref. [9].

Overall, labelling of the O(6) hyperspherical harmonics with labels K, Q, L and m corresponds to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$. Yet, these four quantum numbers are in general insufficient to uniquely specify an $SO(6)$ hyperspherical harmonic and an additional quantum number must be introduced to account for the remaining multiplicity. This is the multiplicity that necessarily occurs when $SU(3)$ unitary irreducible representations are labelled w.r.t. the chain $SO(2) \subset SO(3) \subset SU(3)$ (where $SO(3)$ is "matrix embedded" into $SU(3)$), and thus is well documented in the literature. In this context the operator:

$$\mathcal{V}_{LQL} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{7}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk} L_{jk}$ and Q_{ij} is given by Eq. (5)) has often been used to label the multiplicity of $SU(3)$ states. This operator commutes both with the angular momentum L_i, and with the "democracy rotation" generator Q:

$$\left[\mathcal{V}_{LQL}, L_i\right] = 0; \quad \left[\mathcal{V}_{LQL}, Q\right] = 0$$

Therefore we demand that the hyperspherical harmonics be eigenstates of this operator:

$$\mathcal{V}_{LQL} \mathcal{Y}_{L,m}^{KQ\nu} = \nu \mathcal{Y}_{L,m}^{KQ\nu};$$

Thus, ν will be the fifth label of the hyperspherical harmonics, beside the (K, Q, L, m).

5 Tables of Hyperspherical Harmonics of Given K, Q, L, M and ν

Below we explicitly list all hyperspherical harmonics for K ≤ 4, labelled by the quantum numbers (K, Q, L, m, ν) (we will not delve here into lengthy details of the derivation of the expressions). We list only the harmonics with $m = L$ and $Q \geq 0$, as the rest can be easily obtained by acting on them with standard lowering operators and by using the permutation symmetry properties of hyperspherical harmonics: $\mathcal{Y}_{L,m}^{\mathrm{K}Q\nu}(\lambda, \rho) = (-1)^{\mathrm{K}-L}\mathcal{Y}_{L,m}^{\mathrm{K}-Q-\nu}(\lambda, -\rho)$. We use the (more compact) spherical complex coordinates: $X_0^{\pm} \equiv \lambda_3 \pm i\rho_3$, $X_{(\pm)}^{\pm} \equiv \lambda_1 \pm i\rho_1 + (\pm)(\lambda_2 \pm i\rho_2)$, $|X^{\pm}|^2 = X_+^{\pm}X_-^{\pm} + (X_0^{\pm})^2$, while we are also explicitly writing out the K ≤ 3 harmonics in terms of Jacobi coordinates.

$$\mathcal{Y}_{0,0}^{0,0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}_{1,1}^{1,1,-1}(X) = \frac{\sqrt{\frac{3}{2}}X_+^+}{\pi^{3/2}R} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)}{\pi^{3/2}\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}}$$

$$\mathcal{Y}_{1,1}^{2,0,0}(X) = \frac{\sqrt{3}\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{\pi^{3/2}R^2} = \frac{2\sqrt{3}\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,0,0}(X) = \frac{\sqrt{3}X_+^+X_+^-}{\pi^{3/2}R^2} = \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{0,0}^{2,2,0}(X) = \frac{\sqrt{2}\left|X^+\right|^2}{\pi^{3/2}R^2}$$
$$= \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,2,-3}(X) = \frac{\sqrt{\frac{3}{2}}\left(X_+^+\right)^2}{\pi^{3/2}R^2} = \frac{\sqrt{\frac{3}{2}}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{1,1}^{3,1,3}(X) = \frac{\sqrt{6}\left(X_+^-\left|X^+\right|^2 - \frac{1}{2}R^2X_+^+\right)}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{6}\left(\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)\left(\left(\lambda_1 + i\rho_1\right)^2 + \left(\lambda_2 + i\rho_2\right)^2 + \left(\lambda_3 + i\rho_3\right)^2\right)\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)} -$$
$$\frac{\frac{\sqrt{6}}{2}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{3,1,-5}(X) = \frac{\sqrt{5}X_+^+\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{\pi^{3/2}R^3}$$
$$= \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,1,-2}(X) = \frac{\sqrt{15}\left(X_+^+\right)^2 X_+^-}{2\pi^{3/2}R^3}$$
$$= \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{1,1}^{3,3,-1}(X) = \frac{\sqrt{3}X_+^+\left|X^+\right|^2}{\pi^{3/2}R^3}$$
$$= \frac{\sqrt{3}(\lambda_1+i(\lambda_2+\rho_1+i\rho_2))\left(2i\lambda_1\rho_1+2i\lambda_2\rho_2+2i\lambda_3\rho_3+\lambda_1^2+\lambda_2^2+\lambda_3^2-\rho_1^2-\rho_2^2-\rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2+\rho_1^2+\rho_2^2+\rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,3,-6}(X) = \frac{\sqrt{5}\left(X_+^+\right)^3}{2\pi^{3/2}R^3} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{0,0}^{4,0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^-\right|^2\left|X^+\right|^2\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,-\sqrt{105}}(X) =$$
$$\frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,\sqrt{105}}(X) =$$
$$\frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,0,0}(X) = \frac{3\sqrt{5}X_+^+X_+^-\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,0,0}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_+^+\right)^2\left(X_+^-\right)^2}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{1,1}^{4,2,2}(X) = \frac{3\left(X_+^-X_0^+ - X_+^+X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}^{4,2,2}_{2,2}(X) = \frac{\sqrt{\frac{3}{7}}X^{+}_{+}\left(5X^{-}_{+}\left|X^{+}\right|^{2} - 2R^{2}X^{+}_{+}\right)}{\pi^{3/2}R^{4}}$$

$$\mathcal{Y}^{4,2,-13}_{3,3}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X^{+}_{+}\right)^{2}\left(X^{-}_{+}X^{+}_{0} - X^{+}_{+}X^{-}_{0}\right)}{2\pi^{3/2}R^{4}}$$

$$\mathcal{Y}^{4,2,-5}_{4,4}(X) = \frac{\sqrt{15}\left(X^{+}_{+}\right)^{3}X^{-}_{+}}{2\pi^{3/2}R^{4}}$$

$$\mathcal{Y}^{4,4,0}_{0,0}(X) = \frac{\sqrt{3}\left|X^{+}\right|^{4}}{\pi^{3/2}R^{4}}$$

$$\mathcal{Y}^{4,4,-3}_{2,2}(X) = \frac{3\sqrt{\frac{5}{14}}\left(X^{+}_{+}\right)^{2}\left|X^{+}\right|^{2}}{\pi^{3/2}R^{4}}$$

$$\mathcal{Y}^{4,4,-10}_{4,4}(X) = \frac{\sqrt{15}\left(X^{+}_{+}\right)^{4}}{4\pi^{3/2}R^{4}}$$

6 Permutation Symmetric Hyperspherical Harmonics

There is a small step remaining from obtaining the hyperspherical harmonics labelled by quantum numbers (K, Q, L, m, ν) to achieving our goal, which is to construct hyperspherical functions with well-defined values of parity $P = (-1)^{\mathrm{K}}$, rotational group quantum numbers (L, m), and permutation symmetry M (mixed), S (symmetric), and A (antisymmetric).[1] In this section we clarify how to obtain the latter as linear combinations of the former.

Properties under particle permutations of the functions $\mathcal{Y}^{KQ\nu}_{J,m}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ are inferred from the transformation properties of the coordinates $X^{\pm}_{i}$: under the transpositions (two-body permutations) $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs of particles (1,2), (2,3) and (3,1), the Jacobi coordinates transform as:

[1] The mixed symmetry representation of the S_3 permutation group being two-dimensional, there are two different state vectors (hyperspherical harmonics) in each mixed permutation symmetry multiplet, usually denoted by M_ρ and M_λ.

$$
\begin{aligned}
\mathcal{T}_{12}:\ & \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho}, \\
\mathcal{T}_{23}:\ & \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \\
\mathcal{T}_{31}:\ & \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.
\end{aligned} \tag{8}
$$

That induces the following transformations of complex coordinates $X_i^{\pm}$:

$$
\begin{aligned}
\mathcal{T}_{12}:\ & X_i^{\pm} \to X_i^{\mp}, \\
\mathcal{T}_{23}:\ & X_i^{\pm} \to e^{\pm\frac{2i\pi}{3}} X_i^{\mp}, \\
\mathcal{T}_{31}:\ & X_i^{\pm} \to e^{\mp\frac{2i\pi}{3}} X_i^{\mp}.
\end{aligned} \tag{9}
$$

None of the quantum numbers K, L and m change under permutations of particles, whereas the values of the "democracy label" Q and multiplicity label ν are inverted under all transpositions: $Q \to -Q$, $\nu \to -\nu$.

Apart from the changes in labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyperspherical harmonic. For values of K, Q, L and m with no multiplicity, we readily derive (Ref. [7]) the following transformation properties of h.s. harmonics under (two-particle) particle transpositions:

$$
\begin{aligned}
\mathcal{T}_{12}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-J} \mathcal{Y}_{L,m}^{K,-Q,-\nu}, \\
\mathcal{T}_{23}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}, \\
\mathcal{T}_{31}:\ & \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{-\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}.
\end{aligned} \tag{10}
$$

There are three distinct irreducible representations of the S_3 permutation group - two one-dimensional (the symmetric S and the antisymmetric A ones) and a two-dimensional (the mixed M one). In order to determine to which representation of the permutation group any particular h.s. harmonic $\mathcal{Y}_{L,m}^{KQ\nu}$ belongs, one has to consider various cases, with and without multiplicity, see Ref. [7]; here we simply state the results of the analysis conducted therein. The following linear combinations of the h.s. harmonics,

$$
\mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}_{L,m}^{K|Q|\nu} \pm (-1)^{K-L} \mathcal{Y}_{L,m}^{K,-|Q|,-\nu} \right). \tag{11}
$$

are no longer eigenfunctions of Q operator but are (pure sign) eigenfunctions of the transposition $\mathcal{T}_{12}$ instead:

$$
\mathcal{T}_{12}: \mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \to \pm \mathcal{Y}_{L,m,\pm}^{K|Q|\nu}.
$$

They are the appropriate h.s. harmonics with well-defined permutation properties:

1. for $Q \not\equiv 0$ (mod 3), the harmonics $\mathcal{Y}^{K|Q|\nu}_{L,m,\pm}$ belong to the mixed representation M, where the $\pm$ sign determines the M_ρ, M_λ component,
2. for $Q \equiv 0$ (mod 3), the harmonic $\mathcal{Y}^{K|Q|\nu}_{L,m,+}$ belongs to the symmetric representation S and $\mathcal{Y}^{K|Q|\nu}_{L,m,-}$ belongs to the antisymmetric representation A.

The above rules define the representation of S_3 for any given h.s. harmonic.

7 Summary

In this paper we have reported on our recent construction of permutation symmetric three-body $SO(6)$ hyperspherical harmonics. In the Sect. 5 we have displayed explicit forms the harmonic functions labelled by quantum numbers K, Q, L, m and ν, postponing explanation of their derivation to [7]. In Sect. 6 we demonstrated that simple linear combinations $\mathcal{Y}^{K|Q|\nu}_{L,m,\pm}$ of these functions have well defined permutation properties. To our knowledge, this is the first time that such hyperspherical harmonics are constructed in full generality.

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Three-particle hyper-spherical harmonics and quark bound states

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Abstract. We construct the three-body permutation symmetric hyperspherical harmonics based on the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$ (and the subalgebra chain $u(1) \otimes so(3)_{rot} \subset u(3) \subset so(6)$). These hyperspherical harmonics represent a natural basis for solving non-relativistic three-body Schrödinger equation in three spatial dimensions. In particular, we apply the calculated three-particle harmonics to the three-quark bound state problem. We consider confining Δ- and Y-string three-quark effective potentials, and then calculate the spectrum of low-lying ($\mathrm{K} \leq 4$) bound states.

1. Introduction

The quantum-mechanical three-body bound-state problem has been addressed by a huge literature, in which the hyperspherical harmonics, Refs. [1, 2, 3, 4], provide one of the most firmly established theoretical tools. Nevertheless, little is known about the general structure of the three-body bound-state spectrum, such as the ordering of states, even in the (simplest) case of three indentical particles. In comparison, the two-body bound state problem is well understood, see Refs. [5, 6, 7, 8], where theorems controlling the ordering of bound states in convex two-body potentials were proven more than 30 years ago. In this paper we make the first significant advance in the three-body problem after the 1990 paper by Taxil & Richard, Ref. [9].

The basic difficulty lay in the absence of a systematic construction of permutation-symmetric three-body wave functions. Classification of wave functions into distinct classes under permutation symmetry in the three-body system, should be a matter of course, and yet permutation symmetric three-body hyperspherical harmonics in three dimensions were known explicitly only in a few special cases, such as those with total orbital angular momentum $L = 0$, see Refs. [3, 10] before the recent progress made in Ref. [11]. In this paper we confine ourselves to the study of factorizable (in the hyper-radius and hyper-angles) three-body potentials for technical reasons: For this class of potentials our method allows closed-form ("analytical") results, at sufficiently small values of the grand angular momentum K (i.e. up to, and including the $\mathrm{K} \leq 8$ shell). Factorizable potentials include homogenous potentials, which in turn include pair-wise sums of two-body power-law potentials, such as the linear (confining) "Δ-string", and the Coulomb ones, as well as the genuine three-body "Y-string" potential [12, 13].

In this paper, we shall: 1) show how the Schrödinger equation for three particles in a homogenous/factorizable potential can be reduced to a single differential equation and an algebraic/numerical problem for their coupling strengths; 2) use this result to explicitly confirm

Richard and Taxil's results, [9], for the ordering of K = 3 shell three-quark states, and thus resolve the controversy with [14]; 3) calculate the K = 4 shell's (purported "universal") spectral splittings in terms of four parameters (lowest hyperspherical harmonics expansion coefficients) that characterize the three-body potential. 4) show that the first manifest differences in the ordering of states in the Y- and Δ-string potentials appear in the K = 3 shell, and then reappear more emphatically in the K = 4 shell.

Our work is based on the recent advances in the construction of three-body wave functions with well-defined permutation symmetry, see Ref. [11].

2. Three-body problem in hyper-spherical coordinates

The three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ can be transcribed from the Euclidean relative position (Jacobi) vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2})$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3})$, into hyper-spherical coordinates as $\Psi(R, \Omega_5)$, where $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyper-radius, and five angles Ω_5 that parametrize a hyper-sphere in the six-dimensional Euclidean space. Three (Φ_i; $i = 1, 2, 3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. As we saw above, one linear combination of the two variables $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$, is already taken by the hyper-radius R, so the shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere.

There are two traditional ways of parameterizing this sphere: 1) the standard Delves choice, [1], of hyper-angles (χ, θ), that somewhat obscures the full S_3 permutation symmetry of the problem; 2) the Iwai, Ref. [4], hyper-angles (α, ϕ): $(\sin\alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right)$, reveal the full S_3 permutation symmetry of the problem: the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions. We shall use the latter choice, as it leads to permutation-symmetric hyperspherical harmonics, see Ref. [11].

We expand the wave function $\Psi(R, \Omega_5)$ in terms of hyper-spherical harmonics $\mathcal{Y}^{\rm K}_{[\rm m]}(\Omega_5)$, $\Psi(R, \Omega_5) = \sum_{{\rm K},[\rm m]} \psi^{\rm K}_{[\rm m]}(R)\mathcal{Y}^{\rm K}_{[\rm m]}(\Omega_5)$, where K together with $[\rm m] = [Q, \nu, L, L_z = m]$ constitute the complete set of hyperspherical quantum numbers: K is the hyper-spherical angular momentum, L is the (total orbital) angular momentum, $L_z = m$ its projection on the z-axis, Q is the Abelian quantum number conjugated with the Iwai angle ϕ, and ν is the multiplicity label that distinguishes between hyperspherical harmonics with remaining four quantum numbers that are identical.

The hyper-spherical harmonics turn the Schrödinger equation into a set of (infinitely) many coupled equations,

$$\begin{aligned} &- \frac{1}{2\mu}\left[\frac{d^2}{dR^2} + \frac{5}{R}\frac{d}{dR} - \frac{{\rm K}({\rm K}+4)}{R^2} + 2\mu E\right]\psi^{\rm K}_{[\rm m]}(R) \\ &+ V_{\rm eff.}(R)\sum_{{\rm K}',[\rm m']} C^{\rm K\ K'}_{[\rm m][\rm m']}\psi^{\rm K'}_{[\rm m']}(R) = 0 \end{aligned} \tag{1}$$

with a hyper-angular coupling coefficients matrix $C^{\rm K\ K'}_{[\rm m][\rm m']}$ defined by

$$\begin{aligned} V_{\rm eff.}(R)C^{\rm K'\ K}_{[\rm m'][\rm m]} &= \langle\mathcal{Y}^{\rm K'}_{[\rm m']}(\Omega_5)|V(R,\alpha,\phi)|\mathcal{Y}^{\rm K}_{[\rm m]}(\Omega_5)\rangle \\ &= V(R)\langle\mathcal{Y}^{\rm K'}_{[\rm m']}(\Omega_5)|V(\alpha,\phi)|\mathcal{Y}^{\rm K}_{[\rm m]}(\Omega_5)\rangle. \end{aligned} \tag{2}$$

In Eq. (1) we used the factorizability of the potential $V(R, \alpha, \phi) = V(R)V(\alpha, \phi)$ to reduce this set to one (common) hyper-radial Schrödinger equation. The hyper-angular part $V(\alpha, \phi)$ can be expanded in terms of O(6) hyper-spherical harmonics with zero angular momenta $L = m = 0$ (due to the rotational invariance of the potential),

$$V(\alpha, \phi) = \sum_{\mathrm{K},Q}^{\infty} v_{\mathrm{K},Q}^{3-\mathrm{body}} \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha, \phi) \tag{3}$$

where $v_{\mathrm{K},Q}^{3-\mathrm{body}} = \int \mathcal{Y}_{00}^{\mathrm{K}Q\nu *}(\Omega_5)\, V(\alpha, \phi)\, d\Omega_{(5)}$ leading to

$$\begin{aligned} V_{\mathrm{eff.}}(R) C_{[\mathrm{m}''][\mathrm{m}']}^{\mathrm{K}''\ \mathrm{K}'} = &\quad V(R) \sum_{\mathrm{K},Q}^{\infty} v_{\mathrm{K},Q}^{3-\mathrm{body}} \\ \langle \mathcal{Y}_{[\mathrm{m}'']}^{\mathrm{K}''}(\Omega_5)| &\quad \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha, \phi) | \mathcal{Y}_{[\mathrm{m}']}^{\mathrm{K}'}(\Omega_5)\rangle \end{aligned} \tag{4}$$

There is no summation over the multiplicity index in Eq. (3), because no multiplicity arises for harmonics with $L < 2$. Here we separate out the $\mathrm{K} = 0$ term and absorb the factor $\frac{v_{00}^{3-\mathrm{body}}}{\pi\sqrt{\pi}}$ into the definition of $V_{\mathrm{eff.}}(R) = \frac{v_{00}^{3-\mathrm{body}}}{\pi\sqrt{\pi}} V(R)$ to find

$$\begin{aligned} C_{[\mathrm{m}''][\mathrm{m}']}^{\mathrm{K}''\ \mathrm{K}'} = \delta_{\mathrm{K}'',\mathrm{K}'} \delta_{[\mathrm{m}''],[\mathrm{m}']} + \pi\sqrt{\pi} \textstyle\sum_{\mathrm{K}>0,Q}^{\infty} \frac{v_{\mathrm{K},Q}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \\ \times \langle \mathcal{Y}_{[\mathrm{m}'']}^{\mathrm{K}''}(\Omega_5) | \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha, \phi) | \mathcal{Y}_{[\mathrm{m}']}^{\mathrm{K}'}(\Omega_5)\rangle. \end{aligned} \tag{5}$$

Homogenous potentials, such as the Δ and Y-string ones, which are linear in R, and the Coulomb one, have first coefficients $v_{00}^{3-\mathrm{body}}$ in the h.s. expansion that are one order of magnitude larger than the rest $v_{\mathrm{K}>0,Q}^{3-\mathrm{body}}$. This reflects the fact that, on the average, these potential energies depend more on the overall size of the system than on its shape, thus justifying the perturbative approach taken in Ref. [9], with the first term in Eq. (5) taken as the zeroth-order approximation.[1]

In such cases Eqs. (1) decouple, leading to zeroth order solutions for $\psi_{0[\mathrm{m}]}^{\mathrm{K}}(R)$ that are independent of $[m]$ and thus have equal energies within the same K shell, and different energies in different K shells. Two known exceptions are potentials with the homogeneity degree $\mathrm{k} = -1, 2$, that lead to "accidental degeneracies" and have to be treated separately.

The first-order corrections are obtained by diagonalization of the block matrices $C_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}}$, $\mathrm{K} = 1, 2, ...,$ while the off-diagonal couplings $C_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}'}, \mathrm{K} \neq \mathrm{K}'$ appear only in the second-order corrections. Rather than calculating perturbative first-order energy shifts, a better approximation is obtained when the diagonalized block matrices are plugged back into Eq. (1), which equations then decouple into a set of (separate) individual ODEs in one variable, that differ only in the value of the effective coupling constant:

$$\left[\frac{d^2}{dR^2} + \frac{5}{R}\frac{d}{dR} - \frac{\mathrm{K}(\mathrm{K}+4)}{R^2} + 2\mu(E - V_{[\mathrm{m_d}]}^{\mathrm{K}}(R))\right] \psi_{[\mathrm{m_d}]}^{\mathrm{K}}(R) = 0, \tag{6}$$

where $V_{[\mathrm{m_d}]}^{\mathrm{K}}(R) = C_{[\mathrm{m_d}]}^{\mathrm{K}} V_{\mathrm{eff.}}(R)$, with $C_{[\mathrm{m_d}]}^{\mathrm{K}}$ being the eigenvalues of matrix $C_{[\mathrm{m}][\mathrm{m}']}^{\mathrm{K}\ \mathrm{K}}$.

The spectrum of three-body systems in homogenous potentials is now reduced to finding the eigenvalues of a single differential operator, just as in the two-body problem with a radial potential. The matrix elements in Eq. (5) can be readily evaluated using the permutation-symmetric O(6) hyper-spherical harmonics and the integrals that are spelled out in Ref. [11].

[1] (note that the h.s. matrix elements $\langle \mathcal{Y}_{[\mathrm{m}'']}^{\mathrm{K}''}(\Omega_5) | \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha, \phi) | \mathcal{Y}_{[\mathrm{m}']}^{\mathrm{K}'}(\Omega_5)\rangle$ under the sum are always less than $\frac{1}{\pi\sqrt{\pi}}$).

This is our main (algebraic) result: combined with the hyperspherical harmonics recently obtained in Ref. [11], it allows one to evaluate the discrete part of the (energy) spectrum of a three-body potential as a function of its shape-sphere harmonic expansion coefficients $v_{\mathrm{K},Q}^{3-\mathrm{body}}$. Generally, these matrix elements obey selection rules: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $m' = m''$. Moreover, Q is an Abelian (i.e. additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$. That reduces the sum in Eq. (5) to a finite one, that depends on a finite number of coefficients $v_{\mathrm{K},Q}^{3-\mathrm{body}}$; for small values of K, this number is also small.

A matrix such as that in Eq. (5) is generally sparse in the permutation-symmetric basis, so its diagonalization is not a serious problem, and, for sufficiently small K values it can even be accomplished in closed form: for example, for $\mathrm{K} \leq 5$, all results depend only on four coefficients $(v_{00}, v_{40}, v_{6\pm6}, v_{80})$, and there is at most three-state mixing, so the eigenvalue equations are at most cubic ones, with well-known solutions. For brevity's sake we confine ourselves to $\mathrm{K} \leq 4$ states here.

3. Results

1) In the $\mathrm{K} = 2$ band/shell of the three-body energy spectrum the eigen-energies depend on two coefficients (v_{00}, v_{40}), and the splittings among various levels depend only on the (generally small, see Table 1) ratio v_{40}/v_{00}. This means that the eigen-energies form a fixed pattern ("ordering") that does not depend on the shape of the three-body potential. The actual size of the $\mathrm{K} = 2$ shell energy splitting depends on the small parameter v_{40}/v_{00}, provided that the potential is permutation symmetric. This fact was noticed almost 40 years ago, Refs. [15, 16], and it suggested that similar patterns might exist in higher-K shells.

The advantage of permutation-symmetric hyperspherical harmonics over the conventional ones is perhaps best illustrated here: the $\mathrm{K} = 2$ shell splittings in the Y- and Δ-string potentials were obtained, after some complicated calculations using conventional hyperspherical harmonics in Ref. [17], whereas here they follow from the calculation of four (simple) hyper-angular matrix elements.

2) Historically, extensions of this kind of calculations to higher ($\mathrm{K} \geq 3$) bands, for general three-body potentials turned out more difficult than expected: Bowler et. al, Ref. [14], published a set of predictions for the $\mathrm{K} = 3, 4$ bands, which were later questioned by Richard and Taxil's [9], $\mathrm{K} = 3$ hyperspherical harmonic calculation; see also Refs. [18]. This controversy had not been resolved to the present day, to our knowledge, so we address that problem first: In the $\mathrm{K} = 3$ case the energies depend on three coefficients $(v_{00}, v_{40}, v_{6\pm6})$, and there is no mixing of multiplets, so all eigen-energies can be expressed in simple closed form that agrees with Ref. [9] and depends on two small parameters $v_{40}/v_{00}, v_{6\pm6}/v_{00}$.

Note that the third coefficient $v_{6\pm6}$ vanishes in the simplified Y-string potential without two-body terms and thus causes the first observable difference between Y- and Δ-string potentials: the splittings between $[20, 1^-]$, and $[56, 1^-]$, as well as between $[20, 3^-]$, and $[56, 3^-]$. The vanishing of $v_{6\pm6}$ implies that the Y-string potential is independent of the Iwai angle ϕ, and that consequently there is a (new) dynamical "kinematic rotations/democracy transformations" O(2) symmetry, [12, 13] associated with it.

3) In the $\mathrm{K} = 4$ band $SU(6)$, or S_3 multiplets have one of the following 12 values of the diagonalized C-matrix $C_{[\mathrm{m_d}]}^{\mathrm{K}} \times \frac{v_{00}}{\pi\sqrt{\pi}}$, from which one can evaluate the eigen-energies. We use the baryon-spectroscopic notation: $[\mathrm{dim.}, L^P]$, where dim. is the dimension of the $SU_{\mathrm{FS}}(6)$ representation and the correspondence with the representations of the permutation group S_3

is given as $70 \leftrightarrow M$, $20 \leftrightarrow A$, $56 \leftrightarrow S$.

$$\begin{aligned}
&[70, 0^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{\sqrt{3}}{2}v_{40} + \frac{1}{2\sqrt{5}}v_{80}\right)\\
&[56, 0^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{2}{\sqrt{5}}v_{80}\right)\\
&[70, 1^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{5}}v_{80}\right)\\
&[70, 2^+] : \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00} + \frac{1}{35}\Big(7\sqrt{3}v_{40} + 2\sqrt{5}v_{80}\\
&-3\sqrt{3v_{40}^2 - 2\sqrt{15}v_{40}v_{80} + 5v_{80}^2 + 120v_{6\pm6}^2}\Big)\Bigg)\\
&\left[70', 2^+\right] : \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00} + \frac{1}{35}\Big(7\sqrt{3}v_{40} + 2\sqrt{5}v_{80}\\
&+3\sqrt{3v_{40}^2 - 2\sqrt{15}v_{40}v_{80} + 5v_{80}^2 + 120v_{6\pm6}^2}\Big)\Bigg)\\
&[56, 2^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{12\sqrt{3}}{35}v_{40} + \frac{\sqrt{5}}{7}v_{80}\right)\\
&[20, 2^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{1}{\sqrt{5}}v_{80}\right)\\
&[20, 3^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{3\sqrt{3}}{14}v_{40} - \frac{\sqrt{5}}{14}v_{80}\right)\\
&[70, 3^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{5\sqrt{3}}{14}v_{40} + \frac{1}{14\sqrt{5}}v_{80}\right)\\
&[56, 4^+] : \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{5\sqrt{3}}{14}v_{40} + \frac{3}{14\sqrt{5}}v_{80}\right)\\
&[70, 4^+] : \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00} + \frac{1}{42\sqrt{5}}\Big(-2v_{80}\\
&-\sqrt{1215v_{40}^2 - 54\sqrt{15}v_{40}v_{80} + 9v_{80}^2 + 1280v_{6\pm6}^2}\Big)\Bigg)\\
&\left[70', 4^+\right] : \frac{1}{\pi\sqrt{\pi}}\Bigg(v_{00} + \frac{1}{42\sqrt{5}}\Big(-2v_{80}\\
&+\sqrt{1215v_{40}^2 - 54\sqrt{15}v_{40}v_{80} + 9v_{80}^2 + 1280v_{6\pm6}^2}\Big)\Bigg).
\end{aligned}$$

Table 1 shows that the ordering of K = 4 states is not universally valid even for these two convex confining potentials. This, of course, is a consequence of different ratios v_{40}/v_{00}, $v_{6\pm6}/v_{00}$ and v_{80}/v_{00}. That goes to show that one cannot expect strongly restrictive ordering theorems

Table 1. Expansion coefficients v_{KQ} of the Y- and Δ-string potentials in terms of O(6) hyper-spherical harmonics $\mathcal{Y}_{0,0}^{K,0,0}$, for K = 0, 4, 8, respectively, and of the hyper-spherical harmonics $\mathcal{Y}_{0,0}^{6,\pm 6,0}$.

(K, Q)	(0,0)	(4,0)	(6,±6)	(8,0)
v_{KQ}(Y)	8.18	-0.44	0	-0.09
$v_{KQ}(\Delta)$	16.04	-0.44	-0.14	-0.06

to hold for three-body systems in general, the way they hold in the two-body problem, Ref. [8]. Nevertheless, even the present results can be useful, as they indicate that certain groups of multiplets are jointly lifted, or depressed in the spectrum, subject to the value of the ratio v_{40}/v_{00}, with ordering within each group being subject to the finer structure of the potential, i.e., to higher coefficients ratios $v_{6\pm 6}/v_{00}$ and v_{80}/v_{00}.

Of course, similar conclusions hold also for K = 3 spectrum splitting, but are less pronounced, as that shell depends only on two numbers: the ratios v_{40}/v_{00} and $v_{6\pm 6}/v_{00}$. As the difference between Δ and Y-string potentials is most pronounced in the value of $v_{6\pm 6}$, that is the case where the distinction between these two potentials is most clearly seen.

4. Summary and Conclusions

We have reduced the non-relativistic (quantum) three-identical-body problem to a single ordinary differential equation for the hyper-radial wave function with coefficients multiplying the linear hyper-radial potential determined by O(6) group-theoretical arguments. That equation can be solved in the same way as the radial Schrödinger equation in 3D. The breaking of the O(6) symmetry by the three-body potential determines the ordering of states in the spectrum.

In three dimensions (3D) the hyper-spherical symmetry group is O(6), and the residual dynamical symmetry of the potential is $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$, where $SO(3)_{rot}$ is the rotational symmetry associated with the (total orbital) angular momentum L. We showed how the energy eigenvalues can be calculated in terms of the three-body potential's (hyper-)spherical harmonics expansion coefficients v_{KQ}.

The ordering of bound states has its most immediate application in the physics of three confined quarks, where the question was raised originally, Refs. [9, 14, 15, 16, 17]. We have used these results to calculate the energy splittings of various SU(6)/S_3 multiplets in the K $\leq$ 4 shells of the Y- and Δ-string potential spectra. The dynamical $O(2)$ dynamical symmetry of the Y-string potential was discovered in Ref. [12], with the permutation group $S_3 \subset O(2)$ as its subgroup. The existence of an additional dynamical symmetry strongly suggested an algebraic approach to this problem, such as that used in two-dimensional space, in Ref. [13]. We have shown that the first clear difference between the spectra of these two models of confinement appears in the K $\geq$ 3 shell. That is also the first explicit consequence of the dynamical O(2) symmetry of the "Y-string" potential. We stress the analytical nature of our results, in contrast to the numerical results of Refs. [18].

The next step would be to apply the method to linear combinations of homogenous potentials, which can only be done numerically, however. Several "realistic" two-body potentials, such as the Lennard-Jones inter-atomic one, as well as the "Coulomb + linear" quark-quark one, are simple linear combinations of (only) two homogenous potentials.

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Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, I\!R)$ and Character Formulae

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Abstract

We continue the study of positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, I\!R)$. We update previous results and present the full list of these UIRs. We give also some character formulae for these representations.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. Until recently only those for $D \leq 6$ were studied since in these cases the relevant superconformal algebras satisfy [1] the Haag-Lopuszanski-Sohnius theorem [2]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [3] (for $N = 1$), [4–7] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [8] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [9].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras $osp(1|2n)$. Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [10]. Later it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [11]. In all cases we need to find first the UIRs of $osp(1|2n, I\!R)$ which study was started in [12] and [13].

In the present paper we intend to finalize unitarity classification of [12] and in addition to provide some character formulae.

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Since this paper is a sequel of [12], where there is extensive literature, and for the lack of space we only update the supersymmetry literature (for $D > 2$) after 2004, cf. [14–61]

2. Representations of the superalgebras $osp(1|2n)$ and $osp(1|2n, I\!R)$

2.1. The setting

Our basic references for Lie superalgebras are [62, 63], although in this exposition we follow [12].

The even subalgebra of $\mathcal{G} = osp(1|2n, I\!R)$ is the algebra $sp(2n, I\!R)$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, a_1\, , ..., a_{n-1}\,] \tag{1}$$

where d is the conformal weight, and $a_1, ..., a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

In [12] were classified (with some omissions to be spelled out below) the positive energy (lowest weight) UIRs of $\mathcal{G}$ following the methods used for the $D = 4, 6$ conformal superalgebras, cf. [4–7, 9], resp. The main tool was an adaptation of the Shapovalov form [64] on the Verma modules V^χ over the complexification $\mathcal{G}^{I\!\!C} = osp(1|2n)$ of $\mathcal{G}$.

2.2. Root systems

We recall some facts about $\mathcal{G}^{I\!\!C} = osp(1|2n)$ (denoted $B(0,n)$ in [62]) as used in [12]. The root systems are given in terms of $\delta_1 \ldots, \delta_n$, $(\delta_i, \delta_j) = \delta_{ij}$, $i, j = 1, ..., n$. The even and odd roots systems are [62]:

$$\begin{aligned} \Delta_{\bar 0} &= \{\pm\delta_i \pm \delta_j\ ,\ 1 \le i < j \le n\ ,\quad \pm 2\delta_i\ ,\ 1 \le i \le n\}\ , \\ \Delta_{\bar 1} &= \{\pm\delta_i\ ,\ 1 \le i \le n\} \end{aligned} \tag{2}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [62]:

$$\Pi = \{\, \delta_1 - \delta_2\, ,\, , \ldots, \delta_{n-1} - \delta_n\, ,\, \delta_n\, \}\ , \tag{3}$$

or introducing standard notation for the simple roots:

$$\begin{aligned} \Pi &= \{\, \alpha_1\, , ..., \,\alpha_n\, \}\ , \\ &\alpha_j = \delta_j - \delta_{j+1}\ ,\quad j = 1, ..., n-1\ ,\quad \alpha_n = \delta_n\ . \end{aligned} \tag{4}$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is:

$$\underset{1}{\circ} \frac{\quad}{} \cdots \frac{\quad}{} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet} \tag{5}$$

The black dot is used to signify that the simple odd root is not nilpotent, otherwise a gray dot would be used [62]. In fact, the superalgebras $B(0,n) = osp(1|\,2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [62].

The corresponding to Π positive root system is:

$$\Delta^+_{\bar 0} = \{\delta_i \pm \delta_j \,,\ 1 \le i < j \le n \,,\ 2\delta_i \,,\ 1 \le i \le n\} \,, \qquad \Delta^+_{\bar 1} = \{\delta_i \,,\ 1 \le i \le n\} \tag{6}$$

We record how the elementary functionals are expressed through the simple roots:

$$\delta_k = \alpha_k + \cdots + \alpha_n \,. \tag{7}$$

From the point of view of representation theory more relevant is the restricted root system, such that:

$$\bar\Delta^+ = \bar\Delta^+_{\bar 0} \cup \Delta^+_{\bar 1} \,, \tag{8}$$

$$\bar\Delta^+_{\bar 0} \equiv \{\alpha \in \Delta^+_{\bar 0} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar 1}\} = \{\delta_i \pm \delta_j \,,\ 1 \le i < j \le n\} \tag{9}$$

The superalgebra $\mathcal{G} = osp(1|\,2n, I\!\!R)$ is a split real form of $osp(1|\,2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram:

$$\underset{1}{\circ} \text{———} \cdots \text{———} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ} \tag{10}$$

Naturally, for the B_n positive root system we drop the roots $2\delta_i$

$$\Delta^+_{\mathrm{B_n}} = \{\delta_i \pm \delta_j \,,\ 1 \le i < j \le n \,,\ \ \delta_i \,,\ 1 \le i \le n\} \cong \bar\Delta^+ \tag{11}$$

This shall be used essentially below.

2.3. Lowest weight through the signature

Besides (1) we shall use the Dynkin-related labelling:

$$(\Lambda, \alpha_k^\vee) = -\,a_k \,, \quad 1 \le k \le n \,, \tag{12}$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k,\alpha_k)$, and the minus signs are related to the fact that we work with lowest weight Verma modules (instead of the highest weight modules used in [63]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [6, 12]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

$$a_n = -\,2d - a_1 - \cdots - a_{n-1} \,. \tag{13}$$

The actual Dynkin labelling is given by:

$$m_k = (\rho - \Lambda, \alpha_k^\vee) \tag{14}$$

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}, \rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (6):

$$\begin{aligned} \rho \doteq \rho_{\bar{0}} - \rho_{\bar{1}} &= (n-\tfrac{1}{2})\delta_1 + (n-\tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n \ , \quad (15) \\ \rho_{\bar{0}} &= n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n \ , \\ \rho_{\bar{1}} &= \tfrac{1}{2}(\delta_1 + \cdots + \delta_n) \ . \end{aligned}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho, \alpha_i^\vee) = 1$, $i = 1, ..., n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ $(\forall k)$ the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$ (and of B_n).

Having in hand the values of Λ on the basis we can recover them for any element of $\mathcal{H}^*$.

We shall need only $(\Lambda, \beta^\vee)$ for all positive roots β as given in [12]:

$$\begin{aligned} (\Lambda, (\delta_i - \delta_j)^\vee) &= (\Lambda, \delta_i - \delta_j) = -a_i - \cdots - a_{j-1} \quad (16) \\ (\Lambda, (\delta_i + \delta_j)^\vee) &= (\Lambda, \delta_i + \delta_j) = 2d + a_1 + \cdots + a_{i-1} - a_j - \cdots - a_{n-1} \\ (\Lambda, \delta_i^\vee) &= (\Lambda, 2\delta_i) = 2d + a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1} \\ (\Lambda, (2\delta_i)^\vee) &= (\Lambda, \delta_i) = d + \tfrac{1}{2}(a_1 + \cdots + a_{i-1} - a_i - \cdots - a_{n-1}) \end{aligned}$$

2.4. Verma modules

To introduce Verma modules we use the standard triangular decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{17}$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned} Z\, v_0 &= 0 \ , \quad Z \in \mathcal{G}^- \\ H\, v_0 &= \Lambda(H)\, v_0 \ , \quad H \in \mathcal{H} \ . \end{aligned} \tag{18}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\, v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [63] (which generalizes the BGG-criterion [65] to the super case) to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [12]:

$$(\rho - \Lambda, \beta^\vee) = m_\beta \ , \qquad \beta \in \Delta^+ \ , \quad m_\beta \in \mathbb{N} \ . \tag{19}$$

If a condition from (19) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m,β : $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v'_0 of $V^{\Lambda'}$ to the **singular vector** $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions:

$$\begin{aligned} X\, v_s^{m,\beta} &= 0\ , \quad X \in \mathcal{G}^-\ , \\ H\, v_s^{m,\beta} &= \Lambda'(H)\, v_0\ , \quad H \in \mathcal{H}\ , \quad \Lambda' \ = \ \Lambda + m\beta\ . \end{aligned} \tag{20}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [6, 66]:

$$v_s^{m,\beta} \ = \ P^{m,\beta}\, v_0\ , \quad P^{m,\beta} \in U(\mathcal{G}^+)\ . \tag{21}$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)\, P^{m,\beta}\, v_0$.

Note that the Casimirs of $\mathcal{G}^{\mathbb{C}}$ take the same values on V^Λ and $V^{\Lambda'}$.

Certainly, (19) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (19) is fulfilled, and let us denote: $\tilde{I}^\Lambda \ \equiv \ \cup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

Further we shall use also the following notion. The singular vector v_1 is called **descendant** of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 \ = \ P_{12}\, v_2$. Clearly, in this case we have: $I^1 \ \subset \ I^2$, where I^k is the submodule generated by v_k .

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [63, 65]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda \ = \ V^\Lambda/I^\Lambda$. (If V^Λ is irreducible then $L_\Lambda = V^\Lambda$.)

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if rank $\mathcal{G} \ \leq 2$, or when (19) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition: [65, 67, 68] *Let* V^Λ *be a reducible Verma module. A vector* $v_{\rm ssv} \ \in \ V^\Lambda$ *is called a* **subsingular vector** *if* $v_{\rm su} \ \notin \ \tilde{I}^\Lambda$ *and the following holds:*

$$X\, v_{\rm su} \ \in \ \tilde{I}^\Lambda\ , \quad \forall X \in \mathcal{G}^- \tag{22}$$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [12] it was established that from (19) follows that the Verma module

$V^{\Lambda(\chi)}$ is reducible if one of the following relations holds (following the order of (16):

$$\begin{aligned}
I\!N \ni m^-_{ij} &= j - i + a_i + \cdots + a_{j-1} && \text{(23a)}\\
I\!N \ni m^+_{ij} &= 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d && \text{(23b)}\\
I\!N \ni m_i &= 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d && \text{(23c)}\\
I\!N \ni m_{ii} &= n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d\ . && \text{(23d)}
\end{aligned}$$

Further we shall use the fact from [12] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (23d) is fulfilled also (23c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j. Naturally, $X^-_j \in \mathcal{G}^-$ corresponds to $-\alpha_j$.

Further, we notice that all reducibility conditions in (23a) are fulfilled. In particular, for the simple roots from those condition (23a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, ..., n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)\, v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}\, v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ we shall consider also the factor-modules:

$$F^\Lambda_c = V^\Lambda / I^\Lambda_c \supset F^\Lambda\ . \tag{24}$$

We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c :

$$(X^+_i)^{1+a_i}\, |\Lambda_c\rangle = 0\ , \quad i = 1, ..., n-1. \tag{25}$$

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (23b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (23b) are obtained for $m^+_{ij} = 1$ and they are:

$$d_{ij} \equiv n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j)\ , \tag{26}$$

the corresponding root being $\delta_i + \delta_j$.

The maximal d coming from the different possibilities in (23c,d), resp., are obtained for $m_i = 1$, $m_{ii} = 1$, resp., and they are:

$$\begin{aligned}
d_i &\equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1})\ , && \text{(27)}\\
d_{ii} &= d_i - \tfrac{1}{2}\ ,
\end{aligned}$$

the corresponding roots being $\delta_i\,, 2\delta_j$, resp.

There are some orderings between these maximal reduction points [12]:

$$\begin{array}{rcl} d_1 & > & d_2 \ > \ \cdots \ > \ d_n \ , \\ d_{i,i+1} & > & d_{i,i+2} \ > \ \cdots \ > \ d_{in} \ , \\ d_{1,j} & > & d_{2,j} \ > \ \cdots \ > \ d_{j-1,j} \ , \\ d_i & > & d_{jk} \ > \ d_\ell \ , \qquad i \le j < k \le \ell \ . \end{array} \tag{28}$$

Obviously the first reduction point is:

$$d_1 \ = \ n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \ . \tag{29}$$

3. Unitarity

The first results on the unitarity were given in [12]. These were not complete so the statement below should be called *Dobrev-Zhang-Salom Theorem.*
Theorem: All positive energy unitary irreducible representations of the superalgebras $osp(1|\,2n, I\!\!R)$ characterized by the signature χ in (1) are obtained for real d and are given as follows:

$$\begin{array}{l} d \ \ge \ n - 1 + \frac{1}{2}(a_1 + \cdots + a_{n-1}) \ = \ d_1 \ , \quad a_1 \ne 0 \ , \qquad (30) \\ d \ \ge \ n - \frac{3}{2} + \frac{1}{2}(a_2 + \cdots + a_{n-1}) \ = \ d_{12} \ , \quad a_1 = 0, \ a_2 \ne 0 \ , \\ d \ = \ n - 2 + \frac{1}{2}(a_2 + \cdots + a_{n-1}) \ = \ d_2 > d_{13} \ , \quad a_1 = 0, \ a_2 \ne 0 \ , \qquad (31) \\ d \ \ge \ n - 2 + \frac{1}{2}(a_3 + \cdots + a_{n-1}) \ = \ d_2 \ = \ d_{13} \ , \quad a_1 = a_2 = 0, \ a_3 \ne 0 \ , \\ d \ = \ n - \frac{5}{2} + \frac{1}{2}(a_3 + \cdots + a_{n-1}) \ = \ d_{23} \ > \ d_{14} \ , \quad a_1 = a_2 = 0, \ a_3 \ne 0 \ , \\ d \ = \ n - 3 + \frac{1}{2}(a_3 + \cdots + a_{n-1}) \ = \ d_3 \ = \ d_{24} \ > \ d_{15} \ , \quad a_1 = a_2 = 0, \ a_3 \ne 0 \ , \\ ... \\ ... \\ d \ \ge \ n - 1 - \kappa + \frac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}) \ , \quad a_1 = ... = a_{2\kappa} = 0, \ a_{2\kappa+1} \ne 0 \ , \\ \qquad\qquad \kappa = \frac{1}{2}, 1, ..., \frac{1}{2}(n-1) \ , \\ d \ = \ n - \frac{3}{2} - \kappa + \frac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}) \ , \quad a_1 = ... = a_{2\kappa} = 0, \ a_{2\kappa+1} \ne 0 \ , \\ ... \\ d \ = \ n - 1 - 2\kappa + \frac{1}{2}(a_{2\kappa+1} + \cdots + a_{n-1}) \ , \quad a_1 = ... = a_{2\kappa} = 0, \ a_{2\kappa+1} \ne 0 \ , \\ ... \\ ... \\ d \ \ge \ \frac{1}{2}(n-1) \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0 \\ d \ = \ \frac{1}{2}(n-2) \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0 \\ ... \\ d \ = \ \frac{1}{2} \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0 \\ d \ = \ 0 \ , \quad a_1 \ = \ ... \ = \ a_{n-1} \ = \ 0 \end{array}$$

Proof: The statement of the Theorem for $d > d_1$ is clear in [12] from the general considerations since this is the First reduction point. For $d = d_1$ (also following [12]) we have the first zero norm state which is naturally given by the corresponding singular vector $v^1_{\delta_1} = \mathcal{P}^{1,\delta_1}\, v_0$. In fact, all states of the embedded submodule $V^{\Lambda+\delta_1}$ built on $v^1_{\delta_1}$ have zero norms. Due to the above singular vector we have the following additional null condition in F^Λ_c :

$$\mathcal{P}^{1,\delta_1}\, \widetilde{|\Lambda\rangle} \ = \ 0 \ . \tag{32}$$

The above condition factorizes the submodule built on $v^1_{\delta_1}$. There are no other vectors with zero norm at $d = d_1$ since by a general result [63], the elementary embeddings between Verma modules are one-dimensional. Thus, F^Λ is the UIR $L_\Lambda = F^\Lambda$.
Below $d < d_1$ there is no unitarity for $a_1 \neq 0$. On the other hand (as shown in [12]) for $a_1 = 0$ the singular vector $v^1_{\delta_1}$ is descendant of the compact root singular vector $X^+_1 v_0$ which is already factored out for $a_1 = 0$. Thus, below we set $a_1 = 0$.

The next reducibility point is $d = d_{12} = n-\frac{3}{2}+\frac{1}{2}(a_2+\cdots+a_{n-1})$. The corresponding root is $\delta_1+\delta_2 = \alpha_1+2\alpha_2+\cdots+2\alpha_n$. The corresponding singular vector is $v^1_{\delta_1+\delta_2} = \mathcal{P}^{1,\delta_1+\delta_2}\, v_0$. All states of the embedded submodule $V^{\Lambda+\delta_1+\delta_2}$ built on $v^1_{\delta_1+\delta_2}$ have zero norms for $d = d_{12}$. Due to the above singular vector we have the following additional null condition in F^Λ_c :

$$\mathcal{P}^{1,\delta_1+\delta_2}\, \widetilde{|\Lambda\rangle} \ = \ 0 \ , \qquad d \ = \ d_{12} \, . \tag{33}$$

The above conditions factorizes the submodule built on $v^1_{\delta_1+\delta_2}$. Thus, F^Λ_c is the UIR $L_\Lambda = F^\Lambda_c$.

Below $d < d_{12}$ there is no unitarity for $a_2 \neq 0$, except at the isolated point: $d_2 = n-2+\frac{1}{2}(a_2+\cdots+a_{n-1})$. At the latter point there is a singular vector $v^1_{\delta_2}$ which must be factored for unitarity. In addition, the previous singular vector is descendant of $v^1_{\delta_2}$ and the compact root singular vector $X^+_1 v_0$.

Further, for for $a_2 = 0$ the singular vectors $v^1_{\delta_1+\delta_2}$ and $v^1_{\delta_2}$ are descendants of the compact root singular vectors $X^+_1 v_0$ and $X^+_2 v_0$ which are factored out for $a_1 = a_2 = 0$. Thus, below we set also $a_2 = 0$ and there would be no obstacles for unitarity until the next reducibility points (coinciding due $a_2 = 0$): $d_2 = d_{13} = n-2+\frac{1}{2}(a_3+\cdots+a_{n-1})$. The singular vector for $d = d_{13}$ and $m = 1$ has weight $\delta_1+\delta_3 = \alpha_1+\alpha_2+2\alpha_3+\cdots 2\alpha_n$ and for $a_1 = 0$ it is a descendant of the compact root singular vector $X_1\, v_0$ [70]. However, at $d_2 = d_{13}$ there is a subsingular vector which must be factored for unitarity. For $d < d_2 = d_{13}$ and $a_3 \neq 0$ the norm of that subsingular vector is negative, and there will be no unitarity except at some lower reducibility points.

For $d_{23} = n - \frac{5}{2} + \frac{1}{2}(a_3 + \cdots + a_{n-1})$ there is singular vector $v^1_{\delta_2+\delta_3}$ of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3 + \cdots 2\alpha_n$ [70] which must be factored for unitarity. The previous subsingular vector is also factored out since it is descendant of $v^1_{\delta_2+\delta_3}$ and compact root singular vectors.

Further on, the Proof goes on similar lines. We list the points at which there are subsingular vectors - these happen when reducibility points coincide due the zero values of some a_i :

$$\begin{aligned}
& d_2 = d_{13} = n - 2 + \tfrac{1}{2}(a_3 + \cdots + a_{n-1}) \ , \quad a_1 = a_2 = 0 \ , \\
& d_{23} = d_{14} = n - 5/2 + \tfrac{1}{2}(a_4 + \cdots + a_{n-1}) \ , \quad a_1 = a_2 = a_3 = 0, \ n > 3, \\
& d_3 = d_{24} = d_{15} = n - 3 + \tfrac{1}{2}(a_5 + \cdots + a_{n-1}) \ , \quad a_1 = a_2 = a_3 = a_4 = 0, \\
& \qquad\qquad\qquad\qquad n > 3, \\
& \ldots \\
& d_j = d_{1,2j-1} = d_{2,2j-2} = \cdots = d_{j-1,j+1} = n - j + \tfrac{1}{2}(a_{2j-1} + \cdots + a_{n-1}) \ , \\
& \qquad a_1 = \cdots = a_{2j-2} = 0, \quad j < n, \\
& d_{j,j+1} = d_{1,2j} = d_{2,2j-1} = \cdots = d_{j-1,j+2} = n - j - \tfrac{1}{2} + \tfrac{1}{2}(a_{2j} + \cdots + a_{n-1}) \ , \\
& \qquad a_1 = \cdots = a_{2j-1} = 0, \quad j < n-1.
\end{aligned} \tag{34}$$

Above it is understood that $a_j \equiv 0$ for $j \geq n$.
At the points of the subsingular vectors the associated singular vectors are factored out automatically. This happens also when the subsingular vectors are inside a continuous part of the unitarity spectrum. ■

The Proof above is not as explicit as we would like it to be, but due to the lack of space we postpone it to [74]. Below we give separately and explicitly the case $n = 3$.

Example: n=3. For $n = 3$ f-la (28) simplifies to:

$$\begin{array}{c}
d_1 \ > \ d_{12} \ > \ d_2 \ > \ d_{23} \ > \ d_3 \\
\hookrightarrow \ > \ d_{13} \ > \ \nearrow
\end{array}$$

The Theorem now reads:

$$\begin{aligned}
d \ &\geq \ 2 + \tfrac{1}{2}(a_1 + a_2) \ = \ d_1 \ , \quad a_1 \neq 0 \ , \\
d \ &\geq \ \tfrac{3}{2} + \tfrac{1}{2}a_2 \ = \ d_{12} \ , \quad a_1 = 0, \ a_2 \neq 0 \ , \\
d \ &= \ 1 + \tfrac{1}{2}a_2 \ = \ d_2 > d_{13} \ , \quad a_1 = 0, \ a_2 \neq 0 \ , \\
d \ &\geq \ 1 \ = \ d_2 = d_{13} \ , \quad a_1 = a_2 = 0 \ , \\
d \ &= \ \tfrac{1}{2} \ = \ d_{23} \ , \quad a_1 = a_2 = 0 \ , \\
d \ &= \ 0 \ = \ d_3 \ , \quad a_1 = a_2 = 0 \ .
\end{aligned} \tag{35}$$

For $d > d_1$ there are no singular vectors and we have unitarity. At $d = d_1$ there is a singular vector:

$$v^1_{\delta_1} = \Big(a_1(a_1+a_2+1)X_{\delta_1} - a_1 X_{\delta_3}\, X_{13} - (a_1+a_2+1)X_{\delta_2}\, X_1^+ + + X_{\delta_3}\, X_2^+\, X_1^+\Big)\, v_0 \tag{36}$$

which is given in PBW basis, where $X_{\delta_j} \in \mathcal{G}^+$ are the vectors corresponding to the weight vectors δ_j, X_{13} is the compact root vector for $\alpha_{13} = \alpha_1 + \alpha_2 = \delta_1 - \delta_3$. This singular vector is non-trivial for $a_1 \neq 0$ and must be eliminated to obtain an UIR. Below $d < d_1$ there is no unitarity for $a_1 \neq 0$. On the other hand for $a_1 = 0$ the singular vector $v^1_{\delta_1}$ is descendant of the compact root singular vector $X_1^+\, v_0$ which is already factored out for $a_1 = 0$. Thus, below we discuss only the cases with $a_1 = 0$.
The singular vector at $d = d_{12}$ corresponding to the root $\delta_1 + \delta_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3$ is:

$$\begin{aligned} v^1_{\delta_1+\delta_2} = \Big(& X_{\delta_3} X_{\delta_2} X_2^+ X_1^+ + \frac{1}{2}\,(X_{\delta_3})^2 (X_2^+)^2 X_1^+ - a_1\,(X_{\delta_3})^2 X_2^+ X_{13} \\ & + 2\,(a_2+1)\; X_{\delta_3} X_{\delta_2} X_{13} - 2\,(a_1+a_2+1)\; X_{\delta_3} X_{\delta_1} X_2^+ \\ & + (a_1+1)\,(a_1+a_2+1)\; X_{\delta_1+\delta_3} X_2^+ + 4a_2\,(a_1+a_2+1)\; X_{\delta_2} X_{\delta_1} \\ & + 2a_2\,(a_1+a_2+1)\; (X_{\delta_2})^2 X_1^+ - \frac{1}{2}\,(a_1+2a_2+1)\; X_{\delta_2+\delta_3} X_2^+ X_1^+ \\ & - (-a_2 a_1 + a_1 + a_2 + 1)\; X_{\delta_2+\delta_3} X_{13} \\ & - 2\,(a_1+1)\,a_2\,(a_1+a_2+1)\,X_{\delta_1+\delta_2}\Big) v_0 \end{aligned} \tag{37}$$

with norm:

$$16\,(2d - a_2 - 3)\,\big(a_1^2 + 2a_1 + 2d - a_2 - 2\big)\,a_2\,(a_2+1)\,(a_1+a_2+1)\,(a_1+a_2+2)\,.$$

For $a_1 = 0$, $a_2 \neq 0$ it is non-trivial and gives rise to a invariant subspace which must be factored out for unitarity. For $d < d_{12}$ the vector (37) has negative norm and there is no unitarity for $a_2 \neq 0$, except at the isolated unitary point $d = 1 + \frac{1}{2}a_2 = d_2 > d_{13}$. At this point there is a singular vector v_{s2}, while the vector (37) is descendant of compact root singular vector $X_1\, v_0$ and v_{s2}.
Further, we consider $a_1 = a_2 = 0$. Then the vector $v^1_{\delta_1+\delta_2}$ is descendant of compact root singular vectors $X_1^+\, v_0$ and $X_2^+\, v_0$, thus, there is no obstacle for unitarity for $1 < d$. The next reducibility points (coinciding here) are $d = d_{13} = d_2 = 1$. The singular vector for $d = d_2$ and $m = 1$ has weight $\delta_2 = \alpha_2 + \alpha_3$ and is given by:

$$v^1_{\delta_2} = \Big(a_2 X_2^+ X_3^+ - (a_2+1) X_3^+ X_2^+\Big) v_0 \tag{38}$$

For $a_2 = 0$ it is a descendant of the compact root singular vector $X_2^+ v_0$. The singular vector for $d = d_{13} = 1$ and $m = 1$ has weight $\delta_1 + \delta_3 = \alpha_1 + \alpha_2 + 2\alpha_3$ [70] :

$$\begin{aligned} v^1_{\delta_1+\delta_3} \quad = \quad & \Big(ha_1\, X_1^+ (X_3^+)^2 X_2^+ + a_1\, X_1^+ X_3^+ X_2^+ X_3^+ - \\ & - h(a_1+1)(X_3^+)^2 X_2^+ X_1^+ \;- \\ & - ha_1\, X_1^+ X_2^+ (X_3^+)^2 - (a_1+1) X_3^+ X_2^+ X_3^+ X_1^+ \;+ \\ & + h(a_1+1) X_2^+ (X_3^+)^2 X_1^+ \Big), \\ & \qquad h = 1 + \tfrac{1}{2}(a_1 + a_2) \end{aligned} \tag{39}$$

The above vector is given in the simple root basis most appropriate for the case. For $a_1 = 0$ it is a descendant of the compact root singular vector $X_1^+ v_0$. However, there is a subsingular vector:

$$v_{ss} \;=\; (X_{\delta_1} X_{\delta_2} X_{\delta_3} - X_{\delta_3} X_{\delta_2} X_{\delta_1}) v_0 \tag{40}$$

with norm: $16d(d-1)(2d-1)$. This must be factorized in order to obtain UR. Then for $\frac{1}{2} < d < 1$ there will be no unitarity due to the last norm.

Finally, at the next reducibility point: $d = d_{23} = \frac{1}{2}$ there is a singular vector of weight $\delta_2 + \delta_3 = \alpha_2 + 2\alpha_3$:

$$v^1_{\delta_2+\delta_3} \;=\; (2X_{\delta_2+\delta_3} - 4X_{\delta_2} X_{\delta_3} + X_{2\delta_3} X_2^+) v_0 \tag{41}$$

It should be factored out to get unitarity. The subsingular vector (40) has zero norm for $d = \frac{1}{2}$ and furthermore it is descendant of $v^1_{\delta_2+\delta_3}$ and the compact root singular vector $X_2^+ v_0$. Finally, for $d < \frac{1}{2}$ there is no unitarity since then the norm of (41) is negative, except at the trivial isolated unitary point $d = 0 = a_1 = a_2$ of one-dimensional irrep. ■

4. Character formulae

4.1. Character formulae: generalities

In the beginning of this subsection we follow [73]. Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, ($\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i,\alpha_i)$). Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$V \;=\; \mathop{\oplus}_{\mu\in\Gamma_+} V_\mu \quad , \qquad V_\mu \;=\; \{u \in V \mid Hu = (\lambda+\mu)(H)u,\ \forall\ H \in \mathcal{H}\} \tag{42}$$

(Note that $V_0 = \mathbb{C} v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C}$, $c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geq \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1$, $e(\mu)e(\nu) = e(\mu + \nu)$.

Then the (formal) character of V is defined by:

$$ch_0\ V \ = \ \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\Lambda + \mu) \ = \ e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\mu) \quad (43)$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has $\dim\ V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim \mathcal{G}_\beta$ ($= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$ch_0\ V^\Lambda \ = \ e(\Lambda) \sum_{\mu\in\Gamma_+} P(\mu) e(\mu) \ = \ e(\Lambda) \prod_{\alpha\in\Delta^+} (1 - e(\alpha))^{-1}. \quad (44)$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) \ = \ \lambda - (\lambda, \alpha^\vee)\alpha\,, \quad \lambda \in \hat{\mathcal{H}}^*\,, \quad \alpha \in \hat{\Delta}. \quad (45)$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch_0\ L_\Lambda \ = \ \sum_{w\in W} (-1)^{\ell(w)}\ ch_0\ V^{w\cdot\Lambda}\,, \quad \Lambda \in -\Gamma_+ \quad (46)$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda - \rho) + \rho$. For future reference we note:

$$s_\alpha \ \cdot\ \Lambda \ = \ \Lambda \ + \ n_\alpha \alpha \quad (47)$$

where

$$n_\alpha \ = \ n_\alpha(\Lambda) \ \doteq \ (\rho - \Lambda, \alpha^\vee) \ = \ (\rho - \Lambda)(H_\alpha)\,, \quad \alpha \in \Delta^+. \quad (48)$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac [63, 71].[1] For all such superalgebras – except

[1]Kac considers highest weight modules but his results are immediately transferable to lowest weight modules.

$osp(1/2n)$ – the character formula for Verma modules is [63, 71]:

$$ch\ V^\Lambda \ = \ e(\Lambda) \left(\prod_{\alpha\in\Delta_{\bar 0}^+} (1-e(\alpha))^{-1} \right) \left(\prod_{\alpha\in\Delta_{\bar 1}^+} (1+e(\alpha)) \right) . \qquad (49)$$

We are however interested exactly in the $osp(1/2n)$ when the Verma module character formula is:

$$ch\ V^\Lambda \ = \ e(\Lambda) \left(\prod_{\alpha\in\bar\Delta^+} (1-e(\alpha))^{-1} \right) \qquad (50)$$

Naturally, the character formula for the finite-dimensional irreducible LWM L_Λ is again (46) using the Weyl group W_n of B_n.

4.2. Multiplets

A Verma module V^Λ may be reducible w.r.t. to many positive roots, and thus there maybe many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

One main ingredient of the approach of [66] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called *multiplets* [69]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

If a Verma module V^Λ is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., $m_k \in \mathbb{N}$ for all k, then the irreducible submodules are isomorphic to the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ [66]. (Actually, this is a condition only for m_n since $m_k \in \mathbb{N}$ for $k = 1, \ldots, n-1$.) In these cases we have the *main multiplets* which are isomorphic to the Weyl group of $\mathcal{G}^{\mathbb{C}}$ [66].

In the cases of non-dominant weight Λ the character formula for the irreducible LWM is [72] :

$$ch\ L_\Lambda \ = \ \sum_{\substack{w\in W\\ w\leq w_\Lambda}} (-1)^{\ell(w_\Lambda w)}\ P_{w,w_\Lambda}(1)\ ch\ V^{w\cdot(w_\Lambda^{-1}\cdot\Lambda)} , \quad \Lambda\in\Gamma \qquad (51)$$

where $P_{y,w}(u)$ are the Kazhdan–Lusztig polynomials $y, w \in W$ [72] (for an easier exposition see [68]), w_Λ is a unique element of W with minimal length such that the signature of $\Lambda_0 = w_\Lambda^{-1}\cdot\Lambda$ is anti-dominant or semi-anti-dominant:

$$\chi_0 \ = \ (m'_1, \ldots, m'_n), \qquad m'_k \ = \ 1 - \Lambda_0(H_k) \ \in \mathbb{Z}_- \ . \qquad (52)$$

Note that $P_{y,w}(1) \in \mathbb{N}$ for $y \leq w$.

When Λ_0 is semi-anti-dominant, i.e., at least one $m'_k = 0$, then in fact W is replaced by a reduced Weyl group W_R.

Most often the value of $P_{y,w}(1)$ is equal to 1 (as in the character formula for the finite-dimensional irreps), while the cases $P_{y,w}(1) > 1$ are related to the appearance of subsingular vectors, though the situation is more subtle, see [68].

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. In the case of d_{ij} (26) and using (13) we have:

$$m_n(d_{ij}) \;=\; 1 - 2m_j - \cdots - 2m_{n-1} - m_i - \cdots - m_{j-1} \;. \tag{53}$$

In the case of d_i (27) we have:

$$m_n(d_i) \;=\; 1 - 2m_i - \cdots - 2m_{n-1} \;. \tag{54}$$

As expected the weights related to positive energy d are not dominant ($m_n(d_{ij}) \in \mathbb{Z}_-$, $m_n(d_i) \in -\mathbb{N}$, $(i < n)$), since the positive energy UIRs are infinite-dimensional. (Naturally, $m_n(d_n) = 1$ falls out of the picture since $d_n < 0$.)

Thus, the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the w_Λ in the character formula (51). This we do in the next subsection in the case $n = 3$.

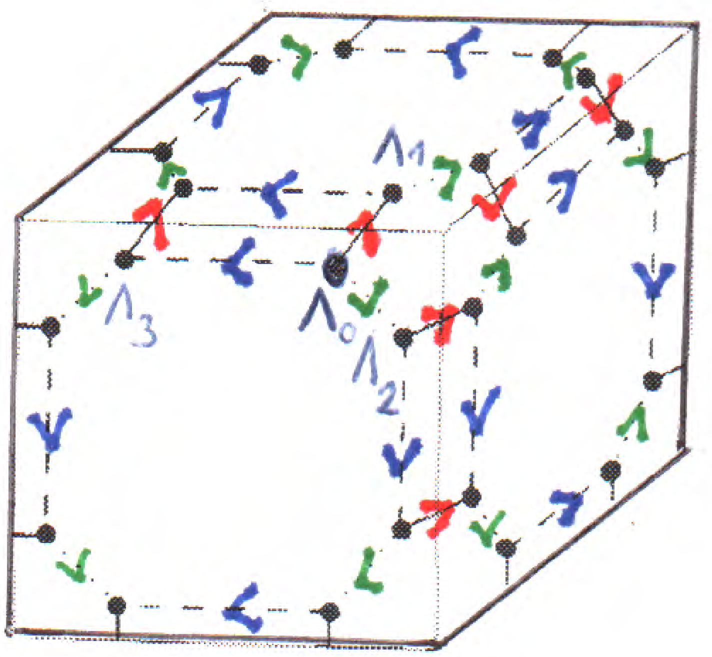
4.3. The case n=3

In order to illustrate what the main ideas we consider the first non-trivial example $n = 3$, i.e., $osp(1/6)$ actually using B_3. The Weyl group W_n of B_n has $2^n n!$ elements, i.e., 48 for B_3. Let $S = (s_1, s_2, s_3)$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

$$s_1^2 = s_2^2 = s_3^2 = e,\; (s_1 s_2)^3 = e,\; (s_2 s_3)^4 = e,\; s_1 s_3 \;=\; s_3 s_1 \,, \tag{55}$$

e being the identity of W_3. The 48 elements may be listed as:

$$
\begin{aligned}
& e\,,\ s_1\,,\ s_2\,,\ s_3 \\
& s_1s_2\,,\ s_1s_3\,,\ s_2s_1\,,\ s_2s_3\,,\ s_3s_2\,, \\
& s_1s_2s_1\,,\ s_1s_2s_3\,,\ s_1s_3s_2\,,\ s_2s_1s_3\,,\ s_2s_3s_2\,,\ s_3s_2s_1\,,\ s_3s_2s_3\,, \\
& s_1s_2s_1s_3\,,\ s_1s_2s_3s_2\,,\ s_1s_3s_2s_1\,,\ s_1s_3s_2s_3\,, \\
& s_2s_3s_2s_1\,,\ s_2s_1s_3s_2\,,\ s_3s_2s_3s_1\,,\ s_3s_2s_3s_2\,, \\
& s_1s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3\,,\ s_1s_2s_1s_3s_2\,,\ s_1s_3s_2s_3s_2\,, \\
& s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_3\,,\ s_3s_2s_3s_1s_2\,,\ s_3s_2s_3s_2s_1\,, \\
& s_1s_3s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3s_2\,,\ s_1s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_1s_3\,, \\
& s_2s_1s_3s_2s_3s_2\,,\ s_3s_2s_3s_1s_2s_1\,,\ s_3s_2s_3s_1s_2s_3\,, \\
& s_2s_1s_3s_2s_3s_2s_1\,,\ s_2s_1s_3s_2s_3s_1s_2\,,\ s_3s_2s_1s_2s_3s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_1s_3\,,\ s_3s_2s_3s_1s_2s_3s_2\,, \\
& s_2s_3s_2s_1s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_1s_2\,,
\end{aligned}
\tag{56}
$$



This Weyl group may be pictorially represented on a cube as in the figure, where we have given only the simple root reflections, namely, continuous (red) arrows represent action of reflection s_1, dashed (blue) arrows represent action of reflection s_2, dotted (green) arrows represent action of reflection s_3. Each face of the cube contains eight elements related by blue and green arrows representing the Weyl group of B_2 generated by s_2 and s_3. The figure contains also eight sextets (around the eight corners of the cube). Each sextet is related by red and green arrows representing the Weyl group of A_2 generated by s_1 and s_2. Finally there are 12 quartets (straddling the edges of the cube). Each quartet is formed by red and blue arrows representing the Weyl group of $A_1 \times A_1$ generated by the commuting reflections s_1 and s_3.

We use the same diagram to depict the main multiplets containing the Verma modules V^{Λ_0} which contain (as factor module) the finite-dimensional irreps of B_3, i.e., with dominant weights Λ_0, i.e., with Dynkin labels (m_1, m_2, m_3), $m_k \in I\!N$. We may do this since these multiplets are isomorphic to the Weyl group, W_3 in our case. On the picture we have indicated the modules, Λ_0 and $\Lambda_k = s_k \cdot \Lambda_0$, $k = 1, 2, 3$. The mentioned isomorphism is fixed by assigning to Λ_0 the identity element e of W_3, and to Λ_k the reflections s_k.

The character formula for the Verma modules in our case is given explicitly by:

$$\begin{aligned} ch\ V^{\Lambda} &= \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)} \times \\ &\times \frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \end{aligned} \tag{57}$$

where $t_j \equiv e(\alpha_j)$.

Now we give the character formulae of the five boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight Λ_0 which determines the main multiplet by (m'_1, m'_2, m'_3), $m'_k \in I\!N$, using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some $m'_k = 0$.

• In the case of $d = d_1 = 2 + \frac{1}{2}(a_1 + a_2)$ there are twelve members of the multiplet which is a submultiplet of a main multiplet. (Remember that that $m_1 > 1$ since $a_1 \neq 0$.) They are grouped into two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_1}}$, where $\Lambda_0^{d_1} = w \cdot \Lambda_0$, $w = w_{\Lambda_0^{d_1}} = s_2 s_1 s_3 s_2 s_3$, with signature:

$$\Lambda_0^{d_1} \ : \ (m_1, m_2, m'_3 = 1 - 2m_{12}) \ , \quad m_1, m_2 \in I\!N \ , \quad m_{12} \equiv m_1 + m_2 \ . \tag{58}$$

The other submultiplet starts from $V^{\Lambda'_0}$ with $\Lambda'_0 = \Lambda_0^{d_1} + \delta_1 = \Lambda_0^{d_1} + \alpha_1 + \alpha_2 + \alpha_3$, with signature: $\Lambda'_0 \ : \ (m_1 - 1, m_2, m'_3 = 1 - 2m_{12})$, $m_1 > 1$.

The character formula is (51) with $w_\Lambda = w_{\Lambda_0^{d_1}}$:

$$\mathrm{ch}\,\Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times \{ \ \mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2) \ - \ t_1t_2t_3\,\mathrm{ch}\,\Lambda_{m_1-1,m_2}(t_1,t_2) \ \} \ , \quad m_1 > 1 \qquad (59)$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2)$ is the normalized character of the finite-dimensional $sl(3)$ irrep with Dynkin labels (m_1,m_2) (and dimension $m_1m_2(m_1+m_2)/2$):

$$\mathrm{ch}\ \Lambda_{m_1,m_2}(t_1,t_2) \ = \ \frac{1-t_1^{m_1}-t_2^{m_2}+t_1^{m_1}t_2^{m_{12}}+t_1^{m_{12}}t_2^{m_2}-t_1^{m_{12}}t_2^{m_{12}}}{(1-t_1)(1-t_2)(1-t_1t_2)} \qquad (60)$$

Naturally, the latter formula is a polynomial in t_1,t_2 , e.g., $\mathrm{ch}\ \Lambda_{1,1}(t_1,t_2) \ = \ 1.$ Note that (59) trivializes for $m_1 = 1$ since the second term disappears by the formal substitution: $\mathrm{ch}\ \Lambda_{0,m_2}(t_1,t_2) \ = \ 0.$

• In the case of $d = d_{12} = \frac{1}{2}(3+a_2)$ which is relevant for unitarity, i.e., $m_1 = 1$, there are again twelve members of the multiplet. The corresponding signature is:

$$\Lambda_0^{d_{12}} \ : \ (1, m_2, m_3' = -2m_2) \ , \quad m_2 \in I\!N \ . \qquad (61)$$

The multiplet is submultiplet of a reduced multiplet with 24 members obtained from a main multiplet for $m_3' = 0$. As above our multiplet consists of two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_{12}}}$, where $\Lambda_0^{d_{12}} = w \cdot \Lambda_0$, $w = w_{\Lambda_0^{d_{12}}} = s_3s_2s_1$. The other submultiplet starts from $V^{\Lambda_0'}$ with $\Lambda_0' = \Lambda_0^{d_{12}} + m_2(\alpha_1 + 2\alpha_2 + 2\alpha_3) = \Lambda_0^{d_{12}} + m_2(\delta_1+\delta_2)$ with signature: $\Lambda_0' : (1, m_2-1, -2m_2)$. The character formula is (51), with $W \mapsto W_R$, (where W_R is a reduced 24-member Weyl group) and with $w_\Lambda = w_{\Lambda_0^{d_{12}}}$:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times\{ \ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) \ - \ (t_1t_2^2t_3^2)^{m_2}\,\mathrm{ch}\,\Lambda_{1,m_2-1}(t_1,t_2) \ \} \ , \quad m_2 > 1 \qquad (62)$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}$ are the $sl(3)$ characters defined in (60).

• In the case of $d = d_2 = 1 + \frac{1}{2}a_2 \geq d_{13}$, i.e., $m_3' = 1-2m_2$, the corresponding signature is:

$$\Lambda_0^{d_2} \ : \ (m_1, m_2, m_3' = 1-2m_2) \ . \qquad (63)$$

We should consider two subcases

$$1 + m_1 - m_2 > 0 \qquad \text{or} \qquad 1 + m_1 - m_2 \le 0$$

We start with the *first subcase* which is relevant when $d = d_2 = d_{13} = 1$ and $a_1 = a_2 = 0$, then $m_1 = m_2 = 1$, and the signature is:

$$\Lambda_0^{d_2=d_{13}} \ : \ (1,1,-1) \ . \tag{64}$$

Our multiplet is a submultiplet of a 12-member reduced multiplet obtained when the signature of Λ_0 is $(m'_1, m'_2, m'_3) = (1,0,1)$, and then $\Lambda_0^{d_2=d_{13}}$ is a submodule with signature (64). Thus, we have $\Lambda_0^{d_2=d_{13}} = s_3 \cdot \Lambda_0$, i.e., $w_{\Lambda_0^{d_2=d_{13}}} = s_3$.

Explicitly, our 12-member multiplet has two $sl(3)$ submultiplets. First we take into account a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}}$ with parameters $(1,1)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}} + \alpha_1 + 2\alpha_2 + 3\alpha_3$ with parameters $(1,1)$. Note that that $\alpha_1 + 2\alpha_2 + 3\alpha_3 = \delta_1 + \delta_2 + \delta_3$ is the weight of the subsingular vector (40).

The character formula is (51), with $W \mapsto W_R$, (where W_R is a reduced 12-member Weyl group) and $w_\Lambda = s_3$:

$$\begin{aligned} &\mathrm{ch}\, \Lambda_0^{d_2=d_{13}} = \\ &= \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \ 1 \ - \ t_1 t_2^2 t_3^3 \ \} \end{aligned} \tag{65}$$

• In the case of $d = d_2 = 1 + \frac{1}{2}a_2 > d_{13} = 1$, i.e., $m_1 = 1$, $m_2 = 1 + a_2 > 1$, thus, this is the subcase $1 + m_1 - m_2 = m_{13} \le 0$. The multiplet has 24 members for $m_2 > 2$ $(m_{13} < 0)$ and starts with $\Lambda_0^{'d_2} = \ = s_3\, s_2\, s_1 \cdot \Lambda_0$, with signatures:

$$\begin{aligned} &\Lambda_0 \ : \ (m_2 - 2, 1, 1) \ , \\ &\Lambda_0^{'d_2} \ : \ (1, m_2, m'_3 = 1 - 2m_2) \ , \quad m_2 \in 1 + I\!N \ . \end{aligned} \tag{66}$$

It has four $sl(3)$ submultiplets. First we take into account a $sl(3)$ sextet starting from $\Lambda_0^{'d_2}$ with parameters $(1, m_2)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{'d_2} + \alpha_{23}$ with parameters $(2, m_2 - 1)$. Then there is a $sl(3)$ sextet with parameters $(2, m_2 - 2)$ starting from a Verma module $V^{\Lambda''}$, $\Lambda'' = \Lambda_0^{'d_2} + \alpha_1 + 3\alpha_{23}$. Finally, there is a $sl(3)$ sextet with parameters $(1, m_2 - 2)$, starting from a Verma module $V^{\Lambda'''}$, $\Lambda''' = \Lambda_0^{'d_2} + 2(\alpha_1 + 2\alpha_2 +$

$2\alpha_3$).
We have the *Conjecture* that the character formula is (51) and $w_\Lambda = s_3\, s_2\, s_1$:

$$\mathrm{ch}\,\Lambda_0^{\prime d_2} = \frac{e(\Lambda_0^{\prime d_2})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$
$$\times \{\ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2)\ -\ t_2t_3\,\mathrm{ch}\,\Lambda_{2,m_2-1}(t_1,t_2)\ +$$
$$+\ t_1t_2^3t_3^3\,\mathrm{ch}\,\Lambda_{2,m_2-2}(t_1,t_2)\ -\ t_1^2t_2^4t_3^4\,\mathrm{ch}\,\Lambda_{1,m_2-2}(t_1,t_2)\ \} \tag{67}$$

When $m_2 = 2$ ($a_2 = 1$, $m_{13} = 0$) the weight Λ_0 is semi dominant, the main multiplet reduces to 24 members, our multiplet reduces to only 12 members, consisting of the first two $sl(3)$ submultiplets mentioned above. The character formula takes this into account by construction since for $m_2 = 2$ the terms in the 2nd row are automatically zero (due to the fact that the $sl(3)$ character formula gives zero: $\mathrm{ch}\,\Lambda_{1,0}(t_1,t_2)\ =\ 0$).

• In the case of $d = d_{23} = \frac{1}{2}$, $a_1 = a_2 = 0$, i.e., $m_1 = m_2 = 1$, and the signature is:

$$\Lambda_0^{d_{23}}\ :\ (1,1,0)\ . \tag{68}$$

This is in fact a multiplet with 24 members which is reduction of the main multiplet starting with the semi dominant weight (68).

The multiplet consists of four $sl(3)$ submultiplets. First there is a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}}$ with parameters $(1,1)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}}\ +\ \alpha_2 + 2\alpha_3$ with parameters $(2,1)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}}\ +\ \alpha_1 + 2\alpha_2 + 4\alpha_3$ with parameters $(1,2)$. Then a $sl(3)$ sextet starting from $\Lambda_0^{d_{23}}\ +\ 2\alpha_1 + 4\alpha_2 + 6\alpha_3$ with parameters $(1,1)$.

The character formula is (51), however, with $W\ \mapsto W_R$, where W_R is the reduced 24-member Weyl group, (generated by $s_1, s_2, s_3s_2s_3$)

and $w_\Lambda = 1$:

$$\begin{aligned}
\mathrm{ch}\,\Lambda_0^{d_{23}} \;&=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \\
&\quad \times \{\; 1 \;-\; t_1t_2^2t_3^4\,\mathrm{ch}\,\Lambda_{1,2}(t_1,t_2) \;+ \\
&\quad +\; t_2t_3^2\,\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2) \;-\; t_1^2t_2^4t_3^6 \;\} \;= \\
=\; &\frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \\
&\quad \times \{\; 1 \;-\; t_1t_2^2t_3^4\,(1+t_2+t_1t_2) \;+ \\
&\quad +\; t_2t_3^2\,(1+t_1+t_1t_2) \;-\; t_1^2t_2^4t_3^6 \;\}
\end{aligned} \tag{69}$$

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Derivation of the trigonometric Gaudin Hamiltonians*

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Abstract

Following Sklyanin's proposal in the rational case, we derive the generating function of the Gaudin Hamiltonians in the trigonometric case. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXZ Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians are obtained as the residues of the generating function.

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1. Introduction

Gaudin models were introduced as interacting spins in a chain [1, 2, 3, 4]. In this approach, these models were obtained as a quasi-classical limit of the integrable quantum chains. Moreover, the Gaudin models were extended to any simple Lie algebra, with arbitrary irreducible representation at each site of the chain [4].

The rational $s\ell(2)$ invariant model was studied in the framework of the quantum inverse scattering method [5]. In his studies, Sklyanin used the $s\ell(2)$ invariant classical r-matrix [5]. A generalization of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang-Baxter equation [6] was relatively straightforward [7, 8]. Therefore, considerable

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attention has been devoted to Gaudin models corresponding to the the classical r-matrices of simple Lie algebras [9, 10] and Lie superalgebras[11, 12]. In the case of the $s\ell(2)$ Gaudin system, its relation to Knizhnik-Zamolodchikov equation of conformal field theory [13, 14, 15] or the method of Gauss factorization [16], provided alternative approaches to computation of correlation functions. The non-unitary r-matrices and the corresponding Gaudin models have been studied recently, see [17, 18] and the references therein. In [19] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms. Moreover, we have implemented the algebraic Bethe ansatz, based on the appropriate non-unitary r-matrices and the corresponding linear bracket, obtaining the spectrum of the generating function and the corresponding Bethe equations [19].

Here, following Sklyanin's proposal in the rational case [5, 19], we derive the generating function of the Gaudin Hamiltonians in the trigonometric case. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXZ Heisenberg spin chain and the central element, the so-called quantum determinant.

2. Inhomogeneous XXZ Heisenberg spin chain

With the aim deriving the Gaudin Hamiltonians in the trigonometric case, we consider the R-matrix of the XXZ Heisenberg spin chain [20, 21, 22]

$$R(\lambda,\eta) = \begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix}. \tag{1}$$

This R-matrix satisfies the Yang-Baxter equation in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ [24, 23]

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu). \tag{2}$$

Here we study the inhomogeneous XXZ spin chain with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ and an inhomogeneous parameter α_m. The Hilbert space of the system is

$$\mathcal{H} = \underset{m=1}{\overset{N}{\otimes}} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{3}$$

We introduce the Lax operator as the following two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$,

$$\mathbb{L}_{0m}(\lambda) = \frac{1}{\sinh(\lambda)} \begin{pmatrix} \sinh\left(\lambda \mathbb{1}_m + \eta S_m^3\right) & \sinh(\eta) S_m^- \\ \sinh(\eta) S_m^+ & \sinh\left(\lambda \mathbb{1}_m - \eta S_m^3\right) \end{pmatrix}. \tag{4}$$

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the R-matrix,

$$\mathbb{L}_{0m}(\lambda) = \frac{1}{\sinh(\lambda)} R_{0m}\left(\lambda - \eta/2\right).$$

Due to the commutation relations (34), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda-\mu)\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0'm}(\mu-\alpha_m) = \mathbb{L}_{0'm}(\mu-\alpha_m)\mathbb{L}_{0m}(\lambda-\alpha_m)R_{00'}(\lambda-\mu). \tag{5}$$

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N)\cdots\mathbb{L}_{01}(\lambda - \alpha_1) \tag{6}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter η and the inhomogeneous parameters $\{\alpha_j, j = 1, \ldots, N\}$. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{7}$$

From RLL-relations (5) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda - \mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda - \mu). \tag{8}$$

The periodic boundary conditions and the RTT-relations (8) imply that the transfer matrix

$$t(\lambda) = \mathrm{tr}_0 T(\lambda), \tag{9}$$

commute at different values of the spectral parameter,

$$[t(\mu), t(\nu)] = 0. \tag{10}$$

The RTT-relations (8) admit a central element [5]

$$\Delta\left[T(\lambda)\right] = \mathrm{tr}_{00'} P^{-}_{00'} T_0\left(\lambda - \eta/2\right) T_{0'}\left(\lambda + \eta/2\right), \tag{11}$$

where

$$P^{-}_{00'} = \frac{-1}{2\sinh(\eta)}\, R_{00'}(-\eta) = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2}, \tag{12}$$

where $\mathbb{1}$ is the identity and $\mathcal{P}$ is the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$. A straightforward calculation shows that $\Delta\left[T(\lambda)\right]$ is a scalar operator

$$\Delta\left[T(\lambda)\right] = \prod_{m=1}^{N} \frac{\sinh\left(\lambda - \alpha_m + \frac{(2s_m+1)\eta}{2}\right)\sinh\left(\lambda - \alpha_m - \frac{(2s_m+1)\eta}{2}\right)}{\sinh\left(\lambda - \alpha_m + \frac{\eta}{2}\right)\sinh\left(\lambda - \alpha_m - \frac{\eta}{2}\right)}, \tag{13}$$

and therefore, it is evidently central,

$$\Big[\Delta\left[T(\lambda)\right]\,,\,T(\nu)\Big] = 0. \tag{14}$$

In the next section we will seek a linear combination of the transfer matrix (9) and the central element (11) whose quasi-classical expansion yields the generating function of the trigonometric Gaudin Hamiltonians in the case when the periodic boundary conditions are imposed.

3. Trigonometric Gaudin model

The Gaudin models were introduce as a quasi-classical limit of the integrable quantum chains [3, 4]. Therefore it is to be expected that the generating function of the trigonometric Gaudin Hamiltonians could be obtained from the quasi-classical expansion of the transfer matrix of the periodic XXZ Heisenberg spin chain. Thus, our first step is to consider the expansion of the monodromy matrix (6) with respect to the quasi-classical parameter η

$$\begin{aligned}
T(\lambda) &= \mathbb{1} + \eta \textstyle\sum_{m=1}^{N} \frac{\sigma_0^3 \otimes \cosh(\lambda-\alpha_m) S_m^3 + \frac{1}{2}\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)}{\sinh(\lambda-\alpha_m)} + \frac{\eta^2}{2} \mathbb{1}_0 \otimes \sum_{m=1}^{N} \left(S_m^3\right)^2 \\
&+ \frac{\eta^2}{2} \textstyle\sum_{\substack{n,m=1 \\ n\neq m}}^{N} \frac{\mathbb{1}_0 \otimes \left(\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \textstyle\sum_{m=1}^{N} \sum_{n<m}^{N} \frac{\sigma_0^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^-\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \textstyle\sum_{m=1}^{N} \sum_{n<m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^-\right)}{2\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \\
&+ \frac{\eta^2}{2} \textstyle\sum_{m=1}^{N} \sum_{n>m}^{N} \frac{\sigma_0^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^-\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} \\
&+ \frac{\eta^2}{2} \textstyle\sum_{m=1}^{N} \sum_{n>m}^{N} \frac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^-\right)}{2\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} + \mathcal{O}(\eta^3).
\end{aligned} \tag{15}$$

It is important to notice that the spin operators S_m^α, with $\alpha = +, -, 3$, on the right hand side of (15) satisfy the usual commutation relations

$$[S_m^3, S_n^\pm] = \pm S_m^\pm\, \delta_{mn}, \quad [S_m^+, S_n^-] = 2S_m^3\, \delta_{mn}. \tag{16}$$

If the Gaudin Lax matrix is defined by

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\sigma_0^3 \otimes \cosh(\lambda-\alpha_m) S_m^3 + \frac{1}{2}\left(\sigma_0^+ \otimes S_m^- + \sigma_0^- \otimes S_m^+\right)}{\sinh(\lambda-\alpha_m)} \tag{17}$$

and the quasi-classical property of the R-matrix (1) [23]

$$\frac{1}{\sinh(\lambda)} R(\lambda) = \mathbb{1} - \eta r(\lambda) + \mathcal{O}(\eta^2), \tag{18}$$

where

$$r(\lambda) = \frac{-1}{2\sinh(\lambda)} \left(\cosh(\lambda)(\mathbb{1} \otimes \mathbb{1} + \sigma^3 \otimes \sigma^3) + \frac{1}{2}\left(\sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+\right) \right), \tag{19}$$

is taken into account, then substitution of the expansion (15) into the RTT-relations (8) yields the so-called Sklyanin linear bracket [5]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]\,. \tag{20}$$

The classical r-matrix (19) has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda), \tag{21}$$

and satisfies the classical Yang-Baxter equation [6]

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)] = 0. \tag{22}$$

Thus the Sklyanin linear bracket (20) is anti-symmetric and it obeys the Jacobi identity. It follows that the entries of the Lax matrix (17) generate a Lie algebra relevant for the Gaudin model.

Using the expansion (15) it is evident that

$$t(\lambda) = 2 + \eta^2 \sum_{m=1}^{N} \Bigg(\left(S_m^3\right)^2 + \textstyle\sum_{n\neq m}^{N} \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \Bigg) + \mathcal{O}(\eta^3). \tag{23}$$

Analogously, we can expand (11) to obtain

$$
\begin{aligned}
&\Delta\left[T(\lambda)\right] = \mathbb{1} + \eta\, \mathrm{tr} L(\lambda) + \tfrac{\eta^2}{2}\left(\mathrm{tr}^2 L(\lambda) - \mathrm{tr}\, L^2(\lambda)\right) \\
&+\eta^2 \textstyle\sum_{m=1}^N \left(\left(S_m^3\right)^2 + \sum_{n\neq m}^N \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right) \\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P_{00'}^- \textstyle\sum_{m=1}^N \sum_{m>n}^N \left(\frac{\sigma_0^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^+\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\,\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right. \\
&+ \left. \tfrac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^+\right)}{2\,\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right) \mathbb{1}_{0'} \\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P_{00'}^- \textstyle\sum_{m=1}^N \sum_{n>m}^N \left(\frac{\sigma_0^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^+\right) + \sigma_0^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\,\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} \right. \\
&+ \left. \tfrac{\sigma_0^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^+\right)}{2\,\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_n)} \right) \mathbb{1}_{0'} \\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P_{00'}^- \mathbb{1}_0 \textstyle\sum_{m=1}^N \sum_{m>n}^N \left(\frac{\sigma_{0'}^3 \otimes \left(S_m^- S_n^+ - S_m^+ S_n^+\right) + \sigma_{0'}^+ \otimes \left(\cosh(\lambda-\alpha_m) S_m^3 S_n^- - \cosh(\lambda-\alpha_n) S_m^- S_n^3\right)}{2\,\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right. \\
&+ \left. \tfrac{\sigma_{0'}^- \otimes \left(\cosh(\lambda-\alpha_n) S_m^+ S_n^3 - \cosh(\lambda-\alpha_m) S_m^3 S_n^+\right)}{2\,\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right) \\
&+\tfrac{\eta^2}{2}\mathrm{tr}_{00'} P_{00'}^- \mathbb{1}_0 \textstyle\sum_{m=1}^N \sum_{n>m}^N \left(\frac{\sigma_{0'}^3 \otimes \left(S_n^- S_m^+ - S_n^+ S_m^+\right) + \sigma_{0'}^+ \otimes \left(\cosh(\lambda-\alpha_n) S_n^3 S_m^- - \cosh(\lambda-\alpha_m) S_n^- S_m^3\right)}{2\,\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_m)} \right. \\
&+ \left. \tfrac{\sigma_{0'}^- \otimes \left(\cosh(\lambda-\alpha_m) S_n^+ S_m^3 - \cosh(\lambda-\alpha_n) S_n^3 S_m^+\right)}{2\,\sinh(\lambda-\alpha_n)\,\sinh(\lambda-\alpha_n)} \right) + \mathcal{O}(\eta^3),
\end{aligned}
\tag{24}
$$

where $L(\lambda)$ is given in (17). The final expression for the expansion of $\Delta\left[T(\lambda)\right]$ is obtained after taking all the traces

$$
\begin{aligned}
&\Delta\left[T(\lambda)\right] = \mathbb{1} + \eta^2 \\
&\times \sum_{m=1}^N \left(\left(S_m^3\right)^2 + \sum_{n\neq m}^N \frac{\cosh(\lambda-\alpha_m)\cosh(\lambda-\alpha_n)\, S_m^3 S_n^3 + \frac{1}{2}\left(S_m^+ S_n^- + S_m^- S_n^+\right)}{\sinh(\lambda-\alpha_m)\,\sinh(\lambda-\alpha_n)} \right) \\
&- \frac{\eta^2}{2}\mathrm{tr}\, L^2(\lambda) + \mathcal{O}(\eta^3).
\end{aligned}
\tag{25}
$$

To obtain the generation function of the Gaudin Hamiltonians notice that (23) and (25) yield

$$
t(\lambda) - \Delta\left[T(\lambda)\right] = \mathbb{1} + \frac{\eta^2}{2}\mathrm{tr}\, L^2(\lambda) + \mathcal{O}(\eta^3). \tag{26}
$$

Therefore

$$
\tau(\lambda) = \frac{1}{2}\mathrm{tr}\, L^2(\lambda) \tag{27}
$$

commute for different values of the spectral parameter,

$$
[\tau(\lambda), \tau(\mu)] = 0. \tag{28}
$$

Moreover, substituting (17) into (27) it is straightforward to obtain the expansion

$$\tau(\lambda) = \sum_{m=1}^{N} \frac{s_m(s_m+1)}{\sinh^2(\lambda-\alpha_m)} + 2\sum_{m=1}^{N} \coth(\lambda-\alpha_m)\, H_m + (S^3_{gl})^2\,, \tag{29}$$

with the Gaudin Hamiltonians

$$H_m = \sum_{n\neq m}^{N} \coth(\alpha_m-\alpha_n)\, S^3_m S^3_n + \frac{1}{2\sinh(\alpha_m-\alpha_n)} \left(S^+_m S^-_n + S^-_m S^+_n\right) \tag{30}$$

and the global generator

$$S^3_{gl} = \sum_{m=1}^{N} S^3_m. \tag{31}$$

The global generator defined above generates the $U(1)$ symmetry

$$\left[S^3_{gl}, H_m\right] = 0, \quad \text{with} \quad m = 1, 2\ldots N. \tag{32}$$

Evidently, we have

$$[H_m, H_n] = 0, \quad \text{with} \quad m, n = 1, 2\ldots N. \tag{33}$$

This shows that $\tau(\lambda)$ is the generating function of Gaudin Hamiltonians (30) when the periodic boundary conditions are imposed [5].

4. Conclusion

Following Sklyanin's proposal [5, 19], we have derive the generating function of the Gaudin Hamiltonians in the trigonometric case by considering the quasi-classical expansion of the linear combination of the transfer matrix of the XXZ Heisenberg spin chain and the corresponding quantum determinant. The Gaudin Hamiltonians are obtained as the residues of the generating function. It would be of considerable interest to generalise these results to the case of non-periodic boundary conditions.

A Basic definitions

We consider the operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations [25]

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{\sinh(2\eta S^3)}{\sinh(\eta)} = [2S^3]_q, \tag{34}$$

with $q = e^{\eta}$. In the space $\mathbb{C}^{2s+1}$ these operators admit the following matrix representation [25, 26, 27]

$$S^3 = \sum_{i=1}^{2s+1} a_i e_{i\,i}, \quad S^+ = \sum_{i=1}^{2s+1} b_i e_{i\,i+1}, \quad \text{and} \quad S^- = \sum_{i=1}^{2s+1} b_i e_{i+1\,i} \tag{35}$$

where

$$(e_{ij})_{kl} = \delta_{i\,k}\delta_{j\,l}, \quad a_i = s{+}1{-}i, \quad b_i = \sqrt{[i]_q\ [2s+1-i]_q} \quad \text{and} \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{36}$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S^\alpha_m = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{37}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

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Creation operators of the non-periodic $s\ell(2)$ Gaudin model*

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Abstract

We define new creation operators relevant for implementation of the algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with the general reflection matrix. This approach is based on the linear bracket corresponding to the relevant non-unitary classical r-matrix.
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1. Introduction

In [1] we have derived the generating function of the $s\ell(2)$ Gaudin Hamiltonians with boundary terms. We have shown that the implementation of the algebraic Bethe ansatz requires an appropriate non-unitary r-matrices and the corresponding linear bracket [1]. The non-unitary r-matrices and the corresponding Gaudin models have been studied recently, see [2, 3] and the references therein. In [1] we have obtained the spectrum of the generating function and the corresponding Bethe equations. However, explicit and compact form of the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, remained open. Our aim here is to propose creation operators which should solve this problem.

2. $s\ell(2)$ Gaudin model with boundary terms

The classical r-matrix relevant for the $s\ell(2)$ Gaudin model is given by [4]

$$r(\lambda) = -\frac{\mathcal{P}}{\lambda}, \tag{1}$$

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where $\mathcal{P}$ is the permutation matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$. This classical r-matrix satisfies the classical Yang-Baxter equation

$$[r_{13}(\lambda), r_{23}(\mu)] + [r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)] = 0, \tag{2}$$

and it has the unitarity property

$$r_{21}(-\lambda) = -r_{12}(\lambda). \tag{3}$$

The general solution of the corresponding classical reflection equation [5, 6, 7]:

$$\begin{aligned} r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu) = \\ = K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda) + K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu), \end{aligned} \tag{4}$$

is give by [1]

$$\widetilde{K}(\lambda) = \left(\begin{array}{cc} \xi - \lambda & \widetilde{\psi}\lambda \\ \widetilde{\phi}\lambda & \xi + \lambda \end{array} \right). \tag{5}$$

An important preliminary step in the implementation of the algebraic Bethe ansatz for the open Gaudin model is to bring the K-matrix (5) to the upper, or lower, triangular form [1]

$$K(\lambda) = U^{-1}\widetilde{K}(\lambda)U = \left(\begin{array}{cc} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{array} \right), \tag{6}$$

where $\psi = \widetilde{\phi} + \widetilde{\psi}$ and

$$U = \left(\begin{array}{cc} -1-\nu & \widetilde{\phi} \\ \widetilde{\phi} & -1-\nu \end{array} \right), \tag{7}$$

with $\nu = \sqrt{1 + \widetilde{\phi}\,\widetilde{\psi}}$.

Here we study the $s\ell(2)$ Gaudin model with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. The Hilbert space of the system is

$$\mathcal{H} = \mathop{\otimes}_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{8}$$

Following [1] we introduce the Lax operator

$$\mathcal{L}_0(\lambda) = \left(\begin{array}{cc} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{array} \right) = \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda) \cdot \vec{S}_m}{\lambda + \alpha_m} \right), \tag{9}$$

with the following local realisation for the entries of the Lax matrix

$$E(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^+}{\lambda - \alpha_m} + \frac{(\xi + \lambda\nu) S_m^+}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right), \tag{10}$$

$$F(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^-}{\lambda - \alpha_m} + \frac{(\xi - \lambda\nu)^2 S_m^- - \lambda^2\psi^2 S_m^+ - 2\lambda\psi(\xi - \lambda\nu) S_m^3}{(\xi + \lambda\nu)(\xi - \lambda\nu)(\lambda + \alpha_m)} \right), \tag{11}$$

$$H(\lambda) = \sum_{m=1}^{N} \left(\frac{S_m^3}{\lambda - \alpha_m} + \frac{\lambda\psi\, S_m^+ + (\xi - \lambda\nu) S_m^3}{(\xi - \lambda\nu)(\lambda + \alpha_m)} \right). \tag{12}$$

Due to the commutation relations (36), it is straightforward to check that the Lax operator (9) satisfies the following linear bracket relations

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r_{00'}^K(\lambda, \mu), \mathcal{L}_0(\lambda) \right] - \left[r_{0'0}^K(\mu, \lambda), \mathcal{L}_{0'}(\mu) \right], \tag{13}$$

where the non-unitary r-matrix is give by

$$r_{00'}^K(\lambda, \mu) = r_{00'}(\lambda - \mu) - K_{0'}(\mu) r_{00'}(\lambda + \mu) K_{0'}^{-1}(\mu). \tag{14}$$

The commutator (13) is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix (14) satisfies the classical Yang-Baxter equation

$$[r_{32}^K(\lambda_3, \lambda_2), r_{13}^K(\lambda_1, \lambda_3)] + [r_{12}^K(\lambda_1, \lambda_2), r_{13}^K(\lambda_1, \lambda_3) + r_{23}^K(\lambda_2, \lambda_3)] = 0. \tag{15}$$

The linear bracket (13) based on the r-matrix $r_{00'}^K(\lambda, \mu)$ (14), corresponding to (6) and the classical r-matrix (1), defines the Lie algebra relevant for the open $s\ell(2)$ Gaudin model.

As it was shown in [1], it is instructive to introduce the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the original ones

$$e(\lambda) = \tfrac{-\xi + \lambda\nu}{\lambda} E(\lambda), \quad h(\lambda) = \tfrac{1}{\lambda} \left(H(\lambda) - \tfrac{\psi\lambda}{2\xi}\, E(\lambda) \right),$$
$$f(\lambda) = \tfrac{1}{\lambda} \left((\xi + \lambda\nu) F(\lambda) + \psi\lambda H(\lambda) \right). \tag{16}$$

The key observation is that in the new basis we have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = [f(\lambda), f(\mu)] = 0. \tag{17}$$

Therefore there are only three nontrivial relations

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2 - \mu^2} \left(e(\mu) - e(\lambda) \right), \tag{18}$$

$$\begin{aligned}[h(\lambda), f(\mu)] &= \frac{-2}{\lambda^2 - \mu^2} \left(f(\mu) - f(\lambda) \right) - \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi} \left(\mu^2 h(\mu) - \lambda^2 h(\lambda) \right) \\ &\quad - \frac{\psi^2}{(\lambda^2 - \mu^2)\xi^2} \left(\mu^2 e(\mu) - \lambda^2 e(\lambda) \right),\end{aligned} \tag{19}$$

$$\begin{aligned}[e(\lambda), f(\mu)] &= \frac{2\psi\nu}{(\lambda^2 - \mu^2)\xi} \left(\mu^2 e(\mu) - \lambda^2 e(\lambda) \right) \\ &\quad - \frac{4}{\lambda^2 - \mu^2} \left((\xi^2 - \mu^2\nu^2) h(\mu) - (\xi^2 - \lambda^2\nu^2) h(\lambda) \right).\end{aligned} \tag{20}$$

In the Hilbert space $\mathcal{H}$ (8), in every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3 \omega_m = s_m \omega_m \quad \text{and} \quad S_m^+ \omega_m = 0. \tag{21}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{22}$$

From the definitions above, the formulas (10) - (12) and (16) it is straightforward to obtain the action of the generators $e(\lambda)$ and $h(\lambda)$ on the vector Ω_+

$$e(\lambda)\Omega_+ = 0, \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \tag{23}$$

with

$$\rho(\lambda) = \frac{1}{\lambda} \sum_{m=1}^{N} \left(\frac{s_m}{\lambda - \alpha_m} + \frac{s_m}{\lambda + \alpha_m} \right) = \sum_{m=1}^{N} \frac{2 s_m}{\lambda^2 - \alpha_m^2}. \tag{24}$$

The generating function of the Gaudin Hamiltonians with boundary terms is given by [1]:

$$\begin{aligned}\tau(\lambda) = \mathrm{tr}_0\, \mathcal{L}_0^2(\lambda) &= 2\lambda^2 \left(h^2(\lambda) + \frac{2\nu^2}{\xi^2 - \lambda^2\nu^2} h(\lambda) - \frac{h'(\lambda)}{\lambda} \right) \\ &\quad - \frac{2\lambda^2}{\xi^2 - \lambda^2\nu^2} \left(f(\lambda) + \frac{\psi\lambda^2\nu}{\xi} h(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2} e(\lambda) - \frac{\psi\nu}{\xi} \right) e(\lambda).\end{aligned} \tag{25}$$

An important initial observation in the implementation of the algebraic Bethe ansatz is that the vector Ω_+ (22) is an eigenvector of the generating function $\tau(\lambda)$. To show this we use the expression (23) and (24):

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2 \left(\rho^2(\lambda) + \frac{2\nu^2\, \rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda} \right) \Omega_+. \tag{26}$$

With the aim of obtaining the explicit and compact form of the Bethe vectors we define the following creation operaters

$$c(\lambda) = f(\lambda) + \frac{\psi\xi}{\nu}\, h(\lambda) + \frac{\psi^2}{4\nu^2}\, e(\lambda). \tag{27}$$

Using the relations (17) - (20) it is straightforward to check that

$$[c(\lambda), c(\mu)] = 0. \tag{28}$$

Consequently, the Bethe vectors generated by the action of the operators (27) on the vector Ω_+ (22) will be symmetric functions of theirs arguments.

Our main aim is to show that the Bethe vector $\varphi_1(\mu)$ has the form

$$\varphi_1(\mu) = c(\mu)\Omega_+ = \left(f(\lambda) + \frac{\psi\xi}{\nu}\, \rho(\lambda) \right) \Omega_+, \tag{29}$$

where $c(\mu)$ is given by (27). The action of the generating function of the Gaudin Hamiltonians reads

$$\tau(\lambda)\varphi_1(\mu) = [\tau(\lambda), c(\mu)]\, \Omega_+ + \chi_0(\lambda)\varphi_1(\mu). \tag{30}$$

Using (25) and the commutation relations (17) - (20) it is evident that

$$[\tau(\lambda), c(\mu)]\, \Omega_+ = [\tau(\lambda), f(\mu)]\, \Omega_+. \tag{31}$$

Then, a straightforward calculation show that

$$\begin{aligned}[\tau(\lambda), f(\mu)]\, \Omega_+ = &-\frac{8\lambda^2}{\lambda^2-\mu^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2 - \lambda^2\nu^2} \right) \varphi_1(\mu) \\ &+ \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)} \left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} \right) \varphi_1(\lambda). \end{aligned} \tag{32}$$

Therefore the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is given by

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)} \left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} \right) \varphi_1(\lambda), \tag{33}$$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{8\lambda^2}{\lambda^2-\mu^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2} \right). \tag{34}$$

The unwanted term in (33) vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} = 0. \tag{35}$$

Thus we have shown that $\varphi_1(\mu)$ (29) is the desired Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_1(\lambda,\mu)$.

3. Conclusion

We have proposed a new creation operators relevant for implementation of the algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with the general reflection matrix. However, explicit and compact form of the relevant Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, requires further studies and will be reported elsewhere. Such a formula would be crucial for the off shell scalar product of the Bethe vectors and these results could lead to the correlations functions of Gaudin model with boundary. Moreover, it would be of considerable interest to establish a relation between Bethe vectors and solutions of the corresponding Knizhnik-Zamolodchikov equations.

A Basic definitions

We consider the spin operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^3, \tag{36}$$

and Casimir operator

$$c_2 = (S^3)^2 + \frac{1}{2}(S^+S^- + S^-S^+) = (S^3)^2 + S^3 + S^-S^+ = \vec{S}\cdot\vec{S}.$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{37}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

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On the Structure of Green's Ansatz

Igor Salom

Abstract It is well known that the symmetric group has an important role (via Young tableaux formalism) both in labelling of the representations of the unitary group and in construction of the corresponding basis vectors (in the tensor product of the defining representations). We show that orthogonal group has a very similar role in the context of positive energy representations of $osp(1|2n, \mathbb{R})$. In the language of parabose algebra, we essentially solve, in the parabosonic case, the long standing problem of reducibility of Green's Ansatz representations.

1 Introduction

The $osp(1|2n, \mathbb{R})$ superalgebra attracts nowadays significant attention, primarily as a natural generalization of the conformal supersymmetry in higher dimensions [1–9]. In the context of space-time supersymmetry, knowing and understanding unitary irreducible representations (UIR's) of this superalgebra is of extreme importance, as these should be in a direct relation with the particle content of the corresponding physical models.

And the most important from the physical viewpoint are certainly, so called, positive energy UIR's, which are the subject of this paper. More precisely, the goal of the paper is to clarify how these representations can be obtained by essentially tensoring the simplest nontrivial positive energy UIR (the one that corresponds to oscillator representation). This parallels the case of the UIR's of the unitary group $U(n)$ constructed within the tensor product of the defining (i.e. "one box") representations. In both cases the tensor product representation is reducible, and while this reduction in the $U(n)$ case is governed by the action of the commuting group of permutations, in the osp case,[1] as we will show, the role of permutations is played by an orthogonal group. We will clarify the details of this reduction.

[1]We will often write shortly $osp(1|2n)$ or osp for the $osp(1|2n, \mathbb{R})$.

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The $osp(1|2n)$ superalgebra is also known by its direct relation to parabose algebra [10, 11]. In the terminology of parastatistics, the tensor product of oscillator UIR's is known as the Green's Ansatz [12]. The problem of the decomposition of parabose Green's Ansatz space to parabose (i.e. $osp(1|2n)$) UIR's is an old one [12], that we here solve by exploiting additional orthogonal symmetry of a "covariant" version of the Green's Ansatz.

2 Covariant Green's Ansatz

Structural relations of $osp(1|2n)$ superalgebra can be compactly written in the form of trilinear relations of odd algebra operators a_α and $a_\alpha^\dagger$:

$$[\{a_\alpha, a_\beta^\dagger\}, a_\gamma] = -2\delta_{\beta\gamma}a_\alpha, \qquad [\{a_\alpha^\dagger, a_\beta\}, a_\gamma^\dagger] = 2\delta_{\beta\gamma}a_\alpha^\dagger, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma], \qquad [\{a_\alpha^\dagger, a_\beta^\dagger\}, a_\gamma^\dagger] = 0, \tag{2}$$

where operators $\{a_\alpha, a_\beta^\dagger\}$, $\{a_\alpha, a_\beta\}$ and $\{a_\alpha^\dagger, a_\beta^\dagger\}$ span the even part of the superalgebra and Greek indices take values $1, 2, \ldots n$ (relations obtained from these by use of Jacobi identity are also implied). This compact notation emphasises the direct connection [11] of $osp(1|2n)$ superalgebra with the parabose algebra of n pairs of creation/annihilation operators [10].

If we (in the spirit of original definition of parabose algebra [10]) additionally require that the dagger symbol † above denotes hermitian conjugation in the algebra representation Hilbert space (of positive definite metrics), then we have effectively constrained ourselves to the, so called, positive energy UIR's of $osp(1|2n)$.[2] Namely, in such a space, "conformal energy" operator $E \equiv \frac{1}{2}\sum_\alpha \{a_\alpha, a_\alpha^\dagger\}$ must be a positive operator. Operators a_α reduce the eigenvalue of E, so the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. If the positive energy representation is irreducible, all vectors from V_0 have the common, minimal eigenvalue ϵ_0 of E: $E|v\rangle = \epsilon_0|v\rangle, |v\rangle \in V_0$. Representations with one dimensional subspace V_0 are called "unique vacuum" representations.

In this paper we will constrain our analysis to UIR's with integer and half-integer values of ϵ_0 (in principle, ϵ_0 has also continuous part of the spectrum—above the, so called, first reduction point of the Verma module). It turns out that all representations from this class can be obtained by representing the odd superalgebra operators a and $a^\dagger$ as the following sum:

$$a_\alpha = \sum_{a=1}^{p} b_\alpha^a\, e^a, \qquad a_\alpha^\dagger = \sum_{a=1}^{p} b_\alpha^{a\dagger}\, e^a. \tag{3}$$

[2] Omitting a short proof, we note that in such a Hilbert space all superalgebra relations actually follow from one single relation—the first or the second of (1).

In this expression integer p is known as the order of the parastatistics, e^a are elements of a real Clifford algebra:

$$\{e^a, e^b\} = 2\delta^{ab} \tag{4}$$

and operators b^a_α together with adjoint $b^{a\dagger}_\alpha$ satisfy ordinary bosonic algebra relations. There are total of $n \cdot p$ mutually commuting pairs of bosonic annihilation-creation operators $(b^a_\alpha, b^{a\dagger}_\alpha)$:

$$[b^a_\alpha, b^{b\dagger}_\beta] = \delta_{\beta\alpha}\delta^{ab}; \quad [b^a_\alpha, b^b_\beta] = 0. \tag{5}$$

Indices $a, b, \ldots$ from the beginning of the Latin alphabet will, throughout the paper, take values $1, 2, \ldots p$. Relation (3) is a slight variation, more precisely, realization, of a more common form of the Green's Ansatz [10, 13].

The representation space of operators (3) can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_a$ of ordinary linear harmonic oscillator in n-dimensions multiplied by the representation space of the Clifford algebra:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p \otimes \mathcal{H}_{CL}. \tag{6}$$

A single factor Hilbert space $\mathcal{H}_a$ is the space of unitary representation of n dimensional bose algebra of operators $(b^a_\alpha, b^{a\dagger}_\alpha), \alpha = 1, 2, \ldots n$: $\mathcal{H}_a \cong \mathcal{U}(b^{a\dagger})|0\rangle_a$, where $|0\rangle_a$ is the usual Fock vacuum of factor space $\mathcal{H}_a$. The representation space $\mathcal{H}_{CL}$ of real Clifford algebra (4) is of dimension $2^{[p/2]}$, i.e. isomorphic with $\mathbb{C}^{2^{[p/2]}}$ (matrix representation). Positive definite scalar product is introduced in usual way in each of the factor spaces, endowing entire space $\mathcal{H}$ also with positive definite scalar product. The space is spanned by the vectors:

$$\mathcal{H} = l.s.\{\mathcal{P}(b^\dagger)|0\rangle \otimes \omega\}, \tag{7}$$

where $\mathcal{P}(b^\dagger)$ are monomials in mutually commutative operators $b^{a\dagger}_\alpha$, $|0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_p$ and $w \in \mathcal{H}_{CL}$.

In the case $p = 1$ (the Clifford part becomes trivial) we obtain the simplest positive energy UIR of $osp(1|2n)$—the n dimensional harmonic oscillator representation. The order p Green's Ansatz representation of $osp(1|2n)$ is, effectively, representation in the p-fold tensor product of oscillator representations [12], with the Clifford factor space taking care of the anticommutativity properties of odd superalgebra operators. It is easily verified that even superalgebra elements act trivially in the Clifford factor space and that their action is simply sum of actions in each of the factor spaces.

The space (6) is highly reducible under action of osp superalgebra. It necessarily decomposes into direct sum of positive energy representations (both unique vacuum and non unique vacuum representations) and thus, from the aspect of osp transformation properties, space $\mathcal{H}$ is spanned by:

$$\mathcal{H} = l.s.\{|(\Lambda, l), \eta_\Lambda\rangle\}, \tag{8}$$

where Λ labels $osp(1|2n)$ positive energy UIR, l uniquely labels a concrete vector within the UIR Λ, and $\eta_\Lambda = 1, 2, \ldots N_\Lambda$ labels possible multiplicity of UIR Λ in the representation space $\mathcal{H}$. If some UIR Λ does not appear in decomposition of $\mathcal{H}$, then the corresponding N_Λ is zero. Label Λ in (8) runs through all (integer and halfinteger positive energy) UIR's of $osp(1|2n)$ such that $N_\Lambda > 0$ and l runs through all vectors from UIR Λ.

3 Gauge Symmetry of the Ansatz

Green's Ansatz in the form (3) possesses certain intrinsic symmetries. First, we note that hermitian operators

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(b_\alpha^{a\dagger} b_\alpha^b - b_\alpha^{b\dagger} b_\alpha^a) + \frac{i}{4}[e^a, e^b] \tag{9}$$

commute with entire osp superalgebra, which immediately follows after checking that $[G^{ab}, a_\alpha] = 0$. Operators G^{ab} themselves satisfy commutation relations of $so(p)$ algebra. The second term in (9) acts in the Clifford factor space, generating a faithful representation of $Spin(p)$ (i.e. spinorial representation of double cover of $SO(p)$ group). Action of the first terms from (9) generate $SO(p)$ group action in the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$. In the entire space $\mathcal{H}$ operators G generate $Spin(p)$ group and all vectors belong to spinorial unitary representations of this symmetry group. The two terms in (9) thus resemble orbital and spin parts of rotation generators and we will often use that terminology. In particular $\mathcal{H} \equiv \mathcal{H}^{\mathbf{o}} \otimes \mathcal{H}^{\mathbf{s}}$, where $\mathcal{H}^{\mathbf{o}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$ and $\mathcal{H}^{\mathbf{s}} = \mathcal{H}_{CL}$. Furthermore, due to existence of operators $I^a \equiv -iexp(i\pi \sum_\alpha b_\alpha^{a\dagger} b_\alpha^a)\overline{e}e^a$ where $\overline{e} \equiv i^{[p/2]} e^1 e^2 \cdots e^p$, for even values of p, the symmetry can be extended to $Pin(p)$ group (the double cover of orthogonal group $O(p)$). We will refer to the symmetry group of the Green's ansatz as the gauge group.

Vectors in space $\mathcal{H}$ carry quantum numbers also according to their transformation properties under the gauge group. As the gauge group commutes with $osp(1|2n)$, these numbers certainly remove at least a part of degeneracy of osp representations in $\mathcal{H}$, in the sense that relation (8) can be rewritten as:

$$\mathcal{H} = l.s.\{|(\Lambda, l), (M, m), \eta_{(\Lambda,M)}\rangle\}, \tag{10}$$

where $(\Lambda,\ l)$ uniquely label vector l within $osp(1|2n)$ positive energy UIR Λ, $(M,\ m)$ uniquely label vector m within finite dimensional UIR M of the gauge group, and $\eta_{(\Lambda,M)} = 1, 2, \ldots N_{(\Lambda,M)}$ labels possible remaining multiplicity of tensor product of these two representations $\mathcal{D}_\Lambda^{osp} \otimes \mathcal{D}_M^{gauge}$ in the space $\mathcal{H}$. Again, if some combination (Λ, M) does not appear in decomposition of $\mathcal{H}$, then the corresponding $N_{(\Lambda,M)}$ is zero.

Important property of the gauge symmetry is that it actually removes all degeneracy in decomposition of $\mathcal{H}$ to $osp(1|2n)$ UIR's, i.e. that the multiplicity of $osp(1|2n)$ UIR's is fully taken into account by labeling transformation properties of the vector w.r.t. the gauge symmetry group. Furthermore, there is one-to-one correspondence between UIR's of $osp(1|2n)$ and of the gauge group that appear in the decomposition, meaning that transformation properties under the gauge group action automatically fix the $osp(1|2n)$ representation. We formulate this more precisely in the following theorem.

Theorem 1. *The following statements hold for the basis (10) of the Hilbert space $\mathcal{H}$:*

1. *All multiplicities $N_{(\Lambda,M)}$ are either 1 or 0.*
2. *Let the $\mathcal{N}$ be the set of all pairs (Λ, M) for which $N_{(\Lambda,M)} = 1$, i.e. $\mathcal{N} = \{(\Lambda, M)|N_{(\Lambda,M)} = 1\}$ and let the $\mathcal{L}$ and $\mathcal{M}$ be sets of all Λ and M, respectively, that appear in any of the pairs from $\mathcal{N}$. Then pairs from $\mathcal{N}$ naturally define bijection from $\mathcal{L}$ to $\mathcal{M}$, $\mathcal{N}: \mathcal{L} \to M$.*

The theorem is proved by explicit construction of the bijection $\mathcal{N}$. First we must go through some preliminary definitions and lemmas.

Corollary 1. *If $osp(1|2n)$ representation Λ appears in the decomposition of the space $\mathcal{H}$, then its multiplicity in the decomposition is given by the dimension of the gauge group representation $\mathcal{N}(\Lambda)$.*

4 Root Systems

At this point we must introduce root systems, both for $osp(1|2n)$ superalgebra and for the $so(p)$ algebra of the gauge group.

We choose basis of a Cartan subalgebra $\mathfrak{h}_{osp}$ of (complexified) $osp(1|2n)$ as:

$$\mathfrak{h}_{osp} = l.s.\left\{\frac{1}{2}\{a_\alpha^\dagger, a_\alpha\}, \alpha = 1, 2, \dots n\right\}. \tag{11}$$

Positive roots, expressed using elementary functionals, are:

$$\begin{aligned}\Delta_{osp}^+ = \{&+\delta_\alpha, 1 \leq \alpha \leq n; +\delta_\alpha + \delta_\beta, 1 \leq \alpha < \beta \leq n;\\ &+\delta_\alpha - \delta_\beta, 1 \leq \alpha < \beta \leq n; +2\delta_\alpha, 1 \leq \alpha \leq n\}\end{aligned} \tag{12}$$

and the corresponding positive root vectors, spanning subalgebra $\mathfrak{g}_{osp}^+$, are (in the same order):

$$\begin{aligned}&\Big\{a_\alpha^\dagger, 1 \leq \alpha \leq n; \{a_\alpha^\dagger, a_\beta^\dagger\}, 1 \leq \alpha < \beta \leq n;\\ &\{a_\alpha^\dagger, a_\beta\}, 1 \leq \alpha < \beta \leq n; \{a_\alpha^\dagger, a_\alpha^\dagger\}, 1 \leq \alpha \leq n\Big\}.\end{aligned} \tag{13}$$

Simple root vectors are:

$$\Big\{\{a_1^\dagger, a_2\}, \{a_2^\dagger, a_3\}, \ldots, \{a_{n-1}^\dagger, a_n\}, a_n^\dagger\Big\}. \tag{14}$$

With this choice of positive roots, positive energy UIR's of $osp(1|2n)$ become lowest weight representations. Thus, we will label positive energy UIR's of $osp(1|2n)$ either by their lowest weight

$$\underline{\lambda} = (\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_n), \tag{15}$$

or by its signature

$$\Lambda = [d; \Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}] \tag{16}$$

related to the lowest weight $\underline{\lambda}$ by $d = \underline{\lambda}_1$, $\Lambda_\alpha = \underline{\lambda}_{\alpha+1} - \underline{\lambda}_\alpha$. Λ_α are nonnegative integers [14] and spectrum of d is positive and dependant of Λ_α values.

As a basis of Cartan subalgebra $\mathfrak{h}_{so}$ of $so(p)$ we take:

$$\mathfrak{h}_{so} = l.s.\Big\{G^{(k)} \equiv G^{2k-1,2k}, k = 1, 2, \ldots q\Big\}, \tag{17}$$

where $q = [p/2]$ is the dimension of Cartan subalgebra (indices $k, l, \ldots$ from the middle of alphabet will take values $1, 2, \ldots, q$). Positive roots in case of even p are:

$$\Delta_{so}^+ = \{+\delta_k + \delta_l, 1 \le k < l \le q; +\delta_k - \delta_l, 1 \le k < l \le q\}, \tag{18}$$

while in the odd case we additionally have $\{+\delta_k, 1 \le k \le q\}$.

In accordance with the choice of Cartan subalgebra $\mathfrak{h}_{so}$ it is more convenient to use the following linear combinations:

$$B_{\alpha\pm}^{(k)\dagger} \equiv \frac{1}{\sqrt{2}}(b_\alpha^{2k-1\dagger} \pm i b_\alpha^{2k\dagger}), \qquad B_{\alpha\pm}^{(k)} = \frac{1}{\sqrt{2}}(b_\alpha^{2k-1} \mp i b_\alpha^{2k}), \tag{19}$$

instead of $b^\dagger$ and b, as $[G^{(k)}, B_{\alpha\pm}^{(l)\dagger}] = \pm\delta^{kl} B_{\alpha\pm}^{(l)\dagger}$ and $[G^{(k)}, B_{\alpha\pm}^{(l)}] = \mp\delta^{kl} B_{\alpha\pm}^{(l)}$. Similarly, we introduce $e_\pm^{(k)} \equiv \frac{1}{\sqrt{2}}(e^{2k-1} \pm i e^{2k})$ that satisfy:

$$[G^{(k)}, e_\pm^{(l)}] = \pm\delta^{kl} e_\pm^{(l)}. \tag{20}$$

Odd superalgebra operators take form:

$$a_\alpha^\dagger = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)\dagger} e_-^{(k)} + B_{\alpha-}^{(k)\dagger} e_+^{(k)}\Big) + \epsilon\, b_\alpha^{p\dagger} e^p, \tag{21}$$

$$a_\alpha = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)} e_+^{(k)} + B_{\alpha-}^{(k)} e_-^{(k)} \Big) + \epsilon\, b_\alpha^p e^p, \tag{22}$$

where $\epsilon = p \mod 2$.

The space $\mathcal{H}$ decomposes to spinorial UIR's of $so(p)$ with the highest weight $\overline{\mu} = (\overline{\mu}^1, \overline{\mu}^2, \dots, \overline{\mu}^q)$ satisfying $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^{q-1} \geq |\overline{\mu}^q| \geq \frac{1}{2}$ with all $\overline{\mu}^q$ taking half-integer values ($\overline{\mu}^q$ can take negative values when p is even). However, since the gauge symmetry group in the case of even p is enlarged to $Pin(p)$ group, any highest weight of UIR of the gauge group satisfies: $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^q \geq 0$. As the gauge group representation in $\mathcal{H}$ is spinorial, all $\overline{\mu}^k$ take half-integer values greater or equal to $\frac{1}{2}$. To label UIR's of the gauge group we will also use signature

$$M = [M^1, M^2, \dots, M^q] \tag{23}$$

with $M^k = \overline{\mu}^k - \overline{\mu}^{k+1}, k < q$ and $M^q = \overline{\mu}^q - \frac{1}{2}$. All M^k are nonnegative integers.

The "spin" factor space $\mathcal{H}^{\mathbf{s}}$ is irreducible w.r.t. action of the gauge group. Gauge group representation in the space $\mathcal{H}^{\mathbf{s}}$ has the highest weight $\overline{\mu}_{\mathbf{s}} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Weight spaces of this representation are one dimensional, meaning that basis vectors can be fully specified by weights $\mu_{\mathbf{s}}$:

$$\mathcal{H}^{\mathbf{s}} = l.s.\{\omega_{\mu_{\mathbf{s}}} \equiv \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \dots, \mu_{\mathbf{s}}^q) | \mu_{\mathbf{s}}^k = \pm\frac{1}{2}\}. \tag{24}$$

An action of operators $e_+^{(k)}, e_-^{(k)}$ and e^p in this basis is given by:

$$e_\pm^{(k)} \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \dots, \mu_{\mathbf{s}}^q) = \sqrt{2} \left(\prod_{l=1}^{k-1} 2\mu_{\mathbf{s}}^l \right) \omega(\mu_{\mathbf{s}}^1, \dots, \mu_{\mathbf{s}}^{k-1}, \mu_{\mathbf{s}}^k \pm 1, \mu_{\mathbf{s}}^{k+1}, \dots, \mu_{\mathbf{s}}^q) \tag{25}$$

and, when p is odd, also:

$$e^p \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \dots, \mu_{\mathbf{s}}^q) = \left(\prod_{l=1}^{q} 2\mu_{\mathbf{s}}^l \right) \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \dots, \mu_{\mathbf{s}}^q). \tag{26}$$

In these definitions it is implied that $\omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \dots, \mu_{\mathbf{s}}^q) \equiv 0$ if any $|\mu_{\mathbf{s}}^k| > \frac{1}{2}$.

Gauge group representation in "orbital" factor space $\mathcal{H}^{\mathbf{o}}$ decomposes to highest weight $\overline{\mu}_{\mathbf{o}}$ UIR's such that all $\overline{\mu}_{\mathbf{o}}^k$ are nonnegative integers. Besides, it is not difficult to verify that, if $n < q$, then

$$\overline{\mu}_{\mathbf{o}}^{n+1} = \overline{\mu}_{\mathbf{o}}^{n+2} = \cdots = \overline{\mu}_{\mathbf{o}}^q = 0 \tag{27}$$

(since maximally n operators (19) can be antisymmetrized).

5 Decomposition of the Green's Ansatz Space

Now we can formulate the following lemma that is the remaining step necessary for the proof of Theorem 1.

Lemma 1. *The vector* $|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda}, \overline{\mu})}\rangle \in \mathcal{H}$ *that is the lowest weight vector of* $osp(1|2n)$ *positive energy UIR* $\underline{\lambda}$ *and the highest weight vector of the gauge group UIR* $\overline{\mu}$ *exists if and only if signatures* Λ *and* M *(16, 23) satisfy:*

$$M_k = \Lambda_{n-k}, \tag{28}$$

where $\Lambda_0 \equiv d - p/2$ *and it is implied that* $M_k = 0, k > q$ *and* $\Lambda_\alpha = 0, \alpha < 0$. *In that case this vector has the following explicit form (up to multiplicative constant) in the basis (7):*

$$|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda}, \overline{\mu})}\rangle = \left(B_{n+}^{(1)\dagger}\right)^{\Lambda_{n-1}} \left(B_{n+}^{(1)\dagger} B_{n-1+}^{(2)\dagger} - B_{n+}^{(2)\dagger} B_{n-1+}^{(1)\dagger}\right)^{\Lambda_{n-2}} \cdots$$

$$\cdot \left(\sum_{k_1, k_2, \ldots k_n = 1}^{\min(n,q)} \varepsilon_{k_1 k_2 \ldots k_n} B_{n+}^{(k_1)\dagger} B_{n-1+}^{(k_2)\dagger} \cdots B_{1+}^{(k_n)\dagger} \right)^{\Lambda_0} |0\rangle \otimes \omega\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right). \tag{29}$$

We will omit a rather lengthy proof of the lemma.

Note that the Lemma 1 also determines whether an osp representation Λ appears or not in the decomposition of Green's Ansatz of order p: UIR Λ appears in the decomposition if and only if the condition (28) can be satisfied by allowed integer values of M_k. However, if q is not sufficiently high, the first $n - q$ of the Λ components $\Lambda_0, \Lambda_1, \ldots \Lambda_{n-q-1}$ are bound to be zero.

Corollary 2. *All (half)integer positive energy UIR's of* $osp(1|2n)$ *can be constructed in space* $\mathcal{H}$ *with* $p \le 2n + 1$.

Proof. Due to relation (28), values $\Lambda_0, \Lambda_1, \ldots \Lambda_{n-1}$ can be arbitrary integers when $q \ge n$: choice $p = 2n$ contains integer values of d UIR's while $p = 2n + 1$ contains half-integer values. That spaces $\mathcal{H}$ for some $p < 2n$ also contain all UIR's with $d < n$, can be verified by checking the list of all positive energy UIR's of $osp(1|2n)$ will be given elsewhere. □

In other words, the above corollary states that no additional (half)integer energy UIR's of $osp(1|2n)$ appear when considering $p > 2n + 1$, i.e. it is sufficient to consider only $p \le 2n + 1$.

The proof of the **Theorem 1** now follows from the Lemma 1.

Proof. Lemma 1 gives the explicit form of the vector that is the lowest weight vector of $osp(1|2n)$ positive energy UIR $\underline{\lambda}$ and the highest weight vector of the gauge group UIR $\overline{\mu}$, when such vector exists. It follows that there can be at most one such vector. Therefore, the multiplicity $N_{(\underline{\lambda}, \overline{\mu})}$ can be either 1 or 0. The relation between $\underline{\lambda}$ and $\overline{\mu}$ is given by (28) and it defines bijection $\mathcal{N}$. □

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Green-Clifford ansatz realization of Parabose representations*

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Abstract

Green's ansatz is a well known method for construction of "unique vacuum" representations of parabose (parafermi) algebra. Exploiting a Clifford algebra variant of the Green's ansatz we construct unitary representations with vacuum state carrying arbitrary $SU(n)$ representation (n being the number of parabose operator pairs).

1. Introduction

Parabose algebra was introduced by H.S. Green [1] long ago, as a generalization of the common bose algebra relations. Following the Green's definition, parabose algebra is algebra of n pairs of mutually hermitian conjugate operators $a_\alpha, a_\alpha^\dagger$, satisfying trilinear relations:

$$[\{a_\alpha, a_\beta^\dagger\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma] = 0, \tag{2}$$

together with relations (additional four) that follow from these by hermitian conjugation and by use of Jacobi identities.[1]

In the same paper [1], Green offered a solution for the above relations, in the terms of sum of operators satisfying "mixed" commutation and anticommutation relations:

$$a_\alpha = \sum_{a=1}^p a_\alpha^a, \tag{3}$$

where a_α^a and $a_\alpha^{a\dagger}$ anticommute for different values of Green's indices a and b:

$$a \neq b \Rightarrow \{a_\alpha^{a\dagger}, a_\alpha^{b\dagger}\} = \{a_\alpha^a, a_\alpha^b\} = \{a_\alpha^a, a_\alpha^{b\dagger}\} = 0 \tag{4}$$

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[1] We note that, in a Hilbert space equipped with positive definite metrics (with respect to which one defines the adjoint $a_\alpha^\dagger$), all algebra relations actually follow from the single relation (1).

and behave as usual bose creation and annihilation operators otherwise:

$$a = b \Rightarrow [a_\alpha^a, a_\beta^{b\dagger}] = \delta_{\beta\alpha},\ [a_\alpha^a, a_\beta^b] = 0. \qquad (5)$$

This construction is nowadays known as the "Green's ansatz" while the integer p is called the "order of parastatistics". Obviously, the case $p = 1$ corresponds to usual bose algebra.

Parabose algebra was originally introduced as an alternative, i.e. generalized method for field quantization, that would correspond to hypothetical particles obeying neither the usual Fermi nor the Bose statistics, but a generalization called parastatistics. In [3] Greenberg and Messiah have concluded that, in this particular context of parastatistics, Green's ansatz suffices for construction of all relevant unitary representations. However, his considerations included two elements special to the parastatistics context: 1) assumption of infinite many degrees of freedom n (i.e. infinite number parabose pairs) that effectively precluded solutions with non-integer p values, and 2) assumption that only "unique vacuum" Fock space representations are of physical interest.

On the other hand, parabose algebra has importance as an algebraic structure in its own right, irrespectively of the parastatistical context. Mathematically, it was realised by Ganchev and Palev [2] that this algebra is equivalent to the orthosymplectic $osp(1|2n)$ superalgebra.[2] In the light of this connection, parabose algebra, or, in other words, $osp(1|2n)$ superalgebra, has its importance in many physical areas/models. Of particular interest are the models where parabose algebra ($osp(1|2n)$ superalgebra) represents the space-time supersymmetry algebra (eg. [4, 5]). In this context number of parabose pairs n depends on the dimensionality of the space-time and ranges usually from $n = 4$ (in the four dimensional case) to $n = 32$ or $n = 64$ (string theory). The assumptions taken by Greenberg and Messiah in [3] here have no longer physical sense: 1) due to finite n, the order of parastatistics p can also take noninteger values from a certain continuum range (p_0, ∞), where p_0 is related to the, so called, first reduction point (Verma module terminology) [6], and 2) Fock vacuum has no more interpretation of "no particle state" but merely represents lowest conformal energy state and thus representations other than "unique vacuum" ones must also be considered. Being not applicable to both of these classes of representations, the basic form of the Green's ansatz construction is therefore no longer sufficient. Of the two, the latter shortcoming is far more serious. Namely, in the context of space-time symmetry, unitary irreducible representations (UIR's) of parabose algebra should be directly related to particle content of the model. Whereas it could be argued that non integer values of order of parastatistics p could be nonphysical, it is not so for the "unique vacuum" representations. On the contrary, in these

[2]This is exactly so if the parabose algebra is defined solely by structural relations, without any mention of Hermitian conjugation. However, if the algebra is introduced as in [1], then, strictly speaking, it is one concrete realization of the $osp(1|2n)$ superalgebra that has only positive energy unitary representations.

representations lowest weight state (i.e. Fock vacuum state) carries nontrivial representation of $SU(n)$ compact subgroup of $osp(1|2n)$ and thus they carry additional quantum numbers – the fact that makes corresponding particles physically highly interesting and important.

2. Unitary representations

In this section we recapitulate [7] classification of unitary irreducible representations of parabose algebra, as defined by (1,2) (results correspond to classification of positive energy UIR's of $osp(1|2n)$). The results were obtained by computer analysis of the lowest weight Verma module structure for cases $n \leq 4$, followed by a straightforward conjecture for the classification for case of arbitrary n.

First we will fix the notation and definitions, which basically follow that of [6].

We consider lowest weight Verma modules $V^\Lambda \cong U(\mathcal{G}^+) \otimes |v_0\rangle$. Here, $\mathcal{G}^+$ denotes subalgebra of positive roots in standard algebra decomposition $\mathcal{G}^\mathbb{C} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$ ($\mathcal{G}$ denotes superalgebra $osp(1|2n)$ and $\mathcal{G}^\mathbb{C}$ its complexification; $\mathcal{H}$ is Cartan subalgebra) and $|v_0\rangle$ is a lowest weight vector of weight Λ:

$$X \in \mathcal{G}^- \Rightarrow X|v_0\rangle = 0, \quad H \in \mathcal{H} \Rightarrow H|v_0\rangle = \Lambda(H)|v_0\rangle. \tag{6}$$

Roots, expressed using elementary functionals, are:

$$\begin{aligned}\Delta \;=\; & \{\pm\delta_\alpha, 1 \leq \alpha \leq n; \pm\delta_\alpha \pm \delta_\beta, 1 \leq \alpha < \beta \leq n; \\ & \pm 2\delta_\alpha, 1 \leq \alpha \leq n\}\end{aligned} \tag{7}$$

(the two signs in $\pm\delta_\alpha \pm \delta_\beta$ not being correlated) and the corresponding root vectors we will denote as (in the same order):

$$\begin{aligned}\mathcal{G}^+ \oplus \mathcal{G}^- \;=\; & \{a^\dagger_{\pm\alpha}, 1 \leq \alpha \leq n; a^\dagger_{\pm\alpha,\pm\beta}, 1 \leq \alpha < \beta \leq n; \\ & a^\dagger_{\pm\alpha,\pm\alpha}, 1 \leq \alpha \leq n\}.\end{aligned} \tag{8}$$

Here we introduced a compact notation for superalgebra elements, that emphasises the parabose connection:

$$a^\dagger_{-\alpha} \equiv a_\alpha, \quad a^\dagger_{\alpha,\beta} \equiv \{a^\dagger_\alpha, a^\dagger_\beta\}. \tag{9}$$

Simple root vectors are:

$$\{a^\dagger_{-2,1}, a^\dagger_{-3,2}, \ldots, a^\dagger_{-n,n-1}, a^\dagger_n\}. \tag{10}$$

We will label representations by the signature

$$\chi = \{s_1, s_2, \ldots, s_{n-1}, d\}, \tag{11}$$

that is connected to the lowest weight Λ in the following way:

$$(\Lambda, \delta_\alpha) = d + \frac{1}{2}(a_1 + a_2 + \cdots + a_{\alpha-1} - a_\alpha - \cdots - a_{n-1}). \qquad (12)$$

Notice that parameters $s_1, s_2, \ldots, s_{n-1}$ define behaviour of the lowest weight state $|v_0\rangle$ under action of the $SU(n)$ subgroup generated by elements $\{a_\alpha, a_\beta^\dagger\}$. The case $s_1 = s_2 = \cdots = s_{n-1} = 0$ corresponds to the "unique vacuum" representations.

We introduce a (Shapovalov) norm on the Verma module via natural involutive antiautomorphism: $\omega : \omega(a_\alpha) = a_\alpha^\dagger$ (compatible with the assumed Hilbert space metric). Right away we note that simple unitarity considerations – calculating norms of vectors $a^\dagger_{-(\alpha+1),\alpha}|v_0\rangle$ and $a_1^\dagger|v_0\rangle$ – result in constraints: $s_\alpha \geq 0$, $d \geq (s_1+s_2+\cdots+s_{n-1})/2$. Parameters $s_1, s_2, \ldots, s_{n-1}$ must be integers, labelling an $SU(n)$ Young tableau with $s_1+s_2+\cdots+s_{n-1}$ boxes in the first row, $s_1 + s_2 + \cdots + s_{n-2}$ boxes in the second and so on, ending with s_1 boxes in the row $n-1$.

For certain values of Λ, submodules appear in the structure of the Verma module V^Λ and the module becomes reducible. Basic case is when this happens due to existence of a singular vector $|v_s\rangle \in V^\Lambda$:

$$X|v_s\rangle = 0, \qquad \forall X \in \mathcal{G}^-. \qquad (13)$$

This singular vector, in turn, generates a submodule $V^{\Lambda'} \cong U(\mathcal{G}^+)|v_s\rangle$ within V^Λ.

To ensure irreducibility, all submodules corresponding to singular vectors must be factored out. However, after factoring out these submodules, new singular vectors may appear in the remaining space – called subsingular vectors. Namely, if the union of all submodules of singular vectors is denoted by $\tilde{I}^\Lambda$ then a vector $|v_{ss}\rangle \in V^\Lambda$ is called a subsingular vector [9] if $|v_{ss}\rangle \notin \tilde{I}^\Lambda$ and:

$$X|v_{ss}\rangle \in \tilde{I}^\Lambda, \qquad \forall X \in \mathcal{G}^-. \qquad (14)$$

Just as singular vectors, subsingular vectors also generate submodules that have to be factored out when looking for irreducible representations.

In the particular case of $osp(1|2n)$ there are always, irrespectively of d value, singular vectors of the form:

$$|v_s^\alpha\rangle \equiv (a^\dagger_{-(\alpha+1),\alpha})^{s_\alpha+1}|v_0\rangle, \quad \alpha = 1, 2, \ldots n-1, \qquad (15)$$

(when considering cases of unitary and therefore finite dimensional $SU(n)$ representations μ, related to integer values of s_α). Of special interest thus are additional d-dependant singular vectors.

Our analysis of the Verma module structure heavily relied on the computer analysis and was carried out in the following general manner (that we just briefly describe). First, Kac determinant of a sufficiently high level was considered as a function of parameter d (for each given class of $SU(4)$ representation μ). In this way it was possible to locate the highest value of d for which the determinant vanishes and the Verma module becomes reducible. The singular or subsingular vector responsible for the singularity of the Kac matrix was then calculated, effectively by solving an (optimized) system of linear equations. Next we would find the norm of this vector and look for possible additional discrete reduction points at (lower) values of d for which the norm also vanishes. If new reduction points with new (sub)singular vectors were found it was also necessary to check that, upon removal of the corresponding submodules, no vectors with zero or negative norm remained. For this, it was enough to check that previously found (sub)singular vectors (i.e. those occurring for higher d values) belonged to the factored-out submodules. Optimized Wolfram Mathematica code was written to perform all these calculations. The analysis was carried out for $n \leq 4$ cases and the results turned out to be readily generalizable to the case arbitrary n. Classification of parabose UIR's is given in the following list, where the allowed values of the parameter d are given for different possible cases of parameters $s_1, s_2, \dots, s_{n-1}$ values:

- $s_1 = s_2 = \cdots = s_{n-1} = 0$, i.e. "unique vacuum" UIR's:

$$\begin{array}{ll} d > (n-1)/2; & \\ d = (n-1)/2, & |v_{ss}^{(1,1,1,\dots,1,1,1,1)}\rangle; \\ d = (n-2)/2, & |v_{ss}^{(0,1,1,\dots,1,1,1,1)}\rangle; \\ & \dots \\ d = 2/2, & |v_{ss}^{(0,0,0,\dots,0,1,1,1)}\rangle; \\ d = 1/2, & |v_{s}^{(0,0,0,\dots,0,0,1,1)}\rangle; \\ d = 0/2, & |v_{s}^{(0,0,0,\dots,0,0,0,1)}\rangle; \end{array} \tag{16}$$

- $s_1 = s_2 = \cdots = s_{n-2} = 0, s_{n-1} > 0$, i.e. single row tableaux UIR's:

$$\begin{array}{ll} d > s_{n-1}/2 + (n-1+1)/2; & \\ d = s_{n-1}/2 + (n-1)/2, & |v_{ss}^{(1,1,1,\dots,1,1,0)}\rangle; \\ d = s_{n-1}/2 + (n-1-1)/2, & |v_{ss}^{(0,1,1,\dots,1,1,0)}\rangle; \\ & \dots \\ d = s_{n-1}/2 + 4/2, & |v_{ss}^{(0,0,\dots,1,1,1,0)}\rangle; \\ d = s_{n-1}/2 + 3/2, & |v_{s}^{(0,0,\dots,0,1,1,0)}\rangle; \\ d = s_{n-1}/2 + 2/2, & |v_{s}^{(0,0,\dots,0,0,1,0)}\rangle; \end{array} \tag{17}$$

- ...
- $s_1 = 0, s_2 > 0$, i.e. $(n-2)$ rows tableaux UIR's:

$$\begin{array}{ll} d > (s_2 + \cdots + s_{n-1})/2 + n - 3/2; & \\ d = (s_2 + \cdots + s_{n-1})/2 + n - 3/2, & |v_{s}^{(1,1,0,\dots,0,0,0)}\rangle; \\ d = (s_2 + \cdots + s_{n-1})/2 + n - 4/2, & |v_{s}^{(0,1,0,\dots,0,0,0)}\rangle; \end{array} \tag{18}$$

- $s_1 > 0$, i.e. $(n-1)$ rows tableaux UIR's:

$$\begin{array}{ll} d > (s_1 + \cdots + s_{n-1})/2 + n - 1; & \\ d = (s_1 + \cdots + s_{n-1})/2 + n - 1, & |v_s^{(1,0,0,\ldots,0,0,0)}\rangle. \end{array} \tag{19}$$

For each allowed value of d, existence of a corresponding singular or subsingular vector is indicated, using the following notation: ss in the lower index stands for "subsingular" whereas s means "singular" vector; in the upper index we give "relative weight" of the vector – if the (sub)singular vector generates Verma submodule of weight Λ' the the relative weight is $\Lambda' - \Lambda$. For UIR's from continuous d range, no (sub)singular vectors appear.

3. Construction of parabose UIR's

In this section we will use a Clifford algebra variant of Green's ansatz, first proposed by Greenberg and Macrae [8], to explicitly construct the listed parabose UIR's. Note, that, whereas Greenberg and Messieah have discussed use of Green's ansatz only for construction of "unique vacuum" UIR's [3], we will demonstrate that Green's ansatz suffices for construction of all discrete UIR's.

The method cannot be applied to UIR's from the continuous spectre, i.e. those UIR's that occur for non (half)integer values of parameter d. However, from the physical viewpoint, representations from the discrete spectre (d taking discrete (half)integer values less or equal to the first reduction point) are of greater significance since only in these cases singular or subsingular vectors appear. It is well known that these vectors turn into important equations of motion (e.g. see [9]). In the particular case of the parabose generalization of supersymmetry, these vectors, for example, turn into Klein-Gordon, Dirac and Maxwell equations [5].

In the same paper where he first introduced parabose (and parafermi) algebra [1], H.S.Green has also offered a way to construct some of the unitary representations using what is nowadays known as the Green's ansatz (3). Greenberg and Macrea in [8] introduced a "gauge-invariant" variant of the Green's ansatz, representing the annihilation parabose operators as the following sum:

$$a_\alpha = \sum_{a=1}^{p} a_\alpha^a \; e_a. \tag{20}$$

In this expression integer p is the order of the parastatistics, e_a are elements of a real Clifford algebra[3]:

$$\{e_a, e_b\} = 2\delta_{ab} \tag{21}$$

and operators a_α^a together with adjoint $a_\alpha^{a\dagger}$ satisfy ordinary bosonic algebra relations. There are total of $n \cdot p$ mutually commuting pairs of annihilation-creation operators $(a_\alpha^a, a_\alpha^{a\dagger})$:

$$[a_\alpha^a, a_\beta^{b\dagger}] = \delta_{\beta\alpha}\delta^{ab}; \quad [a_\alpha^a, a_\beta^b] = 0, \tag{22}$$

[3]Greenberg has also considered using complex Clifford algebra instead of real one, but that case requires altering of parabose algebra relations.

where $a, b = 1, 2, \ldots p$ and $\alpha, \beta = 1, 2, \ldots n$.

The overall Green's ansatz representation space of order p can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_a$ of ordinary linear harmonic oscillator in n-dimensions multiplied by the representation space of matrix representation of the Clifford algebra:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p \otimes \mathbb{C}^{2^{[p/2]}}. \tag{23}$$

A single factor Hilbert space $\mathcal{H}_a$ is the space of unitary representation of n dimensional bose algebra of operators $(a^a_\alpha, a^{a\dagger}_\alpha), \alpha = 1, 2, \ldots n$: $\mathcal{H}_a \cong U(a^{a\dagger}_\alpha)|0\rangle_a$, where $|0\rangle_a$ is the usual Fock vacuum of factor space $\mathcal{H}_a$.

It is clear that no negative or zero norm states appear in this space. Therefore, if we can find, in this framework, a lowest weight vector $|v_0\rangle$ of a proper weight (corresponding to UIR's classified in the previous section) then the vectors of the form $\mathcal{P}(X)|v_0\rangle, \mathcal{P}(X) \in U(\mathcal{G}^+)$ will span that representation space.

The unique vacuum representations of order p are constructed upon lowest weight vector $|v_0^{\{0,\ldots,0,d\}}\rangle$ of the form:

$$|0_p, w_0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_p \otimes w_0, \tag{24}$$

where w_0 is an arbitrary (column) vector from $\mathbb{C}^{2^{[p/2]}}$ of unit norm (scalar product in $\mathbb{C}^{2^{[p/2]}}$ is defined in usual way). All representations with (half) integer d from the class (17) of the UIR's classification can be constructed in this manner. The order of parastatistics has, for this class, the following connection with the UIR signature: $p = 2d$.

However, construction of the "unique vacuum", i.e. $s_1 = s_2 = \cdots = s_{n-1} = 0$ representations within Green's ansatz was known already to Green and Greenberg [1, 3]. The nontrivial part is construction of other representations, in which the lowest weight state carries nontrivial representation of the $SU(n)$ subgroup. A key step toward this end is a specific "pairing" of factor spaces. We define operators:

$$A^{(k)\dagger}_{\alpha\pm} \equiv \tfrac{1}{\sqrt{2}}(a^{2k-1\dagger}_\alpha \pm ie_{(k)}a^{2k\dagger}_\alpha), \tag{25}$$

where $e_{(k)} \equiv -ie_{2k-1}e_{2k}$ are mutually commuting ($[e_{(k)}, e_{(l)}] = 0$) and hermitian, and, by a convention, Green's index put in brackets enumerates "pairs" of factors spaces.

We note the following important relations satisfied by the operators (25):

$$[A^{(k)}_{\alpha\pm}, A^{(l)\dagger}_{\beta\pm}] = \delta^{kl}\delta_{\alpha\beta}, \qquad [A^{(k)}_{\alpha\pm}, A^{(l)\dagger}_{\beta\mp}] = 0, \tag{26}$$

where $A^{(k)}_{\alpha\pm} = (A^{(k)\dagger}_{\alpha\pm})^\dagger = \frac{1}{\sqrt{2}}(a^{2k-1}_\alpha \mp ie_{(k)}a^{2k}_\alpha)$. In other words, operators $(A^{(k)\dagger}_{\alpha+}, A^{(k)}_{\alpha+})$ and $(A^{(k)\dagger}_{\alpha-}, A^{(k)}_{\alpha-})$ are two independent sets of bose creation-annihilation operators.

Expressed using these operators, the parabose operator a_α has the following form:

$$a_\alpha = \sum_{k=1}^{[p/2]} \sqrt{2} e_{2k-1} A^{(k)}_{\alpha -} + \epsilon\, e_p a^p_\alpha, \tag{27}$$

where $\epsilon \equiv (p \mod 2)$. This form directly follows from the definitions (20, 25) and relation $e_{2k-1} A^{(k)}_{\alpha\pm} e_{2k-1} = A^{(k)}_{\alpha\mp}$. The last term is simply a remainder left after the pairing, that exists when p is odd.

From (27) it immediately follows that parabose operators a_α will annihilate any state built by acting of $A^{(l)\dagger}_{\beta +}$ operators upon the Fock vacuum of order p:

$$a_\alpha \mathcal{P}(A^{(k)\dagger}_{\beta +}) |0_p, w_0\rangle = 0, \tag{28}$$

with $\mathcal{P}(A^{(k)\dagger}_{\beta +})$ denoting arbitrary polynomial of the operators (25).

On the other hand, such states transform nontrivially under action of $SU(n)$ subgroup, which is readily seen from:

$$\{a^\dagger_\alpha, a_\beta\} = p\delta_{\alpha\beta} + 2\sum_{k=1}^{[p/2]} (A^{(k)\dagger}_{\alpha +} A^{(k)}_{\beta +} + A^{(k)\dagger}_{\alpha -} A^{(k)}_{\beta -}) + \epsilon\, 2 a^{p\dagger}_\alpha a^p_\beta. \tag{29}$$

It is this combination of properties that allows us to easily construct lowest weight states of non-unique vacuum representations by using operators (25).

The discrete UIR's that correspond to single row Young tableaux (17) are constructed upon the lowest weight state of the form (up to normalization):

$$|v_0^{\{0,\ldots,0,s_{n-1},d\}}\rangle \sim (A^{(1)\dagger}_{n+})^{s_{n-1}} |0_p, w_0\rangle, \tag{30}$$

where $p = 2d - s_{n-1}$. Note that such UIR's are obtainable for $p \geq 2$.

Those discrete UIR's corresponding to double rows Young tableaux are constructed by using antisymmetrized products of two $A^{(k)\dagger}_{\alpha +}$ operators:

$$|v_0^{\{0,\ldots,0,s_{n-2},s_{n-1},d\}}\rangle \sim (A^{(1)\dagger}_{n+} A^{(2)\dagger}_{n-1+} - A^{(1)\dagger}_{n-1+} A^{(2)\dagger}_{n+})^{s_{n-2}} (A^{(1)\dagger}_{n+})^{s_{n-1}} |0_p, w_0\rangle, \tag{31}$$

where $p = 2d - s_{n-2} - s_{n-1}$. Such UIR's are obtainable for $p \geq 4$, that is, at least two pairs of factor spaces are needed.

Construction of UIR's that correspond to Young tableaux with more rows follows the same obvious pattern.

By inspecting the classification of parabose UIR's (16-19) it is evident that all representations for which $2d \in \mathcal{N}$ can be constructed in this manner, in particular all representations corresponding to appearance of additional (sub)singular vectors.

4. On a symmetry of the ansatz

We note that this Clifford variant of the Green's ansatz possesses an intrinsic $SO(p)$ symmetry generated by the following hermitian operators:

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(a_\alpha^{a\dagger} a_\alpha^b - a_\alpha^{b\dagger} a_\alpha^a) + \frac{i}{4}[e^a, e^b], \tag{32}$$

where $e^a = e_a$. Note that two terms in (32) resemble orbital and spin parts of rotation generators and that all vectors from the space (23) of Green's ansatz belong to spinorial representations of this symmetry group. These generators commute with entire parabose algebra:

$$[G^{ab}, a_\alpha] = 0, \tag{33}$$

and this fact can help to solve problem of the reducibility of the Green's ansatz space (23) for a given p. Namely, due to this commutativity, all states from (23) are, apart from $osp(1|2n)$ quantum numbers, also labelled by quantum numbers of some (spinorial) UIR of $SO(p)$. Besides, behaviour of the vectors from (23) under action of $SO(p)$ group (32) is determined solely by transformation properties of the corresponding lowest weight vector $|v_0\rangle$. This is easily seen as all vectors belonging to a parabose UIR determined by the lowest weight vector $|v_0\rangle$ can be written as

$$\mathcal{P}(X)|v_0\rangle, \mathcal{P}(X) \in U(\mathcal{G}^+), \tag{34}$$

while

$$G^{ab}\mathcal{P}(X)|v_0\rangle = \mathcal{P}(X)G^{ab}|v_0\rangle. \tag{35}$$

With a suitable choice of positive root system of the $so(p)$ algebra, it can be shown that $osp(1|2n)$ lowest weight vectors of the form (30-31) are, at the same time, the highest (lowest) weight vectors of certain $SO(p)$ UIR's.

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Generalization of the Gell–Mann Decontraction Formula for $sl(n,\mathbb{R})$ and Its Applications in Affine Gravity

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Abstract The Gell–Mann Lie algebra decontraction formula was proposed as an inverse to the Inonu–Wigner contraction formula. We considered recently this formula in the content of the special linear algebras $sl(n)$, of an arbitrary dimension. In the case of these algebras, the Gell–Mann formula is not valid generally, and holds only for some particular algebra representations. We constructed a generalization of the formula that is valid for an arbitrary irreducible representation of the $sl(n)$ algebra. The generalization allows us to explicitly write down, in a closed form, all matrix elements of the algebra operators for an arbitrary irreducible representation, irrespectively whether it is tensorial or spinorial, finite or infinite dimensional, with or without multiplicity, unitary or nonunitary. The matrix elements are given in the basis of the $Spin(n)$ subgroup of the corresponding $SL(n,R)$ covering group, thus covering the most often cases of physical interest. The generalized Gell–Mann formula is presented, and as an illustration some details of its applications in the Gauge Affine theory of gravity with spinorial and tensorial matter manifields are given.

1 Introduction

The Inönü–Wigner contraction [7] is a well known transformation of algebras (groups) with numerous applications in various fields of physics. Just to mention a few: contractions from the Poincaré algebra to the Galilean one; from the Heisenberg algebras to the Abelian ones of the same dimensions (a symmetry background of a transition processes from relativistic and quantum mechanics to classical mechanics); contractions in the Kaluza–Klein gauge theories framework;

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from (Anti-)de Sitter to the Poincaré algebra; various cases involving the Virasoro and Kac–Moody algebras; relation of strong to weak coupling regimes of the corresponding theories; relation of geometrically curved to "less curved" and/or flat spaces….

However, existence of a transformation (i.e. algebra deformation) inverse to the Inönü–Wigner contraction, so called the "Gell–Mann formula" [1,3,5,6], is far less known. The aim of the formula is to express the elements of the starting algebra as explicitly given expressions containing elements of the contracted algebra. In this way, a relation between certain representations of the two algebras is also established. This, in turn, can be very useful since, by a rule, various properties of the contracted algebras are much easier to explore (e.g. construction of representations [8], decompositions of a direct product of representations [5], etc.).

Before we write down the Gell–Mann formula in the general case, some notation is in order. Let $\mathscr{A}$ be a symmetric Lie algebra $\mathscr{A} = M + T$ with a subalgebra $\mathscr{M}$ such that:

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{T}] \subset \mathscr{T}, \quad [\mathscr{T},\mathscr{T}] \subset \mathscr{M}. \tag{1}$$

Further, let $\mathscr{A}'$ be its Inönü–Wigner contraction algebra w.r.t its subalgebra $\mathscr{M}$, i.e. $\mathscr{A}' = \mathscr{M} + \mathscr{U}$, where

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{U}] \subset \mathscr{U}, \quad [\mathscr{U},\mathscr{U}] = \{0\}. \tag{2}$$

The Gell–Mann formula states that the elements $T \in \mathscr{T}$ can be in certain cases expressed in terms of the contracted algebra elements $M \in \mathscr{M}$ and $U \in \mathscr{U}$ by the following rather simple expression:

$$T = i\frac{\alpha}{\sqrt{U \cdot U}}[C_2(\mathscr{M}),U] + \sigma U. \tag{3}$$

Here, $C_2(\mathscr{M})$ and $U \cdot U$ denote the second order Casimir operators of the $\mathscr{M}$ and $\mathscr{A}'$ algebras respectively, while α is a normalization constant and σ is an arbitrary parameter. For a mathematically more strict definition, cf. [3].

Probably the main reason why this formula is not widely known—in spite of its potential versatility—is the lack of its general validity. Namely, there is a number of references dealing with the question when this formula is applicable [1, 5, 6, 14]. Apart form the case of (pseudo) orthogonal algebras where, loosely speaking, the Gell–Mann formula works very well [17], there are only some subclasses of representations when the formula can be applied [5, 6]. To make the things worse, the question of its applicability is not completely resolved.

Recently, we studied the $\overline{SL}(n,\mathbb{R})$ group cases, contracted w.r.t the maximal compact $Spin(n)$ subgroups. By $\overline{SL}(n,\mathbb{R})$ we denote the double cover of $SL(n,\mathbb{R})$. Note that there faithful spinorial representations are always infinite dimensional and physically correspond to fermionic matter. In these cases the Gel–Mann formula does not hold as a general operator expression and its validity depends heavily on the

$sl(n,\mathbb{R})$ algebra representation space. An exhaustive list of the cases for which the Gell–Mann formula for $sl(n,\mathbb{R})$ algebras hold was obtained [14]. In particular, we have shown that the Gell–Mann formula is not valid for any spinorial representation, nor for any representation with nontrivial $Spin(n)$ multiplicity, rendering the Gell–Mann formula here useless for most of physical applications.

There were some attempts to generalize the Gell–Mann formula for the "decontracted" algebra operators of the complex simple Lie algebras g with respect to decomposition $g = k + ik = k_c$ [9, 19], that resulted in a form of relatively complicated polynomial expressions. Recently we have managed to obtain a generalized form of this formula, first in the concrete case of $sl(5,\mathbb{R})$ algebra, and then also in the case of $sl(n,\mathbb{R})$ algebra, for any n.

In this paper we shall consider the obtained generalized expressions and illustrate applicability of the formula in the context of affine theory of gravity. In particular, we analyze the five dimensional affine gravity models.

2 Generalized Formula

The $sl(n,\mathbb{R})$ algebra operators, i.e. the $SL(n,\mathbb{R})$, $\overline{SL}(n,\mathbb{R})$ group generators, can be split into two subsets: M_{ab}, $a,b = 1,2,\ldots,n$ operators of the maximal compact subalgebra $so(n)$ (corresponding to the antisymmetric real $n\times n$ matrices, $M_{ab} = -M_{ba}$), and the, so called, sheer operators T_{ab}, $a,b = 1,2,\ldots,n$ (corresponding to the symmetric traceless real $n\times n$ matrices, $T_{ab} = T_{ba}$). The $sl(n,\mathbb{R})$ commutation relations, in this basis, read:

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), \tag{4}$$

$$[M_{ab}, T_{cd}] = i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), \tag{5}$$

$$[T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{6}$$

The Inönü–Wigner contraction of $sl(n,\mathbb{R})$ with respect to its maximal compact subalgebra $so(n)$ is given by the limiting procedure:

$$U_{ab} \equiv \lim_{\varepsilon\to 0}(\varepsilon T_{ab}), \tag{7}$$

which leads to the following commutation relations:

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) \tag{8}$$

$$[M_{ab}, U_{cd}] = i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) \tag{9}$$

$$[U_{ab}, U_{cd}] = 0. \tag{10}$$

Therefore, the Inönü–Wigner contraction of $sl(n,\mathbb{R})$ gives a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra, where $r_{\frac{n(n+1)}{2}-1}$ is an Abelian subalgebra (ideal) of "translations" in $\frac{n(n+1)}{2}-1$ dimensions.

The generalized Gell–Mann formula for $sl(n,\mathbb{R})$, obtained in [15], reads:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}. \tag{11}$$

Operators T_{ab} live in the space $\mathscr{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ manifold and it is known that this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra representations [2]. A natural discrete orthonormal basis in this space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$\left\{ \left| \begin{matrix} \{J\} \\ \{k\}\{m\} \end{matrix} \right\rangle \equiv \int \sqrt{dim(\{J\})} D^{\{J\}}_{\{k\}\{m\}}(g^{-1}) dg \, |g\rangle \right\},$$

$$\left\langle \begin{matrix} \{J'\} \\ \{k'\}\{m'\} \end{matrix} \middle| \begin{matrix} \{J\} \\ \{k\}\{m\} \end{matrix} \right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{k'\}\{k\}}\delta_{\{m'\}\{m\}}, \tag{12}$$

where dg is an (normalized) invariant Haar measure and $D^{\{J\}}_{\{k\}\{m\}}$ are the $Spin(n)$ irreducible representation matrix elements:

$$D^{\{J\}}_{\{k\}\{m\}}(g) \equiv \left\langle \begin{matrix} \{J\} \\ \{k\} \end{matrix} \middle| R(g) \middle| \begin{matrix} \{J\} \\ \{m\} \end{matrix} \right\rangle. \tag{13}$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $dim(D^{\{J\}})$ representation basis vectors.

In the basis (12) sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the $Spin(n)$ subgroup: $\{J\}$ label irreducible representation of $Spin(n)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ serve to enumerate $Spin(n)$ multiplicity of representation $\{J\}$ within the given representation of $\overline{SL}(n,\mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathscr{L}^2(Spin(n))$.

Operators $U_{ab}^{(cc)}$ appearing in (11) are concrete (normalized) representations (in $\mathscr{L}^2(Spin(n))$ space) of the Inönü–Wigner contractions of shear generators T_{ab}. In basis (12) these operators act in the following way:

$$\left\langle \begin{matrix} \{J'\} \\ \{k'\}\{m'\} \end{matrix} \middle| U_{ab}^{(cd)} \middle| \begin{matrix} \{J\} \\ \{k\}\{m\} \end{matrix} \right\rangle = \sqrt{\frac{dim(\{J\})}{dim(\{J'\})}} C^{\{J\}\square\square\{J'\}}_{\{k\}(cd)\{k'\}} C^{\{J\}\square\square\{J'\}}_{\{m\}(ab)\{m'\}}, \tag{14}$$

where $\square\square$ denotes $Spin(n)$ representation that corresponds to second order symmetric tensors (shear generators, as well as their Inönü–Wigner contractions, transform in this way w.r.t. $Spin(n)$ subgroup) and C stands for Clebsch–Gordan coefficients of $Spin(n)$.

In (11) we also used notation $C_2(so(c)_K) \equiv \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$, where K_{ab} are generators of $Spin(n)$ group left action in basis (12). These operators behave exactly as the rotation generators M_{ab}, but, instead of acting on right-hand $\{m\}$ indices, they act on the lower left-hand side indices $\{k\}$ that label multiplicity:

$$\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| K_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{m'\}\{m\}}\sqrt{C_2(\{J\})}\, C^{\{J\}\ \square\ \{J'\}}_{\{k\}\,(ab)\,\{k'\}}. \tag{15}$$

Finally, the set of $n-1$ indices $\sigma_2,\sigma_3,\dots\sigma_n$ in (11) label the particular representation of the $\overline{SL}(n,\mathbb{R})$. The formula (11) covers all cases: infinite and finite dimensional representations, spinorial and tensorial, with and without multiplicity, unitary and non unitary.

We note that the term $c=n$ in (11) is, essentially, the original Gell–Mann formula, since $C_2(so(n)_K) = C_2(so(n)_M)$. The rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some particular representations thus yielding the original formula.

An immediate mathematical benefit of the generalized formula is the expression for matrix elements of shear generators in basis (12) [15]:

$$\begin{aligned}\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| T_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle &= \frac{i}{2}\sqrt{\frac{dim(\{J\})}{dim(\{J'\})}}\, C^{\ \{J\}\ \square\square\ \{J'\}}_{\ \{m\}\ ab\ \{m'\}}\\ \times \sum_{c=2}^{n}\sqrt{\frac{c-1}{c}}\Big(C_2(so(c)_{\{k'\}}) &- C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c\Big)\, C^{\ \{J\}\,(\square\square)^{n-c+1}\,\{J'\}}_{\ \{k\}\quad (0)^{c-2}\quad \{k'\}}.\end{aligned} \tag{16}$$

In order to demonstrate application of this result in the context of five dimensional affine gravity models, we introduce a concrete $n=5$ adapted notation (for all $n=5$ notation we adhere to that of our paper [13]). As a basis for $Spin(5)$ representations we pick vectors:

$$\left\{ \left| \begin{matrix}\bar{J}_1 & \bar{J}_2\\ J_1 & J_2\\ m_1 & m_2\end{matrix} \right\rangle, \bar{J}_i = 0, \frac{1}{2}, \dots; \bar{J}_1 \geq \bar{J}_2; m_i = -J_i, \dots J_i \right\}. \tag{17}$$

with respect to decomposition $so(5) \supset so(4) = so(3) \oplus so(3)$. Basis of $\overline{SL}(5,\mathbb{R})$ representation space, corresponding to (12) is then given by vectors:

$$\left\{ \left| \begin{matrix}\bar{J}_1 & \bar{J}_2 & & \\ K_1 & K_2 & J_1 & J_2\\ k_1 & k_2 & m_1 & m_2\end{matrix} \right\rangle \right\}. \tag{18}$$

The reduced matrix elements of the $sl(5,\mathbb{R})$ shear (noncompact) operators, derived from an alternative form of Gell–Mann formula that we have given in the paper [13], read:

$$
\begin{aligned}
&\left\langle \begin{smallmatrix} \bar{J}'_1 \bar{J}'_2 \\ K'_1 K'_2 \\ k'_1 k'_2 \end{smallmatrix} \middle\| T \middle\| \begin{smallmatrix} \bar{J}_1 \bar{J}_2 \\ K_1 K_2 \\ k_1 k_2 \end{smallmatrix} \right\rangle = \sqrt{\tfrac{dim(\bar{J}_1,\bar{J}_2)}{dim(\bar{J}'_1,\bar{J}'_2)}} \\
&\times \Bigg(\Big(\sigma_1 + i\sqrt{\tfrac{4}{5}}(\bar{J}'_1(\bar{J}'_1+2)+\bar{J}'_2(\bar{J}'_2+1)-\bar{J}_1(\bar{J}_1+2)-\bar{J}_2(\bar{J}_2+1))\Big) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \overline{1}\overline{1} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 00 & K'_1 K'_2 \\ k_1 k_2 & 00 & k'_1 k'_2 \end{smallmatrix} \\
&+ i\big(\sigma_2 + K'_1(K'_1+1)+K'_2(K'_2+1)-K_1(K_1+1)-K_2(K_2+1)\big) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \overline{11} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1 k_2 & 00 & k'_1 k'_2 \end{smallmatrix} \\
&- i(\delta_1 + k_1 - k_2) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \overline{11} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1 k_2 & 1\,-1 & k'_1 k'_2 \end{smallmatrix} - i(\delta_1 - k_1 + k_2) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \bar{1}\ \bar{1} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1 k_2 & -1\,1 & k'_1 k'_2 \end{smallmatrix} \\
&+ i(\delta_2 + k_1 + k_2) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \overline{11} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1 k_2 & 11 & k'_1 k'_2 \end{smallmatrix} + i(\delta_2 - k_1 - k_2) C\begin{smallmatrix} \bar{J}_1 \bar{J}_2 & \bar{1}\ \bar{1} & \bar{J}'_1 \bar{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1 k_2 & -1\,-1 & k'_1 k'_2 \end{smallmatrix} \Bigg),
\end{aligned}
\tag{19}
$$

where $dim(\bar{J}_1,\bar{J}_2) = (2\bar{J}_1 - 2\bar{J}_2 + 1)(2\bar{J}_1 + 2\bar{J}_2 + 3)(2\bar{J}_1 + 2)(2\bar{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation characterized by $(\bar{J}_1,\bar{J}_2)$. In this notation, $\overline{SL}(5,\mathbb{R})$ irreducible representations are labelled by parameters $\sigma_1, \sigma_2, \delta_1$ and δ_2, that appear in the formula (19).

3 Gauge Affine Action

The space-time symmetry of the affine models of gravity (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $GL(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of affine gravity models. On the other hand, the essential nontrivial representation determining part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup (R_+ is subgroup of dilatations). We will make use of the $\overline{SL}(n,\mathbb{R})$ matrix elements expression in order to obtain coefficients for some of the gauge field–matter interaction vertices.

A standard way to introduce interactions into affine gravity models is by localization of the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part L_g is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field φ (that ensures action invariance under local dilatations). In the case of the standard Metric Affine [4], i.e. Gauge Affine Gravity [10], the gravitational potentials are tetrads $e^a{}_\mu$, metrics g_{ab} and affine connection $\Gamma^a_{b\mu}$, so that we can write: $L_g = L_g(e, \partial e, g, \partial g, \Gamma, \partial\Gamma, \varphi)$. More precisely, due to action invariance under local affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e, g, T, R, N, \varphi)$, where $T^a_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a_{b\nu} + \Gamma^c_{b\mu}\Gamma^a_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b^a$, where Q_b^a denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of affine fields ϕ^I and their covariant derivatives, together with metrics and tetrads (affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of affine Lagrangians, in arbitrary number of dimensions n, of the form:

$$
\begin{aligned}
&L(e_\mu{}^a, \partial_\nu e_\mu{}^a, \Gamma_{b\mu}{}^a, \partial_\nu \Gamma_{b\mu}{}^a, g_{ab}, \Psi_A, \partial_\nu \Psi_A, \Phi_A, \partial_\nu \Phi_A, \varphi, \partial_\nu \varphi) \; = \\
&e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 + \\
&\bar{\Psi} i g^{ab}\gamma_a e_b{}^\mu D_\mu \Psi + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu (D_\mu \Phi)^+ (D_\nu \Phi) + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu D_\mu \varphi D_\nu \varphi\Big].
\end{aligned}
\tag{20}
$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. affine transformations (dilatational invariance is obtained with the aid of field φ, of mass dimension $n/2-1$). Here T^2 and N^2 stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields. For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n,\mathbb{R})$, $n{\geq}3$, transformations, are written in the second row. The field Ψ denotes a spinorial $\overline{GL}(n,\mathbb{R})$ field—components of that field transform under some appropriate spinorial $\overline{GL}(n,\mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n,\mathbb{R})$ representations are necessarily infinite dimensional [11], and thus the field Ψ will have infinite number of components. The concrete spinorial irreducible representation of field Ψ is given by a set of $n-1$ $\overline{SL}(n,\mathbb{R})$ labels $\{\sigma_c^\Psi\}$ together with the dilatation charge d_Ψ. The field Φ is a representative of a tensorial $\overline{GL}(n,\mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n,\mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^\Phi\}$ and d_Φ. Since, as it is briefly argued later, the noncompact $\overline{SL}(n-1,\mathbb{R})$ affine subgroup is to be represented unitarily, the tensorial field Φ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field φ is scalar with respect to $\overline{SL}(n,\mathbb{R})$ subgroup, and thus has only one component.

Interaction of affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1,n-1)$ subgroup of $\overline{GL}(n,\mathbb{R})$. Such a labelling allows, in principle, to identify affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1,n-1) \subset T^n \wedge \overline{GL}(n,\mathbb{R})$. Therefore, we consider the field Ψ (and similarly for Φ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\ \{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} \left|{}^{\{J\}}_{\{k\}\{m\}}\right\rangle.$$

The interaction term connecting fields g^{cd}, $e_d^{\ \mu}$, Γ^{ab}_{μ}, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd} e_d^{\ \mu} \Gamma^{ab}_{\mu} \bar{\Psi}^{\{J\}}_{\{k\}\{m\}} \Psi^{\{J'\}}_{\{k'\}\{m'\}} \sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \left\langle {}^{\{J\}}_{\{k\}\{m\}} \right| \gamma_c \left| {}^{\{J''\}}_{\{k''\}\{m''\}} \right\rangle \left\langle {}^{\{J''\}}_{\{k''\}\{m''\}} \right| Q_{ab} \left| {}^{\{J'\}}_{\{k'\}\{m'\}} \right\rangle, \quad (21)$$

while the interaction of tensorial field with connection is given by:

$$-\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\nu} \partial_\mu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \Phi^{\{J'\}}_{\{k'\}\{m'\}} \left\langle {}^{\{J\}}_{\{k\}\{m\}} \right| Q_{ab} \left| {}^{\{J'\}}_{\{k'\}\{m'\}} \right\rangle + \quad (22)$$

$$\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\nu} \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \left\langle {}^{\{J'\}}_{\{k'\}\{m'\}} \right| Q_{ab} \left| {}^{\{J\}}_{\{k\}\{m\}} \right\rangle^* + \quad (23)$$

$$\tfrac{1}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_{\mu} \Gamma^{a'b'}_{\nu} \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \cdot$$

$$\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \left\langle {}^{\{J\}}_{\{k\}\{m\}} \right| Q_{ab} \left| {}^{\{J''\}}_{\{k''\}\{m''\}} \right\rangle \left\langle {}^{\{J''\}}_{\{k''\}\{m''\}} \right| Q_{a'b'} \left| {}^{\{J'\}}_{\{k'\}\{m'\}} \right\rangle. \quad (24)$$

The scalar dilaton field interact only with the trace of affine connection:

$$\tfrac{1}{2} g^{ab} e_a^{\ \mu} e_b^{\ \nu} (\partial_\mu - i\Gamma_{a\ \mu}^{\ a} d_\varphi)\varphi (\partial_\nu - i\Gamma_{a\ \nu}^{\ a} d_\varphi)\varphi, \quad (25)$$

where d_φ denotes dilatation charge of φ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n,\mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1,n-1)$. The dilatation generator (that is, the trace Q^a_a) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n,\mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (21); more on these matrices can be found in papers of Šijački [18].) However, before presenting examples of the matrix elements evaluations, and thus calculations of the vertex coefficients, it is due to note that the correct physical interpretation of the $\overline{SL}(n,\mathbb{R})$ representations requires these representations to be unitary w.r.t. its $\overline{SL}(n-1,\mathbb{R})$ subgroup and to be nonunitary w.r.t. its lorentz-like $Spin(1,n-1)$ subgroup. It turns out that these requirements can be properly satisfied by making use of the so called deunitarizing automorphism [11].

4 Gauge Affine Symmetry Vertex Coefficients Evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (21)–(24), and, to do that in $n=5$ case we will use expression (19). However, this formula is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1,n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (16) and replacing the remaining Clebsch–Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group $Spin(1,n-1)$ [12].

As the first example, let the field Φ correspond to an unitary multiplicity free $\overline{SL}(5,\mathbb{R})$ representation, defined by labels $\sigma_2=-4, \delta_1=\delta_2=0$, with σ_1 arbitrary real. The representation space is spanned by vectors (18) satisfying $\overline{J}_1=\overline{J}_2=\overline{J}\in\mathbb{N}_0+\frac{1}{2}; K_1=K_2=0; J_1=J_2=J\leq\overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a=1\dots5$ the five Φ components with $\overline{J}_1=\overline{J}_2=\frac{1}{2}$ (in this sense Φ^a corresponds to a Lorenz 5-vector) then the interaction vertex (22) connecting fields $\Phi^{a\dagger}$, $\partial_\mu\Phi^d$ and affine shear connection Γ_ν^{bc} is:

$$\frac{i}{2}g^{ef}e_e^{\ \mu}e_f^{\ \nu}\Phi^{a\dagger}\Gamma_\nu^{bc}\partial_\mu\Phi^d\frac{\sqrt{5}}{14}\,\sigma_1(\eta_{ab}\eta_{dc}+\eta_{ac}\eta_{db}-\frac{2}{n}\eta_{ad}\eta_{bc}). \tag{26}$$

To obtain this result we used an easily derivable formula for Clebsch–Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$C^{L\square\,\square\square\,\square}_{\ a\ (bc)\ d}=\sqrt{\tfrac{n}{2(n+2)(n-1)}}(\eta_{ab}\eta_{dc}+\eta_{ac}\eta_{db}-\frac{2}{n}\eta_{ad}\eta_{bc}), \tag{27}$$

where we labelled $Spin(1,n-1)$ irreducible representations by Young diagrams, as in [15]. More importantly, we also used value of the reduced matrix element:

$$\left\langle \begin{matrix}\overline{\tfrac{1}{2}\,\tfrac{1}{2}}\\ 0\,0\\ 0\,0\end{matrix}\right\|Q\left\|\begin{matrix}\overline{\tfrac{1}{2}\,\tfrac{1}{2}}\\ 0\,0\\ 0\,0\end{matrix}\right\rangle=\sqrt{\frac{2}{7}}\sigma_1, \tag{28}$$

that we obtained by using formula (19) (based on this formula, a Mathematica program was generated that directly calculates $sl(5,\mathbb{R})$ matrix elements [12], taking into account $Spin(5)$ Clebsch–Gordan coefficients found in [16]).

It is no more difficult to obtain coefficients of the vertices of the form (24). Lagrangian term (24) connecting Lorenz 5-vector Φ components Φ_5, $\Phi_5^\dagger$ and affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15}\left(\sigma_1^2-25\right)g^{cd}e_c^{\ \mu}e_d^{\ \nu}\Gamma_\mu^{55}\Gamma_\nu^{55}\Phi_5^\dagger\partial_\mu\Phi_5. \tag{29}$$

Next we will consider an example where Φ field corresponds to a representation with multiplicity. Let us, again, consider 5-vector component $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ of Φ, only this time without any restriction to the values of $\sigma_1, \sigma_2, \delta_1, \delta_2$. In general, this will correspond to a representation with non trivial multiplicity. Quantum numbers $\{k\} = (K_1, K_2, k_1, k_2)$, that label multiplicity, now can take values: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $(0,0,0,0)$. Therefore, this a priori corresponds to 5 observable 5-vector fields, differentiated by the $\{k\}$ values, and these five vector fields mutually interact by gravitational interaction. Part of the Lagrangian term (22), responsible for this interaction, has the form:

$$\frac{i}{2} g^{ef} e_e^{\ \mu} e_f^{\ \nu} \Phi^{a\dagger}_{\{k'\}} \Gamma_\nu^{bc} \partial_\mu \Phi^d_{\{k\}} \left\langle \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ K_1' K_2' \\ k_1'\, k_2' \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ K_1 K_2 \\ k_1\, k_2 \end{smallmatrix} \right\rangle \frac{\sqrt{5}}{\sqrt{56}} \left(\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{5}\eta_{ad}\eta_{bc}\right). \tag{30}$$

The reduced matrix element is obtained from the generalized Gell–Mann formula:

$$\left\langle \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ \frac{1}{2}\, \frac{1}{2} \\ k_1' k_2' \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ \frac{1}{2}\, \frac{1}{2} \\ k_1 k_2 \end{smallmatrix} \right\rangle =$$

$$\begin{aligned} &\frac{1}{4\sqrt{14}} \Bigg(-2\sigma_1 C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_1\,0\,k_1'} C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_2\,0\,k_2'} + 15\sigma_2 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,0\,k_1'} C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,0\,k_2'} - \\ &-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,-1\,k_1'} \left((k_1 + k_2 - \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k_2'} + (-k_1 + k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k_2'} \right) \\ &-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,1\,k_1'} \left((k_1 - k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k_2'} - (k_1 + k_2 + \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k_2'} \right) \Bigg), \end{aligned}$$

$$\left\langle \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ K_1 K_2 \\ k_1\, k_2 \end{smallmatrix} \right\rangle = 0, \qquad \left\langle \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}}\, \overline{\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{7}}\,\sigma_1, \tag{31}$$

where C_3 denotes an usual $Spin(3)$ Clebsch–Gordan coefficient.

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DIFFERENTIATING BETWEEN Δ- AND Y-STRING CONFINEMENT: CAN ONE SEE THE DIFFERENCE IN BARYON SPECTRA?*

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We use $O(4) \simeq O(3) \times O(3)$ algebraic methods to calculate the energy-splitting pattern of the $K = 2, 3$ excited states of the Y-string in two dimensions. To this purpose we use the dynamical O(2) symmetry of the Y-string in the shape space of triangles and compare our results with known results in three dimensions and find qualitative agreement.

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1. Introduction

QCD seems to demand a genuine three-quark confining potential: the so-called Y-junction string three-quark potential, defined by

$$V_{\mathrm{Y}} = \sigma \min_{\boldsymbol{x}_0} \sum_{i=1}^{3} |\boldsymbol{x}_i - \boldsymbol{x}_0| \,, \tag{1}$$

or, explicitly

$$V_{\mathrm{string}} \;=\; V_{\mathrm{Y}} = \sigma \sqrt{\tfrac{3}{2}\left(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho} \times \boldsymbol{\lambda}|\right)} \,. \tag{2}$$

The complete Y-string potential contains "additional" two-body terms that are valid only in certain parts of the three-particle configuration space, and which we shall ignore here. The $|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$ term is proportional to the area of the triangle subtended by the three quarks. The Y-string potential was proposed as early as 1975, see Refs. [1, 2] and the first schematic calculation (using perturbation theory) of the baryon spectrum for $K \leq 2$ followed soon

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thereafter, Ref. [3]. References [4–6] elaborated on this. The first non-perturbative calculations (variational approximation) of the $K = 3$ band with the Y-string potential were published in the early 1990s, Ref. [7] and extended to the $K = 4$ band later in that decade, Ref. [8]. Yet, some of the most basic properties, such as the ordering of the low-lying states in the spectrum of this potential, without the "QCD hyperfine interaction" and/or relativistic kinematics, remain unknown.

The first systematic attempt to solve the Y-string spectrum, albeit only for the $K \leq 2$ states, can be found in Ref. [9]. That paper used the hyper-spherical harmonics formalism, where the Y-string potential can be written as a function of hyper-angles

$$V_{\mathrm{Y}} \;=\; \sigma\sqrt{\tfrac{3}{2}R^2\left(1+\sin 2\chi|\sin\theta|\right)}\,. \tag{3}$$

This led to the discovery, see Ref. [10], of a new dynamical O(2) symmetry in the Y-string potential, with the permutation group $S_3 \subset O(2)$ as the subgroup of the dynamical O(2) symmetry. That symmetry was further elaborated in Ref. [11]. The present report is a continuation of that line of work.

The three-body sum of two-body potentials has only the three-body permutation group S_3 as its symmetry. When one changes variables from the hyper-angles (χ,θ) to $z^{'} = z = \cos 2\chi$ (vertical axis), and $x^{'} = x\sqrt{1-z^2} = \cos\theta\sin 2\chi$, one can see the full S_3 symmetry, Fig. 1. The area of the triangle $\frac{\sqrt{3}}{2}|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$ and the hyper-radius R are related to the Smith–Iwai variables α, ϕ as follows

$$(\cos\alpha)^2 \;=\; \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2 , \tag{4}$$

$$\tan\phi \;=\; \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2-\boldsymbol{\lambda}^2}\right) . \tag{5}$$

The Y-string potential becomes

$$V_{\mathrm{Y}} \;=\; \sigma\sqrt{\tfrac{3}{2}R^2\left(1+|\cos\alpha|\right)}\,. \tag{6}$$

Independence of the potential on the variable ϕ is equivalent to its invariance under (infinitesimal) "kinematic rotation" O(2) transformations $\delta x^{'} = 2\varepsilon z^{'}, \delta z^{'} = -2\varepsilon x^{'}$ or, in terms of the original Jacobi variables, $\delta\boldsymbol{\rho} = \varepsilon\boldsymbol{\lambda}, \delta\boldsymbol{\lambda} = -\varepsilon\boldsymbol{\rho}$, in the six-dimensional hyper-space. This invariance leads to the new integral of motion $G_3 = \frac{1}{2}\left(\boldsymbol{p}_\rho\cdot\boldsymbol{\lambda} - \boldsymbol{p}_\lambda\cdot\boldsymbol{\rho}\right)$, References [10, 11], associated with the dynamical symmetry (Lie) group $O(2)$ that is a subgroup of the (full hyper-spherical) $O(6)$ Lie group.

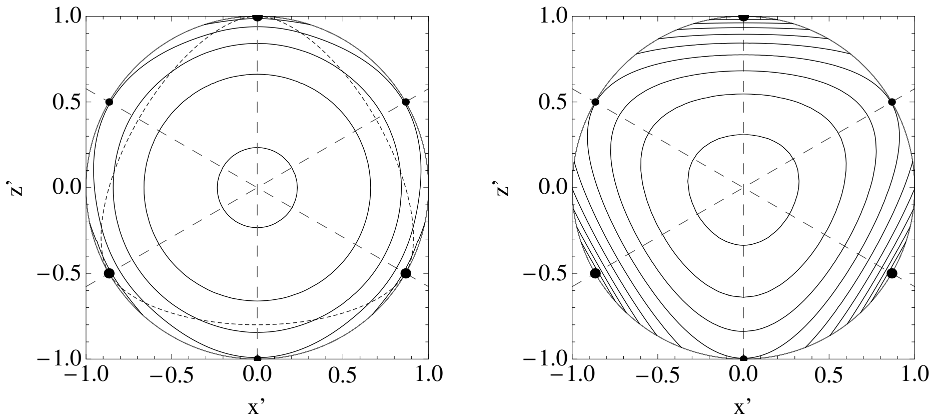

Fig. 1. Left: The equipotential contours for the central Y-string potential (black solid), and the boundary between the central Y-string and two-string potentials (blue dashes). Right: The equipotential contour plot of the Δ-string potential as functions of $z^{'} = z = \cos 2\chi$ (vertical axis), and $x^{'} = x\sqrt{1-z^2} = \cos\theta \sin 2\chi$ (horizontal axis). The three straight lines (red long dashes) of reflection symmetry correspond to the three binary permutations, or "transpositions" S_2 subgroups of S_3. The rotations through $\phi = \pm\frac{2\pi}{3}$ correspond to two cyclic three-body permutations. The rotation symmetry of the Y-string potential (left panel) about the axis pointing out of the plane of the figure should be visible to the naked eye.

Of course, the sums of two-body potentials, such as the Δ-string potential, are invariant only under the finite rotations through $\phi = \pm\frac{2\pi}{3}$, that correspond to cyclic permutations, as well as under reflections about the three symmetry axes. In that case, this generalized hyper-angular momentum G_3 is not an exact integral of motion, but an approximate one. The precise consequences in the energy spectra of systems with such a broken (approximate) symmetry will be shown below.

2. The O(4) algebraic method

The existence of an additional dynamical symmetry strongly suggests an algebraic approach, such as those used in Refs. [12–15]. A careful perusal of Ref. [12, 13] shows, however, that an O(2) group had been used as an enveloping structure for the (discrete) permutation group $S_3 \subset O(2)$, but was not interpreted as a (possible) dynamical symmetry. References [14, 15] did not use this symmetry, however. For the sake of technical simplicity, we confine ourselves to the two spatial dimensions here. In two dimensions (2D), the non-relativistic three-body kinetic energy is a quadratic form of the two Jacobi two-vector velocities, $\dot{\boldsymbol{\rho}}, \dot{\boldsymbol{\lambda}}$, so its "hyper-spherical symmetry" is O(4), and the residual dynamical symmetry of the Y-string potential is $O(2) \otimes O_L(2) \subset O(4)$, where $O_L(2)$ is the (orbital) angular momentum. As the $O(4)$ Lie group can be "factored" in two mutually commuting O(3) Lie groups: $O(4) \simeq O(3) \otimes O(3)$, one may use for our purposes many of the O(3) group results, such as the Clebsch–Gordan coefficients. The 3D case is more complicated than the 2D one; for reasons of simplicity, we limit ourselves to the two-dimensional case in this report.

We (re)formulate the problem in terms of $O(4)$ symmetric variables and then bring the potential into a form that can be (exactly) solved, *i.e.* we expand it in O(4) hyperspherical harmonics $\mathcal{Y}^{JJ}_{LM}$. The energy spectrum is a function of the O(4) hyperspherical expansion coefficients for the potential, and of the O(4) Clebsch–Gordan coefficients, that are products of the ordinary O(3) Clebsch–Gordan coefficients. As the potential is $O_L(2)$ "rotation-symmetric", we have an additional constraint on the allowed hyperspherical harmonics and one finds that for values of $K \leq 3$ one needs only three terms: (1) the "hyper-spherical average", *i.e.* the $\mathcal{Y}^{00}_{00}$ matrix element, (2) the area-term containing the O(4) hyperspherical harmonic $\mathcal{Y}^{22}_{00}$ (which is related to the O(3) spherical harmonic $Y_{20}(\alpha, \phi)$ of the shape space (hyper)spherical angles (α, ϕ), *i.e.*, the V_4 term in the notation of Richard and Taxil [17]) that is present in both the two-body and the Y-string potentials; and (3) the O(2) symmetry-breaking term containing $\mathcal{Y}^{33}_{0\pm3} \simeq Y_{3\pm3}(\alpha, \phi)$, *i.e.*, the V_6 term in the notation of Richard and Taxil [17], that is important in the two-body potentials, and not at all in the Y-string potential Eq. (2).

3. Results

We have evaluated the $K = 2$, 3 bands' splittings in 2D, Ref. [16] and compare them with the 3D case, Ref. [17]:

(1) The only difference between the 2D and 3D $K = 2$ states' splittings is that the $[70, 0^+]$ and $[56, 2^+]$ states are degenerate in 2D, whereas in 3D they are split by one half of the energy difference between $[70, 2^+]$ and $[70, 0^+]$. This shows that the 2D case does relate fairly closely to the 3D one.

(2) We compare our 2D Y-string potential $K = 3$ results with the 3D $K = 3$ two-body potential results of Ref. [17] and find certain similarities, and a few distinctions. There are six SU(6) multiplets in the $K = 3$ sector (other than the hyper-radial excitation $[70, 1^-]''$ of the $K = 1$ state): $[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$, $[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$ in 3D. The main difference between the 2D and 3D is that the $[70, 2^-]$ state disappears in 2D.

In 3D two-body potential the energy splittings can be divided in two parts in Ref. [17]: (a) those due to the V_4 perturbation; and (b) due to the V_6 perturbation. This corresponds to our Y_{20} and $Y_{3\pm3}$ terms, respectively.

(a) In the $V_4 \neq 0$, $V_6 \to 0$ limit, the states can be (roughly) divided in two groups: the $[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$ which are pushed down, and the $[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$ which are pushed up by the V_4 perturbation. Two pairs of states are left degenerate: ($[20, 1^-]$, $[56, 1^-]$) in the lower set and ($[56, 3^-]$, $[20, 3^-]$) in the upper set. In this limit, in 2D we find complete degeneracy of all three members of the lower- ($[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$) and upper levels ($[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$), Fig. 2 (b).

(b) In the $V_4 \neq 0$, $V_6 \neq 0$ case, the remaining degeneracy of states is removed in 3D: the $[20, 1^-]$ and the $[56, 1^-]$ are split in the "lower set" and the $[56, 3^-]$ and the $[20, 3^-]$ in the "upper set". In 2D, we find the same pattern of splitting, and a similar ratio of strengths, Fig. 2 (b).

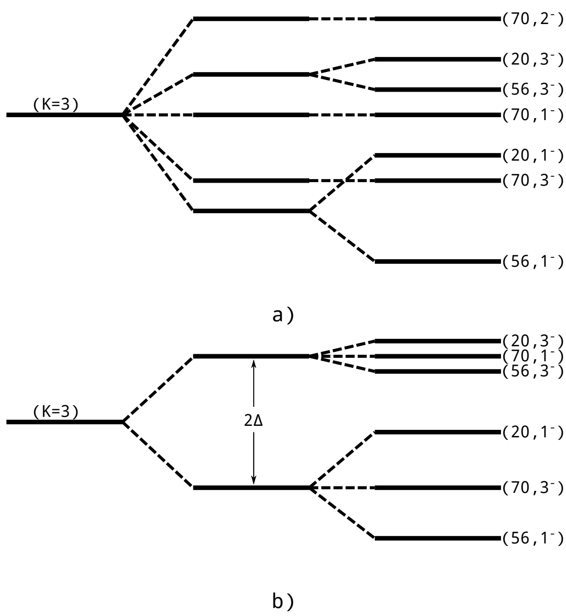

Fig. 2. Schematic representation of the $K = 3$ band in the energy spectrum of the Δ-string potential in (a) three dimensions, following Ref. [17]; and (b) two dimensions (present calculation). The sizes of the two splittings (the v_{20}^{Δ}-induced Δ and the subsequent $v_{3\pm3}^{\Delta}$-induced splitting) are not on the same scale, the latter having been increased, so as to be clearly visible. The Δ here is the same as the Δ in the $K = 2$ band.

So, in the $K = 2, 3$ bands, one sees similarities of dynamical symmetry-breaking patterns in 2D and 3D. This lends credence to the belief that this similarity may persist at higher values of K, where there are not known 3D results, at present.

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Mitja Rosina

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Contents

Preface

The activities and achievements of our encounter at Bled 2012 can be easily summarized. We were caught in a resonance! However, there was also no shortage of interesting "background".

We learned many things about the development of analytical methods for the search of resonances and for implementing the influence of nearby thresholds and interferences in the determination of resonance parameters. In many cases, the interplay between quark and mesonic degrees of freedom is particularly significant, for example in the $\Delta(1700)$ resonance. Resonance parameters, especially poles in the complex energy plane, can sometimes be better determined from the calculation of the time delay instead of the shape of the resonance curve. It was a great encouragement that nowadays Lattice QCD can be used as well to calculate phase shifts for the pion-meson scattering, even in the resonance region.

New mesonic resonances deserved particular attention. One of our experimental colleagues from Belle reported the results on the hypothetical excited states of charmonium, dimesons and tetraquarks. We heard also the prediction that some higly excited states of D and B mesons might be tetraquarks, as well as proposals for its experimental verification. There is also some interesting progress in speculations about "hadronic molecules" DN and BN.

New double polarizaton measurements in Jefferson Lab have clarified several features of the spin structure of ^{3}He and of resonance parameters in electroproduction of pions on nucleons.

The description of baryon spectra, their electroweak structure and decay widths has been successfully extended to hyperons and charmed baryons. There has been noticeable progress in the classification of high-lying baryonic multiplets $(N = 3)$ using the expansion in $1/N_c$ and the $O(3) \times SU(4)$ symmetry; in the description of pionic and photonic reactions by using coupled channels; in the scattering of a superheavy hadron which might be a candidate for dark matter.

As a surprise came the recently discovered magnetar which is too heavy to be supported by quark gas or plasma in its core. This suggests that the three-quark cluster should be stable also at five times the nuclear density, extending the nuclear equation of state into this region.

As you can see, there is no shortage of problems and surprises, but also the strategy of a sequence of small steps can be fruitful! Therefore we hope that our traditional "Bled Workshops" will be continued.

Ljubljana, November 2012

M. Rosina
B. Golli
S. Širca

Predgovor

Aktivnosti in dosežke našega letošnjega blejskega druženja zlahka povzamemo. Ujeli smo se v resonanco! Pa tudi zanimivega "ozadja" ni manjkalo.

Seznanili smo se z razvojem analitskih metod za iskanje resonanc in študijem vpliva bližnjih pragov in interferenc na določitev resonančnih parametrov. Ponekod je izrazito pomembna povezava med kvarkovskimi in mezonskimi prostostnimi stopnjami, na primer pri resonanci $\Delta(1700)$. Resonančne parametre, zlasti pole v kompleksni energijski ravnini, včasih boljše določimo z izračunom časovnega zamika kot z obliko resonančne krivulje. Prijetno nas je vzpodbudilo, da lahko dandanes s kromodinamiko na mreži že računamo fazne premike za sipanje piona na mezonih, celo v območju resonanc.

Posebno pozornost smo namenili novim mezonskim resonancam. Naš eksperimentalni sodelavec v laboratoriju Belle je poročal o rezultatih, ki zadevajo domnevna vzbujena stanja čarmonija, dimezonov in tetrakvarkov. Slišali pa smo tudi napoved, da so nekatera visoka stanja mezonov D in B tetrakvarki ter predloge za eksperimente, s katerimi bi lahko te trditve preverili. Zanimiv je tudi napredek pri špekulacijah o "hadronskih molekulah" DN in BN.

Nove meritve z dvojno polarizacijo v laboratoriju Jefferson Lab so razjasnile nekatere značilnosti spinske strukture jeder ^{3}He ter resonančnih parametrov pri elektroprodukciji pionov na nukleonih.

Opis spektrov barionov, njihove elektro-šibke strukture in razpadnih širin se je uspešno razširil na hiperone in čarobne barione. Napredek je opazen tudi pri razvrščanju visokih barionskih multipletov ($N = 3$) z razvojem po recipročnem številu barv in s simetrijo $O(3) \times SU(4)$; pri obravnavanju pionskih in fotonskih reakcij s sklopljenimi kanali; pri sipanju supertežkega hadrona, ki je morda tudi kandidat za temno snov.

Presenečenje predstavlja nedavno odkriti magnetar, ki je pretežak, da bi njegovo jedro pojasnili s kvarkovskim plinom oziroma plazmo. Zdi se, da so trikvarkovske gruče obstojne tudi pri petkratni jedrski gostoti in tudi velja kar jedrska enačba stanja.

Problemov in presenečenj torej ne zmanjka, in tudi zaporedje majhnih korakov očitno tvori plodno pot! Upamo torej, da se bodo naše tradicionalne "Blejske delavnice" še nadaljevale.

Ljubljana, novembra 2012

M. Rosina
B. Golli
S. Širca

Workshops organized at Bled

- ▷ *What Comes beyond the Standard Model* (June 29–July 9, 1998),
  Vol. **0** (1999) No. 1
- ▷ *Hadrons as Solitons* (July 6–17, 1999)
- ▷ *What Comes beyond the Standard Model* (July 22–31, 1999)
- ▷ *Few-Quark Problems* (July 8–15, 2000), Vol. **1** (2000) No. 1
- ▷ *What Comes beyond the Standard Model* (July 17–31, 2000)
- ▷ *Statistical Mechanics of Complex Systems* (August 27–September 2, 2000)
- ▷ *Selected Few-Body Problems in Hadronic and Atomic Physics* (July 7–14, 2001),
  Vol. **2** (2001) No. 1
- ▷ *What Comes beyond the Standard Model* (July 17–27, 2001), Vol. **2** (2001) No. 2
- ▷ *Studies of Elementary Steps of Radical Reactions in Atmospheric Chemistry*
- ▷ *Quarks and Hadrons* (July 6–13, 2002), Vol. **3** (2002) No. 3
- ▷ *What Comes beyond the Standard Model* (July 15–25, 2002), Vol. **3** (2002) No. 4
- ▷ *Effective Quark-Quark Interaction* (July 7–14, 2003), Vol. **4** (2003) No. 1
- ▷ *What Comes beyond the Standard Model* (July 17–27, 2003),
  Vol. **4** (2003) Nos. 2-3
- ▷ *Quark Dynamics* (July 12–19, 2004), Vol. **5** (2004) No. 1
- ▷ *What Comes beyond the Standard Model* (July 19–29, 2004), Vol. **5** (2004) No. 2
- ▷ *Exciting Hadrons* (July 11–18, 2005), Vol. **6** (2005) No. 1
- ▷ *What Comes beyond the Standard Model* (July 18–28, 2005), Vol. **6** (2005) No. 2
- ▷ *Progress in Quark Models* (July 10–17, 2006), Vol. **7** (2006) No. 1
- ▷ *What Comes beyond the Standard Model* (September 16–29, 2006),
  Vol. **7** (2006) No. 2
- ▷ *Hadron Structure and Lattice QCD* (July 9–16, 2007), Vol. **8** (2007) No. 1
- ▷ *What Comes beyond the Standard Model* (July 18–28, 2007), Vol. **8** (2007) No. 2
- ▷ *Few-Quark States and the Continuum* (September 15–22, 2008),
  Vol. **9** (2008) No. 1
- ▷ *What Comes beyond the Standard Model* (July 15–25, 2008),
  Vol. **9** (2008) No. 2
- ▷ *Problems in Multi-Quark States* (June 29–July 6, 2009), Vol. **10** (2009) No. 1
- ▷ *What Comes beyond the Standard Model* (July 14–24, 2009),
  Vol. **10** (2009) No. 2
- ▷ *Dressing Hadrons* (July 4–11, 2010), Vol. **11** (2010) No. 1
- ▷ *What Comes beyond the Standard Model* (July 12–22, 2010),
  Vol. **11** (2010) No. 2
- ▷ *Understanding hadronic spectra* (July 3–10, 2011), Vol. **12** (2011) No. 1
- ▷ *What Comes beyond the Standard Model* (July 11–21, 2011),
  Vol. **12** (2011) No. 2
- ▷ *Understanding hadronic spectra* (July 1–8, 2012), Vol. **13** (2012) No. 1
- ▷ *What Comes beyond the Standard Model* (July 9–19, 2012),
  Vol. **13** (2012) No. 2

Also published in this series

- ▷ Book of Abstracts, *XVIII European Conference on Few-Body Problems in Physics*,
  Bled, Slovenia, September 8–14, 2002, Edited by Rajmund Krivec, Bojan Golli,
  Mitja Rosina, and Simon Širca, Vol. **3** (2002) No. 1–2


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The Road to Extraction of S-Matrix Poles from Experimental Data *

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Abstract. By separating data points close to a resonance into intervals, and fitting all possible intervals to a simple pole with constant coherently added background, we obtained a substantial number of convergent fits. After a chosen set of statistical constraints was imposed, we calculated the average of a resonance pole position from the statistically acceptable results. We used this method to find pole positions of Z boson.

Breit-Wigner (BW) parameters are often used for the description of unstable particles (see *e.g.*, *Review of Particle Physics* [1]), although shortcomings of such choice have been pointed out on numerous occasions. For example, Sirlin showed that the BW parameters of the Z boson were gauge dependent [2]. To resolve this issue he redefined BW parameters, but also suggested usage of the S-matrix poles as an alternative, since poles are fundamental properties of the S-matrix and therefore gauge independent by definition. In a somewhat different study, Höhler advocated using S-matrix poles for characterization of nucleon resonances [3] in order to reduce confusion that arises when different definitions of BW parameters are used [4]. However, loosely defined [5] BW parameters of mesons and baryons are still being extracted from experimental analyses, compared among themselves [1], and used as input to QCD-inspired quark models [6] and as experiment-to-theory matching points for lattice QCD [7].

Our group has been very interested in reducing human and model dependence from resonance parameters' extraction procedures (from scattering matrices). We developed the regularization method for pole extraction from S-matrix elements [8]. Its main disadvantage is that it needs very dense data, one that is attainable only after an energy dependent partial-wave analysis. The other method was the K-matrix pole extraction method [9] which needed the whole unitary S-matrix to begin with, making it impossible to use on any single reaction. Both of those methods were purely mathematical, and the only assumption were that there is a pole in the complex energy plane of an S-matrix. We had no physical input into our procedures. Therefore, we proclaimed these procedures model-independent. The only thing missing, was a method which could be applied directly to the experimental data, *e.g.,* total cross sections.

In this proceeding, we illustrate a method for model-independent extraction of S-matrix pole positions directly from the data.

* Talk delivered by S. Ceci

The first step in devising a method for extraction of the pole parameters from the experimental data is to set up an appropriate parameterization. The parameterization presented here is based on the assumption that close to a resonance, the T matrix will be well described with a simple pole and a constant background. The similar assumption was used in Höhler's speed plot technique [3]. The speed plot is a method used for the pole parameter extraction from the known scattering amplitudes. It is based on calculating the first order energy derivative of the scattering amplitude, with the key assumption that the first derivative of the background is negligible.

The T matrix with a single pole and constant background term is given by

$$T(W) = r_p \, \frac{\Gamma_p/2}{M_p - W - i\,\Gamma_p/2} + b_p, \tag{1}$$

where W is center-of-mass energy, r_b and b_b are complex, while M_p and Γ_p are real numbers. Total cross section is then proportional to $|T|^2/q^2$, where q is the initial center-of-mass momentum. Equation (1), as well as other similar forms (see e.g. [1]), are standardly called Breit-Wigner parameterizations, which can be somewhat misleading since M_p and Γ_p are generally not Breit-Wigner, but pole parameters (hence the index p). The square of the T matrix defined in Eq. (1) is given by

$$|T(W)|^2 = T_\infty^2 \, \frac{(W - M_z)^2 + \Gamma_z^2/4}{(W - M_p)^2 + \Gamma_p^2/4}, \tag{2}$$

where, for convenience, we simplified the numerator by combining the old parameters into three new real-valued ones: T_∞, M_z, and Γ_z. Pole parameters M_p and Γ_p are retained in the denominator.

With such a simple parameterization, it is crucial to use only data points close to the resonance peak. To avoid picking and choosing the appropriate data points by ourselves, we analyzed the data from a wider range around the resonance peak, and fitted localy the parameterization (2) to each set of seven successive data points (seven data points is minimum for our five-parameter fit). Then we increased the number of data points in the sets to eight and fitted again. We continued increasing the number of data points in sets until we fitted the whole chosen range. Such procedure allowed different background term for each fit, which is much closer to reality than assuming a single constant background term for the whole chosen data set (see e.g. discussion on the problems with speed plot in Ref. [8]). In the end, we imposed a series of statistical constraints to all fits to distinguish the good ones. The whole analysis was done in Wolfram Mathematica 8 using NonlinearModelFit routine [11].

Having defined the fitting strategy, we tested the method by applying it to the case of the Z boson. The data set we used is from the PDG compilation [1], and shown in Fig. 1. Extracted pole masses are shown in the same figure: filled histogram bins show pole masses from the good fits, while the empty histogram bins are stacked to the solid ones to show masses obtained in the discarded fits. Height of the pole-mass histogram in Fig. 1 is scaled for convenience.

Extracted S-matrix pole mass and width of Z boson are given in Table 1. The pole masses are in excelent agreement, while the pole widths are reasonably close.

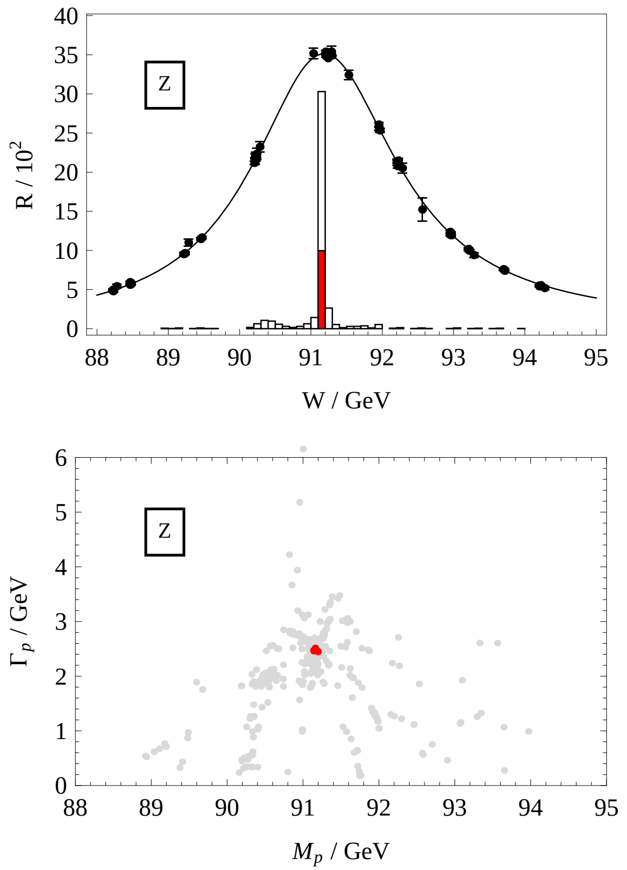

Fig. 1. [Upper figure] PDG compilation of Z data [1] and histogram of obtained pole masses. Line is the fit result with the lowest reduced χ^2 (just for illustration). Dark (red online) colored histogram bins are filled with statistically preferred results. [Lower figure] Pole masses vs. pole widths. Dark (red online) circles show statistically preferred results we use for averages.

It is important to stress that the difference between the pole and BW mass of the Z boson is fundamental and statistically significant. Distribution of discarded and good results is shown in the lower part of Fig. 1.

Table 1. Pole parameters of Z obtained in this work. PDG values of pole and BW parameters are given for comparison.

	Pole	Pole PDG [1]	BW PDG [1]
M/MeV	91159 ± 8	91162 ± 2	91188 ± 2
Γ/MeV	2484 ± 10	2494 ± 2	2495 ± 2

In conclusion, we have illustrated here a model-independent method for extraction of resonance pole parameters from total cross sections and partial waves. Very good estimates for Z boson pole position were obtained.

We are today witnessing the dawn of ab-initio calculations in low-energy QCD. In order to compare theoretical predictions with experimentally determined resonance states, we need first to establish proper point of comparison. We hope that our method, once it becomes fully operational, will help connecting experiment and theory.

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Unified Model for Light- and Heavy-Flavor Baryon Resonances ⋆

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Abstract. We report on the construction of a relativistic constituent-quark model capable of describing the spectroscopy of baryons with all flavors u, d, s, c, and b. Some selective spectra are presented, where comparisons to experimental data are yet possible.

1 Introduction

Over the decades the constituent-quark model (CQM) has ripened into a stage where its formulation and solution are well based on a relativistic (i.e. Poincaré-invariant) quantum theory (for a thorough review of relativistic Hamiltonian dynamics see ref. [1]). In such an approach one relies on an invariant mass operator $\hat{M}$, where the interactions are introduced according the so-called Bakamjian-Thomas construction [2]. If the conditions of the Poincaré algebra are fulfilled by $\hat{M}$, this leads to relativistically invariant mass spectra.

Relativistic constituent-quark models (RCQM) have been developed by several groups, however, with limited domains of validity. Of course, it is desirable to have a framework as universal as possible for the description of all kinds of hadron processes in the low- and intermediate-energy regions. This is especially true in view of the advent of ever more data on heavy-baryon spectroscopy from present and future experimental facilities.

We have developed a RCQM that comprises all known baryons with flavors u, d, s, c, and b within a single framework [3]. There have been only a few efforts so far to extend a CQM from light- to heavy-flavor baryons. We may mention, for example, the approach by the Bonn group who have developed a RCQM, based on the 't Hooft instanton interaction, along a microscopic theory solving the Salpeter equation [4] and extended their model to charmed baryons [5], still not yet covering bottom baryons. A further quark-model attempt has been undertaken by the Mons-Liège group relying on the large-N_c expansion [6,7], partially extended to heavy-flavor baryons [8]. Similarly, efforts are invested to expand other approaches to heavy baryons, such as the employment of Dyson-Schwinger equations together with either quark-diquark or three-quark calculations [9, 10].

⋆ Talk delivered by J. P. Day

Also an increased amount of more refined lattice-QCD results has by now become available on heavy-baryon spectra (see, e.g., the recent work by Liu et al. [11] and references cited therein).

2 The Model

Our RCQM is based on the invariant mass operator

$$\hat{M} = \hat{M}_{\text{free}} + \hat{M}_{\text{int}} \,, \tag{1}$$

where the free part corresponds to the total kinetic energy of the three-quark system and the interaction part contains the dynamics of the constituent quarks Q. In the rest frame of the baryon, where its three-momentum $\mathbf{P} = \sum_i^3 \mathbf{k}_i^2 = 0$, we may express the terms as

$$\hat{M}_{\text{free}} = \sum_{i=1}^{3} \sqrt{\hat{m}_i^2 + \hat{\mathbf{k}}_i^2} \,, \tag{2}$$

$$\hat{M}_{\text{int}} = \sum_{i<j}^{3} \hat{V}_{ij} = \sum_{i<j}^{3} (\hat{V}_{ij}^{\text{conf}} + \hat{V}_{ij}^{\text{hf}}) \,. \tag{3}$$

Here, the $\hat{\mathbf{k}}_i$ correspond to the three-momentum operators of the individual quarks with rest masses m_i and the Q-Q potentials $\hat{V}_{ij}$ are composed of confinement and hyperfine interactions. By employing such a mass operator $\hat{M}^2 = \hat{P}^\mu \hat{P}_\mu$, with baryon four-momentum $\hat{P}_\mu = (\hat{H}, \hat{P}_1, \hat{P}_2, \hat{P}_3)$, the Poincaré algebra involving all ten generators $\{\hat{H}, \hat{P}_i, \hat{J}_i, \hat{K}_i\}$, $(i = 1, 2, 3)$, or equivalently $\{\hat{P}_\mu, \hat{J}_{\mu\nu}\}$, $(\mu, \nu = 0, 1, 2, 3)$, of time and space translations, spatial rotations as well as Lorentz boosts, can be guaranteed. The solution of the eigenvalue problem of the mass operator $\hat{M}$ yields the relativistically invariant mass spectra as well as the baryon eigenstates (the latter, of course, initially in the standard rest frame).

We adopt the confinement depending linearly on the Q-Q distance r_{ij}

$$V_{ij}^{\text{conf}}(\mathbf{r}_{ij}) = V_0 + C r_{ij} \tag{4}$$

with the strength $C = 2.33$ fm^{-2}, corresponding to the string tension of QCD. The parameter $V_0 = -402$ MeV is only necessary to set the ground state of the whole baryon spectrum, i.e., the proton mass; it is irrelevant for level spacings.

The hyperfine interaction is most essential to describe all of the baryon excitation spectra. In a unified model the hyperfine potential must be explicitly flavor-dependent. Otherwise, e.g., the N and Λ spectra with their distinct level orderings could not be reproduced simultaneously. Therefore we have advocated for the hyperfine interaction of our universal RCQM the $SU(5)_F$ GBE potential

$$V_{\text{hf}}(\mathbf{r}_{ij}) = \left[V_{24}(\mathbf{r}_{ij}) \sum_{a=1}^{24} \lambda_i^a \lambda_j^a + V_0(\mathbf{r}_{ij}) \lambda_i^0 \lambda_j^0 \right] \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,. \tag{5}$$

Here, we take into account only its spin-spin component, which produces the most important hyperfine forces for the baryon spectra; the other possible force components together play only a minor role for the excitation energies [16]. While $\boldsymbol{\sigma}_i$ represent the Pauli spin matrices of $SU(2)_S$, the λ_i^a are the generalized Gell-Mann flavor matrices of $SU(5)_F$ for quark i. In addition to the exchange of the pseudoscalar 24-plet also the flavor-singlet is included because of the $U(1)$ anomaly. The radial form of the GBE potential resembles the one of the pseudoscalar meson exchange

$$V_\beta(\mathbf{r}_{ij}) = \frac{g_\beta^2}{4\pi}\frac{1}{12m_i m_j}\left[\mu_\beta^2 \frac{e^{-\mu_\beta r_{ij}}}{r_{ij}} - 4\pi\delta(\mathbf{r}_{ij})\right] \tag{6}$$

for $\beta = 24$ and $\beta = 0$. Herein the δ-function must be smeared out leading to [13, 14]

$$V_\beta(\mathbf{r}_{ij}) = \frac{g_\beta^2}{4\pi}\frac{1}{12m_i m_j}\left[\mu_\beta^2 \frac{e^{-\mu_\beta r_{ij}}}{r_{ij}} - \Lambda_\beta^2 \frac{e^{-\Lambda_\beta r_{ij}}}{r_{ij}}\right]. \tag{7}$$

Contrary to the earlier GBE RCQM [13], which uses several different exchange masses μ_γ and different cut-offs Λ_γ, corresponding to $\gamma = \pi, K$, and η=η_8 mesons, we here managed to get along with a universal GBE mass μ_{24} and a single cut-off Λ_{24} for the 24-plet of $SU(5)_F$. Only the singlet exchange comes with another mass μ_0 and another cut-off Λ_0 with a separate coupling constant g_0. Consequently the number of open parameters in the hyperfine interaction could be kept as low as only three (see Tab. 1).

Table 1. Free parameters of the present GBE RCQM determined by a best fit to the baryon spectra.

Free Parameters		
$(g_0/g_{24})^2$	Λ_{24} [fm^{-1}]	Λ_0 [fm^{-1}]
1.5	3.55	7.52

Table 2. Fixed parameters of the present GBE RCQM predetermined from phenomenology and not varied in the fitting procedure.

Fixed Parameters						
Quark masses [MeV]				Exchange masses [MeV]		Coupling
$m_u = m_d$	m_s	m_c	m_b	μ_{24}	μ_0	$g_{24}^2/4\pi$
340	480	1675	5055	139	958	0.7

All other parameters entering the model have judiciously been predetermined by existing phenomenological insights. In this way the constituent quark masses have been set to the values as given in Tab. 2. The 24-plet Goldstone-boson (GB) mass has been assumed as the value of the π mass and similarly the singlet mass as the one of the η'. The universal coupling constant of the 24-plet has been chosen according to the value derived from the π-N coupling constant via the Goldberger-Treiman relation.

3 Results for Baryon Spectra

We have calculated the baryon spectra of the relativistically invariant mass operator $\hat{M}$ to a high accuracy both by the stochastic variational method [17] as well as the modified Faddeev integral equations [18, 19]. The present universal GBE RCQM produces the spectra in the light and strange sectors with similar or even better quality than the previous GBE RCQM [13]. In Figs. 1 and 2 we show the ground states and the first two excitations of $SU(3)_F$ singlet, octet, and decuplet baryons up to $J = \frac{7}{2}$, for which experimental data of at least three stars are quoted by the PDG [15] and J^P is known. Evidently a good overall description is achieved. Most importantly, the right level orderings specifically in the N, Δ, and Λ spectra as well as all other $SU(3)_F$ ground and excited states are reproduced in accordance with phenomenology. The reasons are exactly the same as for the previous GBE RCQM, what has already been extensively discussed in the literature [12–14]. Unfortunately, the case of the $\Lambda(1405)$ excitation could still not be resolved. It remains as an intriguing problem that can possibly not be solved by RCQMs relying on $\{QQQ\}$ configurations only; an explicit coupling to the K-N decay channel whose threshold lies nearby might be needed.

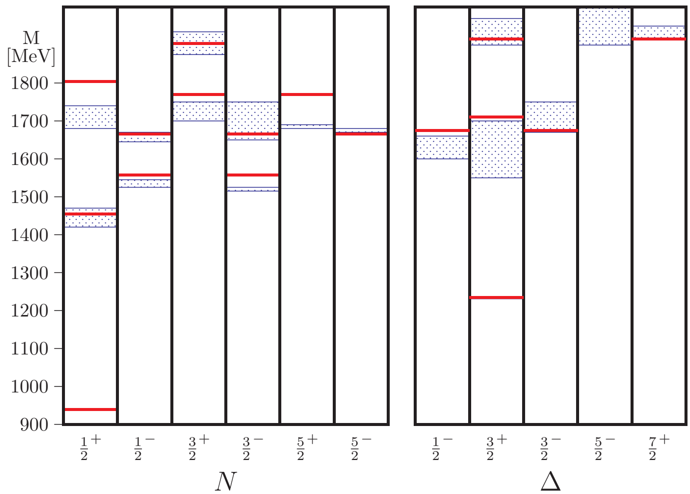

Fig. 1. Nucleon and Δ excitation spectra (solid/red levels) as produced by the universal GBE RCQM in comparison to phenomenological data [15] (the gray/blue lines and shadowed/blue boxes show the masses and their uncertainties).

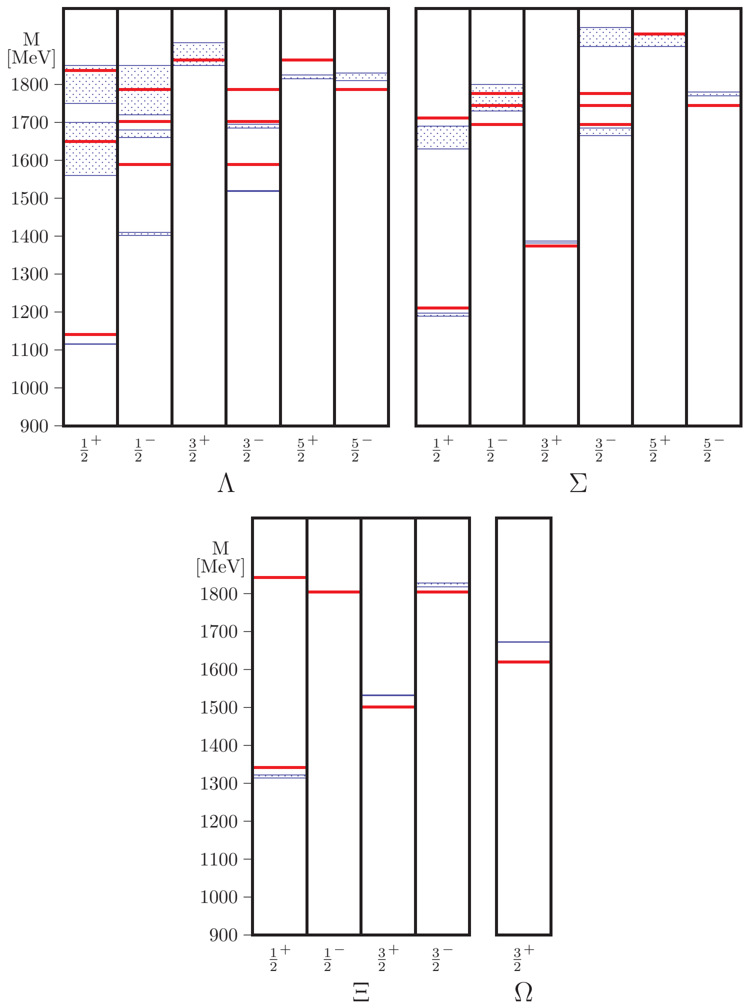

Fig. 2. Same as Fig. 1 but for the strange baryons.

What is most interesting in the context of the present work are the very properties of the light-heavy and heavy-heavy Q-Q hyperfine interactions. Can the GBE dynamics reasonably account for them? In Figs. 3 and 4 we show the spectra of all charm and bottom baryons that experimental data with at least three- or four-star status by the PDG exist for [15]. As is clearly seen, our universal GBE RCQM can reproduce all levels with respectable accuracy. In the Λ_c and Σ_c spectra some experimental levels are not known with regard to their spin and parity J^P. They are shown in the right-most columns of Fig. 3. Obviously they could easily be accommodated in accordance with the theoretical spectra, once their J^P's are determined. Furthermore the model predicts some additional excited states for charm and bottom baryons that are presently missing in the phenomenological data base.

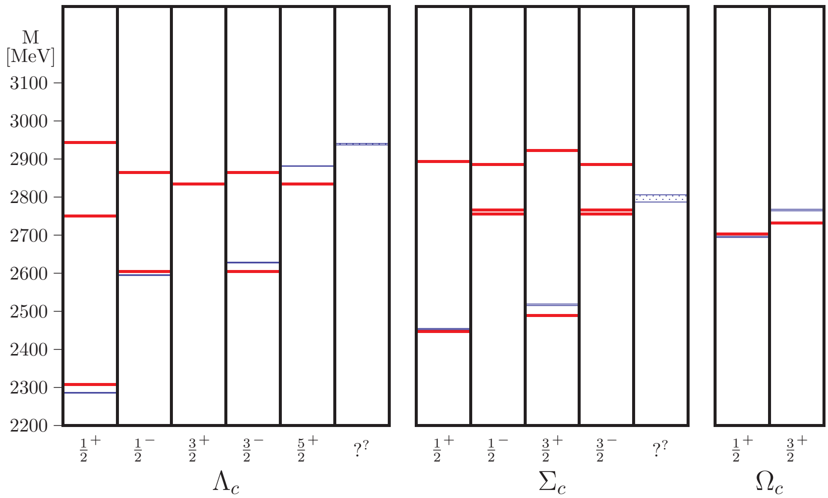

Fig. 3. Same as Fig. 1 but for charm baryons.

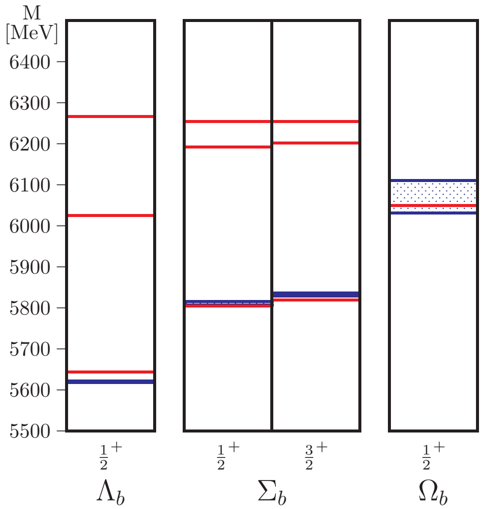

Fig. 4. Same as Fig. 1 but for bottom baryons.

Of course, the presently available data base on charm and bottom baryon states is not yet very rich and thus not particularly selective for tests of effective Q-Q hyperfine forces. The situation will certainly improve with the advent of further data from ongoing and planned experiments. Beyond the comparison to experimental data, we note that the theoretical spectra produced by our present GBE RCQM are also in good agreement with existing lattice-QCD results for heavy-flavor baryons. This is especially true for the charm baryons vis-à-vis the recent work by Liu et al. [11].

4 Discussion and Conclusion

We emphasize that the most important ingredients into the present RCQM are relativity, specifically Poincaré invariance, and a hyperfine interaction that is derived from an interaction Lagrangian built from effective fermion (constituent quark) and boson (Goldstone boson) fields connected by a pseudoscalar coupling [12]. It appears that such kind of dynamics is quite appropriate for constituent quarks of any flavor.

As a result we have demonstrated by the proposed GBE RCQM that a universal description of all known baryons is possible in a single model. Here, we have considered only the baryon masses (eigenvalues of the invariant mass operator $\hat{M}$). Beyond spectroscopy the present model will be subject to further tests with regard to the baryon eigenstates, which are simultaneously obtained from the solution of the eigenvalue problem of $\hat{M}$. They must prove reasonable in order to make the model a useful tool for the treatment of all kinds of baryons reactions within a universal framework.

5 Acknowledgement

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A chiral theory of scalar heavy-light (D, D_s) and (B, B_s) mesons

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Abstract. The talk with the above title delivered at the Bled 2012 workshop was based on the paper "Chiral symmetry of heavy-light scalar mesons with $U_A(1)$ symmetry breaking" that has been published in the meantime as Phys. Rev. D **86**, 016006 (2012). For all details we refer the interested reader to that publication.

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Low-lying states of the Y-string in two dimensions ⋆

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Abstract. We use $SU(2)\times SU(2)$ algebraic methods to calculate the energy-splitting pattern of the K=2,3 excited states of the Y-string in two dimensions. To this purpose we use the dynamical O(2) symmetry of the Y-string in the shape space of triangles and compare our results with known results in three dimensions and find qualitative agreement.

1 Introduction

The three-quark confinement problem has been attacked in many ways: 1) by way of the harmonic oscillator models with some non-harmonic two-body potential components [1–3]; 2) by way of Y-string three-body potentials, Refs. [4–15]; 3) by way of the hyperspherical formalism applied to two-quark potentials, Refs. [16, 17] and 4) by way of dynamical symmetry Lie-algebraic methods, Refs. [18–23], with some success for the low-lying bands of states (up to $K \leq 3$). The higher-lying bands ($K \leq 4$) have generally not been studied systematically (to our knowledge), only individual states with highest values of the orbital angular momentum, for purposes of Regge analyses, with one significant exception ($K = 4$), the Ref. [11].

QCD seems to demand a genuine three-body confining potential: the so-called Y-junction string three-quark potential, defined by

$$V_Y = \sigma \min_{\mathbf{x}_0} \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_0|, \tag{1}$$

or, explicitly

$$V_{\text{string}} = V_Y = \sigma\sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho}\times\boldsymbol{\lambda}|)}. \tag{2}$$

The $|\boldsymbol{\rho}\times\boldsymbol{\lambda}|$ term is proportional to the area of the triangle subtended by the three quarks. The Y-string potential was proposed as early as 1975, see Refs. [4, 5] and the first schematic calculation (using perturbation theory) of the baryon spectrum up to K$\leq$ 2 followed soon thereafter, Ref. [6]. Refs. [7–9], elaborated on this. The first non-perturbative calculations (variational approximation) of the K=3 band with the Y-string potential were published in the early 1990's, Ref. [10] and extended to the K=4 band later in that decade, Ref. [11]. Yet, some of the most basic

⋆ Talk delivered by V. Dmitrašinović

properties of this potential, such as the ordering of the low-lying states in the spectrum, without the "QCD hyperfine interaction" and/or relativistic kinematics remain unknown.

The first systematic attempt to solve the Y-string spectrum, albeit only up to the K=2 band, can be found in Ref. [12]. That paper used the hyperspherical harmonics formalism, which led to the discovery of a new dynamical O(2) symmetry in the Y-string potential, with the permutation group $S_3 \subset O(2)$ as the subgroup of the dynamical O(2) symmetry, see Ref. [13]. That symmetry was further elaborated in Ref. [15]. The present work is a continuation of that line, which has also been represented in this series of workshops [14]. The three-body sum of two-body potentials has only the permutation group S_3 as its symmetry.

2 O(4) algebraic method

The existence of an additional dynamical symmetry strongly suggests an algebraic approach, such as those used in Refs. [18–23]. A careful perusal of Ref. [18,19] shows, however, that an O(2) group had been used as an enveloping structure for the (discrete) permutation group $S_3 \subset O(2)$, but was not interpreted as a (possible) dynamical symmetry. Refs. [20–23] did not use this symmetry, however. We start an algebraic study of Y-string-like potentials with this in mind. For the sake of technical simplicity we confine ourselves to the two-dimensional case here. We say here "Y-string-like potentials", rather than the Y-string potential, because the complete Y-string potential contains "additional" two-body terms that are valid only in certain parts of the tree-particle configuration space (a.k.a. triangle "shape space") and that do not have the O(2) dynamical symmetry. This wider class of three-body potentials has the same dynamical O(2) symmetry in shape space as the Y-string potential, thus making them equivalent in the algebraic sense. We must therefore first establish the basic properties of the dynamical symmetry of the Y-string potential.

In two dimensions (2D) the non-relativistic three-body kinetic energy is a quadratic form of the two Jacobi two-vector velocities, $\dot{\boldsymbol{\rho}}, \dot{\boldsymbol{\lambda}}$, so its "hyper-spherical symmetry" is O(4), and the residual dynamical symmetry of the Y-string potential is $O(2) \otimes O_L(2) \subset O(4)$, where $O_L(2)$ is the (orbital) angular momentum. As the $O(4)$ Lie group can be "factored" in two mutually commuting O(3) Lie groups: $O(4) \simeq O(3) \otimes O(3)$, one may use for our purposes many of the O(3) group results, such as the Clebsch-Gordan coefficients. The 3D case is substantially more complicated than the 2D one: the three-body "hyper-spherical symmetry" is O(6), and the residual dynamical symmetry of the Y-string potential is $O(2) \otimes O_L(3) \subset O(6)$. The $O(6)$ Lie group cannot be "factored" in two mutually commuting O(3) Lie groups and one cannot simply reduce this problem to one in the O(3) group. For these reasons we limit ourselves to the two-dimensional case in this paper.

Thus we are looking for the "chain" of symmetries $O(2) \otimes O_L(2) \subset O(3) \otimes O_L(2) \subset O(4)$. Rather than parametrize the energy E as a function of corresponding Casimir operators, and thus calculate the spectrum, as was done in Refs. [20–23], we reformulate the problem in terms of $O(4)$ variables and then bring the potential into a form that can be (exactly) solved, i.e. we expand it in O(4) hyperspherical harmonics. As the potential must be spherically symmetrical, this imposes and additional constraint on the allowed hyperspherical harmonics and one ends up with only a few (leading) terms: 1) the area-term containing the O(4) hyperspherical harmonic $\mathcal{Y}^{22}_{00}$, which, in turn is related to the O(3) spherical harmonic $Y_{20}(\alpha, \phi)$ of the shape space (hyper)spherical angles (α, ϕ), i.e., the V_4 term in the notation of Richard and Taxil [16]), that is present in both the two-body and the Y-string potentials; and 2) the O(2) symmetry-breaking term containing $\mathcal{Y}^{33}_{0\pm3} \simeq Y_{3\pm3}(\alpha, \phi)$, i.e., the V_6 term in the notation of Richard and Taxil [16], that is important in the two-body potential, and less so in the "complete" Y-string potential and not at all in Eq. (2). The energy spectrum is a function of the O(4) hyperspherical expansion coefficients for the potential, and of the O(4) Clebsch-Gordan coefficients, that are products of the ordinary O(3) Clebsch-Gordan coefficients.

3 Results

Next we proceed to evaluate the K=2,3 bands' splittings and compare them with the 3D case:

1) At the K=2 level, there are four SU(6) multiplets (other than the hyper-radial excitation $[56, 0^+]'$ of the K=0 state): $[70, 0^+]$, $[56, 2^+]$, $[70, 2^+]$, $[20, 1^+]$ in 3D. The main difference between the 2D and 3D is that the $[20, 0^+]$ state has vanishing orbital angular momentum in 2D, rather than unity, as in the 3D state $[20, 1^+]$.

The only difference between the 2D and 3D K=2 states' splittings is that the $[70, 0^+]$ and $[56, 2^+]$ states are degenerate in 2D, whereas in 3D they are split by one half of the energy difference between $[70, 2^+]$ and $[70, 0^+]$. This shows that the 2D case does relate fairly closely to the 3D one.

2) The energy splittings in the K=3 band, for the Y-string potential in 3D has not been worked out analytically, as yet, to our knowledge. Therefore, we compare our 2D Y-string potential K=3 results with the 3D K=3 two-body potential results of Ref. [16] and find certain similarities, and a few distinctions. There are six SU(6) multiplets in the K=3 sector (other than the hyper-radial excitation $[70, 1^-]''$ of the K=1 state): $[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$, $[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$ in 3D. The main difference between the 2D and 3D is that the $[70, 2^-]$ state disappears in 2D.

In 3D two-body potential the energy splittings have been divided in two parts in Ref. [16]: a) those due to the V_4 perturbation; and b) due to the V_6 perturbation. This corresponds to our Y_{20} and $Y_{3\pm3}$ terms, respectively.

a) In the $V_4 \neq 0$, $V_6 \to 0$ limit, the states are roughly divided in two groups: the $[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$ which are pushed down, and the $[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$ which are pushed up by the V_4 perturbation. Two pairs of states are left degenerate: ($[20, 1^-]$, $[56, 1^-]$) in the lower set and ($[56, 3^-]$, $[20, 3^-]$) in the upper set. In this limit in 2D we find complete degeneracy of all three members of the lower- ($[20, 1^-]$, $[56, 1^-]$, $[70, 3^-]$) and upper levels ($[56, 3^-]$, $[70, 2^-]$, $[20, 3^-]$).

b) In the $V_4 \neq 0$, $V_6 \neq 0$ case, the remaining degeneracy of states is removed in 3D: the $[20, 1^-]$ and the $[56, 1^-]$ are split in the "lower set" and the $[56, 3^-]$ and the $[20, 3^-]$ in the "upper set". In 2D we find the same sort of splitting, and in almost the same ratio of strengths.

So, in the K=2,3 bands, one sees similarities of dynamical symmetry-breaking patterns in 2D and 3D. This lends credence to the belief that this similarity may persist at higher values of K, where there are no known 3D results, at present.

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Exotic molecules of heavy quark hadrons

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Abstract. We discuss hadronic molecules containing both heavy and light quarks. The interactions are provided by meson exchanges between light quarks in the constituent hadrons. The tensor force in the one-pion exchange potential mixes states of different spins and angular momenta. This provides attraction and generates rich structure in exotic channels in the heavy quark sectors. The method has been applied to exotic baryons with a $\bar{c}$ or $\bar{b}$ quark, and exotic mesons containing $b\bar{b}$ including the recently found Z_b's.

Recent interest in hadron physics has been largely motivated by the observations of candidates for exotic multi-quark states which are not (easily) explained by the conventional quark model [1–4]. Many of them appear near the threshold region of their possible decay channels. The finding of the twin Z_b's is perhaps the most striking in that they appear very close to the BB^* and B^*B^* thresholds [4–6].

Strictly, multiquarks does not make much sense for light flavors especially for u and d quarks when the quark number is not a conserved quantity. In fact, they interact strongly at the energy scale of Λ_{QCD}, creating $q\bar{q}$ pairs and generating massive constituent quarks. It is known that it is a consequence of spontaneous breaking of chiral symmetry. In the low energy region we expect that such constituent quarks become active degrees of freedom as almost on-shell particles, forming exotic multi-quark states. Contrary to the light flavor sector heavy quarks such as c and b with mass $M \gg \Lambda_{QCD}$ conserve their quark number. Thus we can treat them as almost on shell particles with non-relativistic kinematics at low energies of typical hadron resonances.

Starting from the conventional quark model picture for orbitally excited states, multiquark configurations can mix with them because the typical excitation energy of about 0.5-1 GeV is sufficient to create a (constituent) $q\bar{q}$ pair. A color singlet multiquark system of more than the minimal number ($\bar{q}q$ or qqq) may form color singlet sub-systems (clusters) of hadrons. Clustering phenomena of multiparticle systems have been extensively studied in nuclear physics for many years [7]. Alpha particles saturate the dominant component of spin and isospin dependent nuclear force. The spin-isospin neutral alpha particles interact rather weakly and can form loosely bound states near the threshold regions of alpha decay.

In QCD, the state corresponding to alpha particle is a hadron which saturates the strong color dependent force. If these hadrons have sufficient amount of attraction (but weak as compared to the color force), they may form a bound or resonant state, which is the hadronic molecule. it must be a rather loosely bound state having an extending spatial structure to retain the identity of hadronic constituents. We expect that the relevant energy scale of binding and resonant states should be sufficiently small as compared to Λ_{QCD} of some hundreds MeV.

To establish exotic states is interesting not only for its own sake, but also because it is expected to reveal important aspects of non-perturbative dynamics of QCD. In this respect, as experimental observations imply, hadrons of light and heavy quarks are interesting, where more candidates of exotic states are observed. There, heavy quark symmetry and chiral symmetry play simultaneously. The former suppresses the spin dependent interactions, leading to degeneracy of different spin states. On the other hand, the latter is responsible for the pion coupling to the light quarks, which provides the source of the strong one pion exchange potential between heavy flavor hadrons. When these two conditions are satisfied, we expect the formation of exotic hadronic molecules. The spin and isospin dependent nature of the pion exchange potential as well as its orientation dependence of the tensor structure are the cause of the rich structure of hadron spectrum.

Based on these ideas, we have studied hadronic molecular states for exotic heavy baryons in Refs. [8–10], and for exotic heavy mesons in Ref. [11–13]. They are exotic not only due to hadronic molecular structure but also due to their exotic quantum numbers which are not accessible by the minimal number of quarks. In forming the hadronic molecular state, the following three points are important; (1) heavy mass which suppresses kinetic energy of constituent hadrons, (2) one pion exchange force of tensor nature which mixes the 0^- and 1^- states (DD^* and BB^*), and (3) degeneracy of 0^- and 1^- states which makes the wider space of coupled channels more effective to gain more attraction.

Hadronic molecules have been also studied for DN systems of ordinary quantum numbers [14,15]. These channels allow even more attraction leading to deeply bound states of a binding energy of order a few hundred MeV with much spatially compact configuration. Here $q\bar{q}$ annihilation is also possible, the treatment of which is more difficult than in the case of exotic channel without $q\bar{q}$ annihilation.

Turning to the exotic channels, employing an interactions between heavy flavor hadrons in a boson exchange model including one pion exchange potential, we find several bound and resonant states near the threshold regions. Many of them with small binding energy of order ten MeV or less have a rather extended size compatible to hadronic molecules. For baryons, we have found bound states of $J^P = 1/2^-$ states of exotic quark content $\bar{c}q$-qqq and $\bar{b}q$-qqq just below the threshold of $\bar{D}N$ and BN, respectively. Other resonant sates are also found for $J^P = 3/2^-, 1/2^+, 3/2^+, 5/2^+$ with similar structure of mass spectrum for c and b quark sectors [9, 10].

For mesons, in the hidden bottom sector, we have found ten $B\bar{B}, B\bar{B}^*, B^*\bar{B}^*$ molecules for low lying spin $J \leq 2$. In particular, the hidden bottom exotic mesons Z_b's are well predicted [11]. Further exotic states of double heavy flavor (charm and bottom) mesons are also found [12]. In Ref. [13], we have estimated the decay and production rates of various states in the limit of heavy quarks which are characteristic to the hadronic molecular structure. These theoretical predictions for rich structure of hadronic molecules can be studied in the facilities such as Belle, JPARC and LHC.

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Spin-Flavor Formalism for the Relativistic Coupled-Channels Quark Model*

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Abstract. The ongoing progress in our group of treating hadron resonances within a relativistic coupled-channels quark model is shortly discussed. Following earlier calculations along a simplified toy model for mesons, now all spin and flavor degrees of freedom are being included. Furthermore the approach is now extended also to baryons considered as genuine three-quark states.

Covariant calculations of properties of hadron resonances, such as hadronic decay widths, with relativistic constituent quark models (RCQM) have so far been limited to treating the resonant states as excited bound states rather than true resonances with finite widths. Corresponding predictions in general have been found to underestimate existing experimental data for hadronic decay widths [1–5]. The shortcomings are probably due to the usage of inadequate wave functions for the hadron resonances within single-channel models, such as the Goldstone-boson-exchange (GBE) RCQM [6,7]. Explicit couplings to mesonic channels might be needed.

We have started a project towards setting up a coupled-channels (CC) RCQM. A corresponding toy model applied to meson-like systems of scalar particles has already produced promising results, hinting to a broadening of the decay widths, when the coupling to the decay channels is included [8, 9]. We are now aiming at more realistic calculations both for meson and baryon resonances including all spin and flavor degrees of freedom. The corresponding formalism has been worked out and the implementation into the computer programs is under way.

For a CC RCQM we start out from an invariant mass operator in matrix form that includes beyond the channel of i particles in addition a channel i+1 with a further degree of freedom, say, the meson produced in a decay process. By eliminating the decay channel according to the Feshbach method one arrives at a complex mass operator, whose eigenvalue equation reads

$$\left[M_i + K\left(m - M_{i+1} + i0\right)^{-1} K^\dagger\right] |\psi_i >= m|\psi_i > \ . \qquad (1)$$

Here, M_i and M_{i+1} are the invariant mass operators of the i-particle and (i+1)-particle systems and $K^\dagger$ describes the transition dynamics (emission of the decay product). It should be noted that the mass eigenvalue m appears both in the

* Talk delivered by R. Kleinhappel

optical-potential term and also on the right-hand side of the eigenvalue equation. It assumes real values for bound states and complex values above the resonance thresholds. In the latter case its imaginary part is the half-width of the decaying resonance.

We exemplify the introduction of spin and flavor degrees of freedom in a CC RCQM along the ω-meson decaying into a ρ and a π. Here, the ω- and ρ-mesons are assumed to be built up by a constituent quark and a constituent antiquark, while the π is considered as a fundamental particle (namely, a Goldstone boson, much in analogy to the RCQM proposed in Refs. [6, 7]). The dynamics is thus mediated by GBE according to the interaction Lagrangian density in $SU(3)_F$

$$\mathcal{L}_I = i g_{PS} \bar{\psi} \gamma^5 \boldsymbol{\lambda}^F \psi \boldsymbol{\Phi}, \tag{2}$$

where $\bar{\psi}$ and ψ represent the (anti)quark fields and $\boldsymbol{\Phi}$ the boson (pseudoscalar meson) fields; $\boldsymbol{\lambda}^F$ are the Gell-Mann flavor matrices.

In the construction of the optical potential in Eq.(1), the first channel thus consists of confined quark-antiquark bound states, whereas the second channel adds the π. The spin and flavor degrees of freedom of the process in question are introduced as follows.

Spin states:

$$\rho, \omega: \begin{cases} |1,1\rangle = |\uparrow\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1,-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

Flavor states:

$$\chi = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \bar{\chi} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$$

$$\omega = -\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}), \qquad \begin{array}{l} \rho^+ = u\bar{d} \\ \rho^0 = \frac{1}{\sqrt{2}}\left(d\bar{d} - u\bar{u}\right) \\ \rho^- = -d\bar{u} \end{array}$$

In the optical potential, the spin degrees of freedom undergo Wigner rotations according to Lorentz boosts, and the flavor degrees of freedom specify the various possible decay modes.

The same process can also be treated at a hadronic level. The decay dynamics is then described by the coupling of the fundamental meson fields $\boldsymbol{\rho}^\beta$, $\boldsymbol{\pi}$, and ω^ν following the Lagrangian density [10]

$$\mathcal{L}_{\omega\rho\pi} = \frac{g_{\omega\pi\rho}}{\sqrt{m_\rho m_\omega}} \epsilon_{\alpha\beta\mu\nu} \left(\partial^\alpha \boldsymbol{\rho}^\beta\right) \cdot \left(\partial^\mu \boldsymbol{\pi}\right) \omega^\nu . \tag{3}$$

Here, the vector notation in the ρ and π cases is related to the isospin degrees of freedom, and $\epsilon_{\alpha\beta\mu\nu}$ denotes the Levi-Civita antisymmetric tensor. The macroscopic approach at the hadronic level relies on the assumption of vertex form factors. By comparing with the calculation at the quark level, a microscopic explanation of these form factors can be obtained.

The same approach can also be applied to baryons as three-quark systems. Here we will first consider the couplings of the N and the Δ to the π. Again the GBE dynamics is furnished by the Lagrangian in Eq. (2).

At the hadronic level the following Lagrangian densities are suggested [11]

$$\mathcal{L}_{NN\pi} = -\frac{f_{NN\pi}}{m_\pi}\bar{\Psi}\gamma_5\gamma^\mu\Psi\partial_\mu\phi\,, \tag{4}$$

$$\mathcal{L}_{\Delta N\pi} = -\frac{f_{\Delta N\pi}}{m_\pi}\bar{\Psi}\Psi^\mu\partial_\mu\phi + h.c.\,, \tag{5}$$

where the Ψ and Ψ^μ now represent N and Δ fields, respectively. The phenomenological vertex form factors needed here, can again be deduced with the help of the microscopic calculation along the CC RCQM, just by comparing the two approaches.

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Resonances in photo-induced reactions

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Abstract. The extraction of baryon resonance parameters from experimental data and their interpretation within QCD are central issues in hadron physics. To achieve these goals it is an essential prerequisite to have a sufficient amount of precision data which allows an unambiguous reconstruction of partial wave amplitudes for different reactions. Over the last years an intense effort has started to study photon-induced meson production. Many single and double spin-observables have been measured for the first time. This experimental progress will be illustrated by means of single and double π^0 photo-production. The focus will be on the impact of the new data for the unambiguous reconstruction of partial wave amplitudes.

1 Introduction

Meson scattering and meson production reactions below 3 GeV distinctively exhibit resonances, clearly organized in terms of flavor content, spin and parity, sitting on top of a non resonant "background. In lack of stringent predictions from strong QCD these resonances are usually interpreted in constituent quark models as excitations of massive quasi-particles bound by a confining potential. However, also the strong meson-baryon and meson-meson interaction could give rise to dynamically generated resonances. Chiral unitary methods and coupled channel calculations provide a theoretical framework to study the importance of resonances without including them explicitly in a model. Furthermore, lattice QCD simulations started to become predictive for dynamical quantities like strong decay widths of resonances and scattering phase shifts [1]. In the past, only calculations of approximate mass spectra in the heavy pion limit, where excited baryons are stable particles, were possible.

Empirically, N^* and Δ^* baryon resonance parameters like mass, width or pole position have been extracted for many years by partial-wave analyses of elastic and charge-exchange pion-nucleon scattering experiments. The most recent analysis of existing πN data has been performed by the George Washington Group [2]. Today there is no running experiment dedicated to study πN scattering anymore. However, options for a new generation of experiments with pion beams at Hades/GSI [3], ITEP [4] and J-PARC [5] are presently under discussion.

Instead of πN scattering, an immense effort started during the last decade to study baryon resonances with electromagnetic probes at various laboratories, mainly ELSA, Graal, JLAB, LEPS, LNS and MAMI. The motivation for this ongoing effort is 2-fold. The initial idea was to substantiate or to disprove the existence of questionable resonances or even to discover new states that couple only weakly to πN. Especially, above 2 GeV an abundance of states is predicted by quark models which are not identified in πN partial wave analyses. This fact is often called the "missing resonance" problem. As historically all information about resonances came from pionic reaction, the hope was to discover new states in e.g. $K\Lambda$, $K\Sigma$, ηN or ωN final states. The PDG lists in their latest edition a couple of new states which have been seen in some analyses of recent data [6]. However, there are still many ambiguities and the discussion is ongoing.

The second objective are high precision measurements of the excitation of established resonances with real and virtual photons in order to relax the model constraints in the analyses and understand the influence of background on the extraction and interpretation of resonance properties. Single and double spin observables turned out to be an indispensable prerequisite to address both issues. Such measurements with sufficient acceptance and statistics became technically feasibly only recently. A brief overview of the facilities is given in section 2.

A completely different approaches to baryon spectroscopy are presently being developed at the BES-III e^+e^- collider , where decays like $J/\psi \to \bar{N}N^* \to \bar{N}N\pi$ have been observed, or at the COMPASS experiment at CERN, where diffractive processes like $pp \to pp\pi\pi$ clearly show resonant structures. One important milestone in future experimental baryon spectroscopy will be the combination of all empirical information from very different experiments in order to identify universal, i.e. process independent, properties of genuine nucleon excitations and to quantify the impact of coupled channel dynamics.

2 Photon beam facilities

During the last ten years we noticed an enormous increase in high precision measurements of many single and double spin observables in photo-induced meson production. The experiments are still ongoing and many results are still preliminary. The reason for this unprecedented development was the combination of high-intensity polarized beams, polarized targets and hermetic detector systems which was technically realized at the CLAS spectrometer in Hall B at Jefferson Lab [7], the Crystal-Barrel experiment at the ELSA stretcher ring [8] and the Crystal Ball experiment at the Mainz Microtron MAMI [9]. CLAS is a large acceptance spectrometer based on a toroidal magnetic field configuration. Tracking chambers and time-of-flight detectors provide charge particle identification and momentum resolution. At CLAS, energy tagged, polarized photon beams with up to 6 GeV can be used. The Crystal Barrel calorimeter consisting of 1230 CsI(Tl) crystals is the core of the experimental setup at ELSA and provides excellent acceptance and resolution for multi-photon final states. The Crystal Ball at MAMI (see Fig. 2) consists of 672 NaI(Tl) crystals covering 93% of the full solid angle with an energy resolution of 1.7% for electromagnetic showers at 1 GeV. For charged particle tracking and identification two layers of coaxial multi-wire proportional chambers and a barrel of 24 scintillation counters surrounding the target are installed. The forward angular range is covered by the TAPS calorimeter consisting of BaF2 detectors and a Cerenkov detector.

The polarized target technique at all labs is based on Dynamic Nucleon Polarization (DNP) of solid-state target materials such as butanol, deuterated butanol, NH_3 or ^6LiD. The material is spin polarized by microwave pumping in an external magnetic field of 2.5T at temperatures of about 100mK. During the measurements, the spin orientation is frozen at temperatures of down to 20mK by a moderate longitudinal or transverse magnetic holding field of about 0.5T. The main technical challenge was the construction of a horizontal cryostat that fits into the detector geometry and keeps a temperature of about 20mK without adding too much material that would limit the particle detection. The underlying concept of the targets presently used at ELSA, JLAB and MAMI was developed in Bonn [10] and was successfully used for the first time in 1998 for measurements of the GDH sum rule in Mainz [11].

Fig. 1. Crystal-Ball detector at MAMI and the horizontal cryostat of the frozen spin target, which keeps temperatures of about 20mK.

3 $\gamma N \rightarrow \pi N$

The photo-production of pseudoscalar mesons has four spin degrees of freedom which define four complex scattering amplitudes for each isospin. These amplitudes manifest themselves in 16 different single and double spin observables, including experiments with polarized target, beam and nucleon recoil polarimetry. It is well known for a long time that the full knowledge of 8 selected observables at each energy and scattering angle completely determines all amplitudes in a mathematical sense. Such a procedure is called a ”Complete Experiment” [12]. It would then allow us to predict all remaining observables. However, in a real situation with statistical and systematic uncertainties this procedure is much more difficult. Furthermore, the goal is not a ”Complete Experiment” and the reconstruction of the 4 helicity amplitudes but an understanding of the underlying dynamics. For this, the knowledge of all relevant partial wave or multipole amplitudes is much more important.

Up to a certain maximum orbital angular momentum l_{max}, all $4l_{max}$ complex multipole amplitudes have to be determined from experiment (see Table 1). It can be shown that even a "Complete Experiment" is only of limited value to reach this goal because of the freedom to choose an angular and energy dependent overall phase [13]. Therefore, one has to determine the relevant multipoles directly from experimental data. Each observable, $O_i(W, \theta)$, can be expanded in term on Legendre polynomials:

$$O_i(W, \theta) = \sin^{\alpha_i} \theta \sum_{k=0}^{k_{max}} a_{ik}(W) P_k(\cos(\theta)), \quad \alpha_i = \{0, 1, 2\}. \tag{1}$$

Here k_{max} is given by the truncation to a certain maximum angular momentum. The coefficients $a_{ik}(W)$ are bilinear combinations of the $4l_{max}$ complex multipole amplitudes which can be reconstructed from the coefficients. For a detailed discussion of the concepts of a "Complete Experiment" and such a truncated partial wave analysis see [13].

Table 1. Multipole decomposition of the pion photo-production amplitude for $l_\pi \leq l_{max} = 2$. For each isospin, $4l_{max}$ complex multipoles have to be determined from experiment.

l_π	0	1			2			
J^p	$\frac{1}{2}^-$	$\frac{1}{2}^+$	$\frac{3}{2}^+$		$\frac{3}{2}^-$		$\frac{5}{2}^-$	
multiplole	E_{0+}	M_{1-}	M_{1+}	E_{1+}	M_{2-}	E_{2-}	M_{2+}	E_{2+}

A direct reconstruction of the relevant partial wave amplitudes was achieved for the first time in the energy region of the $\Delta(1232)$ resonance using a truncation to s- and p-waves ($l_{max} < 2$) and additional theoretical constraints [14].

At higher enegies this procedure requires precision measurements of several spin observables with a sufficiently fine energy binning, e.g. 10 MeV, and a full angular coverage. Below $E_{CM} \sim 2$ GeV, where a truncation to F- or G-wave ($l_{max} < 3$ or 4) is possible, already the measurement of 4-6 spin and double-spin observables could provide sufficient constraints for such a direct reconstruction. This has been shown in [15] using generated pseudo-data with realistic uncertainties that will be achieved with the Crystal-Ball experiment at MAMI within the next years. Preliminary results for many new target and beam-target asymmetries from ELSA, JLAB and MAMI have been presented e.g. at the last NSTAR conference [16]. However, the direct reconstruction of multiploles has not yet been achived above the $\Delta(1232)$ resonance region and one has to rely on fits using models for the energy-dependent amplitudes. Figure 3 summarizes the current status of such model dependent analyses in the case of the important lowest order multipole amplitudes, $J^p = 1/2^+(M_{1-})$ and $J^p = 1/2^-(E_{0+})$. Even at relatively low energies in the second resonance region there are significant deviations between different models. A summary of our current knowledge of multipole amplitudes for different flavor states can be found in [21].

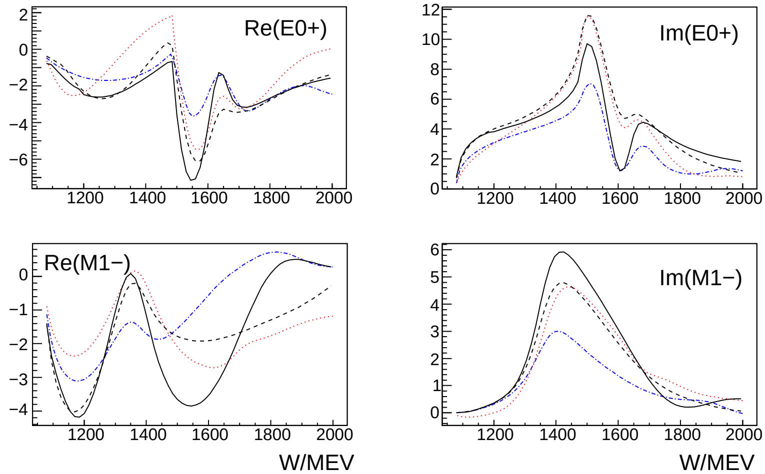

Fig. 2. Lowest order multipole amplitudes of the $\gamma p \to \pi^0 p$ reaction in units of $10^{-3}/M_\pi$. The curves are derived from fits of different models to existing data. The black solid and dashed lines represent the SAID 2011 and the SAID Chew-Mandelstam fits [17, 18], the MAID analysis gives the red dotted line [19]. Finally, the blue dashed-dotted curve is derived from the Bonn-Gatchina analysis [20].

In the case of the $\gamma p \to \pi^0 p$ reaction close to threshold a direct reconstruction of the amplitudes is more simple as the dynamics is dominated only by one s-wave, E_{0+} and 3 p-waves, M_{1-}, M_{1+} and E_{1+}. Furthermore, these multipoles are real between the $\pi^0 p$ and $\pi^+ n$ production thresholds. Above the $\pi^+ n$ threshold the E_{0+} amplitude becomes complex and shows a strong energy dependence due to the unitary cusp [22]. The imaginary parts of the p-waves remain negligible below ~ 180 MeV. With this truncation, the real parts of the multipoles can be reconstructed from measurements of two observables only, namely the differential cross section and the photon beam asymmetry

$$\Sigma = \frac{\sigma_\perp - \sigma_{||}}{\sigma_\perp + \sigma_{||}}. \tag{2}$$

Here $\sigma_\perp$ and $\sigma_{||}$ denote the differential cross sections with the photon polarization vector perpendicular and parallel to the $p\pi^0$ reaction plane. Both observables have recently been measured from threshold up to the Δ resonance region with unprecedented accuracy at the Crystal-Ball experiment at MAMI [23]. Fig. 3 show as an example the results of these measurements at the CM angle of 90^o as function of the incoming photon energy. The new data are compared to existing data and ChPT calculations with updated low-energy parameters [25] as well as the 2001 version of the DMT dynamical model [26]. The reconstruction of the multipoles is almost final and will be published soon [24].

With all relevant multipoles fixed by experiment the additional measurement of target (T) and beam-target (F) spin asymmetries will provide sensitivity to the charge exchange $\pi^+ n \to \pi^0 p$ scattering length from the unitary cusp which enters directly in the imaginary part of the E_{0+} amplitude. Therefore, threshold

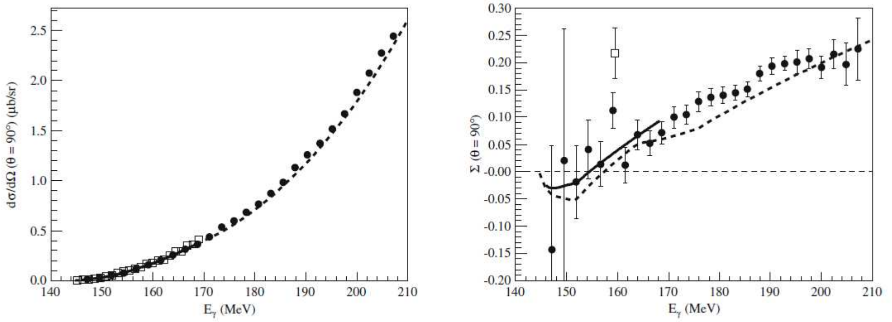

Fig. 3. Preliminary results from Crystal Ball at MAMI (solid circles) of the differential cross section and photon asymmetry for the $\gamma p \rightarrow \pi^o p$ reaction at pion CM angle of 90° compared to the older data from MAMI ([22], open squares) as well as some theory calculations. The solid lines are preliminary ChPT fits to the new data [25] and the dashed lines are a dynamical model [26].

π^0photo-production will enable us to study strong and electromagnetic isospin breaking in πN scattering by comparing the charge exchange scattering lengths for $\pi^+ n \rightarrow \pi^0 p$ and $\pi^- p \rightarrow \pi^0 n$ [23]. The ladder has recently been measured in pionic hydrogen [27].

4 $\gamma N \rightarrow \pi\pi N$

When looking at the production of meson pairs like $\pi\pi$ of $\pi\eta$ it is obvious that the dynamics can be much more complex and an analysis will be even more model dependent than in the case of single meson photo-production. Nevertheless, $\pi\pi N$ and $\pi\eta N$ finals states have attracted a lot of interest during the last years. These processes allow us to study resonances which have no significant branching ratio for a direct decay into the nucleon ground state. This is possible via sequential decays which involve intermediate excited states like $R \rightarrow R'\pi \rightarrow N\pi\pi$. Here R and R' denote nucleon resonances. Such decay chains are a phenomenon that can be observed in other quantum systems like atoms or nuclei as well. The theoretical interpretation is usually based on isobar models or effective field theories [28–32]. Typically, the reaction amplitude is constructed as a sum of background and resonance contributions. The background part contains nucleon Born terms as well as meson exchange in the t channel. The resonance part is a coherent sum of s-channel resonances decaying into $\pi\pi N$ via intermediate formation of meson-nucleon and meson-meson states ("isobars"). Despite significant differences between the models, all of them provide an acceptable description of the existing data. This observation clearly demonstrates, that further experimental and theory studies are necessary.

With the Crystal-Ball at MAMI we have recently studied the $\gamma N \rightarrow \pi^0\pi^0 N$ reactions by measurements of cross sections [33] and beam helicity asymmetries [34, 35].

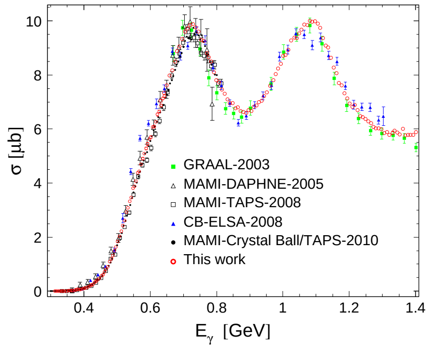

Fig. 4. Total cross section for the $\gamma p \to \pi^0\pi^0 p$ reaction as function of the incoming photon energy. The open circles show the precision that has been obtained at MAMI. Further information and references can be found in [33].

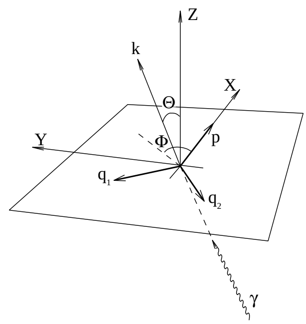

Fig. 5. The coordinate system is fixed by the momenta of the incident-photon, $\mathbf{k}$, out-going proton, $\mathbf{p}$ and the two pions, $\mathbf{q}_1$, and $\mathbf{q}_2$, in the center of mass system.

Fig. 4 shows the existing data for the total cross section. It is widely accepted that the $D_{13}(1520)$ resonance decaying to $\pi\Delta$ channel is responsible for the first peak at $E_\gamma \approx 730$ MeV. However, the underlying dynamics down to threshold as well as the behavior at higher energies have not been well understood so far. E.g., the minimum at $W = 1.6$ GeV and the second maximum at $W = 1.7$ GeV are described in Ref. [32] by the destructive interference between D_{13} and D_{33} partial wave amplitudes. In other models this behavior is explained by different resonance contributions, e.g. in the F_{15} partial wave. The high accuracy of the MAMI new data allowed us to make first steps towards a model independent partial wave analysis for the first time. In case of meson pair production the helicity amplitudes depend on the incoming photon energy, E_γ, the meson energies, ω_1 and ω_2 (Dalitz-Plot) and two angles, Θ and Φ, which are explained in Fig. 5. The angular distributions normalized to the total cross section, $W(E_\gamma, \omega_1, \omega_2, \Theta, \Phi) = \frac{1}{\sigma} \cdot \frac{d\sigma}{d\Omega}$ can now be expanded in terms of spherical harmonics $Y_{LM}(\Theta, \Phi)$. In a first step, we average the distributions over the meson energies, ω_1, ω_2:

$$W(E_\gamma, \Theta, \Phi) \equiv \frac{1}{\sigma} \int d\omega_1 d\omega_2 \frac{d\sigma}{d\Omega} = \sum_{L \geq 0} \sum_{M=-L}^{L} \sqrt{\frac{2J+1}{4\pi}} W_{LM}(E_\gamma) \cdot Y_{LM}(\Theta, \Phi) \tag{3}$$

This expansion determines the general structure of an angular distribution analogous to the expansion of the cross section for single-meson photo-production in terms of the Legendre polynomials (see Eq. 1). The moments $W_{LM}(E_\gamma)$ are bilinear combinations of the partial wave amplitudes. The exact relations have been worked out explicitly by Fix and Arenhoevel in ref. [36]. With the high precision data from MAMI it was possible to determine the moments $W_{LM}(E_\gamma)$ for the first time. The results are shown in Fig. 6. In case of the production of two

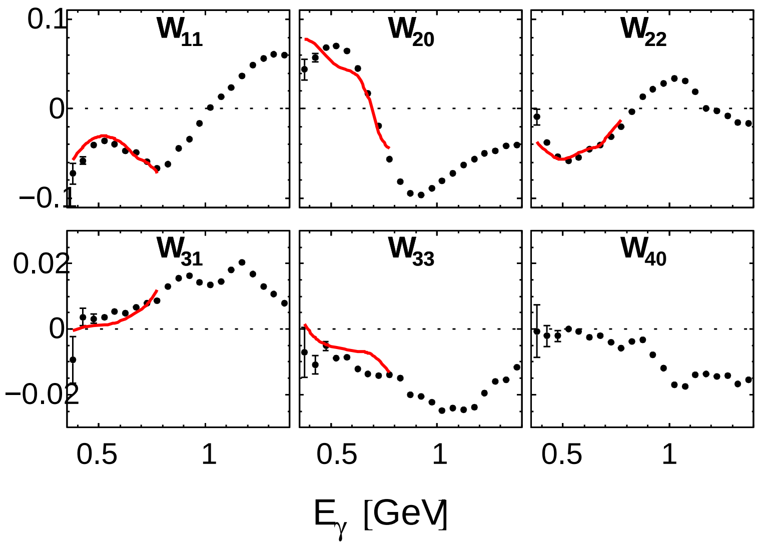

Fig. 6. First moments W_{LM} (normalized such that $W_{00} = 1$) as a function of the incident-photon energy. Our experimental results are shown by filled circles. The solid lines show the results of an isobar model fit [37].

identical particles, e.g. $\gamma p \to \pi^0\pi^0 p$, it can be shown, that the imaginary parts vanish exactly ($\mathrm{Im}(W_{LM}) = 0$). Already at low energies, the quantities W_{20} and W_{22}, which are given by an incoherent sum $J^P = 3/2^-$ and $3/2^+$ partial wave amplitudes, achieve relatively large values. This observation indicates an additional strong $3/2^-$ contribution, interfering with the $D_{13}(1520)$ resonance. This could support the dynamics found in Ref. [32] where a strong contribution from the $D_{33}(1700)$ resonance was found. Of course, the analysis of the moments $W_{LM}(E_\gamma)$ is only a very first step towards a full partial wave analysis of meson pair production processes. Nevertheless, it shows that data with very high precision, which will be available also for other observables in the future, will allow us to reduce the model dependence in the analysis procedures even for more complex final states significantly.

5 Conclusion

During the last decade an immense effort started to study baryon resonances in photo-induced meson production at various laboratories, mainly ELSA, Graal, JLAB, LEPS, LNS and MAMI. New high precision data for many spin observables are expected in the near future. A prerequisite for an unambiguous, model-independent extraction of resonance parameters is the reconstruction of partial wave or multipole amplitudes from experimental data. Resonances as well as effects from coupled channel dynamics manifest themselves in the analytic properties of these amplitudes. The upcoming data will allow us to minimze the model dependence in the determination of partial wave amplitudes in a systematic way.

This goal has already been achieved in π^0 photo-production close to threshold. The methods will be extended to higher photon energies and other final states (ηN, $K\Lambda$, $\pi\pi N$, etc.).

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Electroweak Form Factors of Baryon Ground States and Resonances*

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Abstract. We report of our ongoing studies of the electroweak structures of baryon ground and resonant states with flavors u, d, and s. Particular emphasis is laid on the comparison of the theoretical predictions of our relativistic constituent-quark model with recent experimental data for individual flavor contributions to the nucleon electromagnetic form factors.

The original results of covariant predictions by the Goldstone-boson-exchange relativistic constituent-quark model (GBE RCQM) [1,2] for the elastic electromagnetic and axial form factors of the nucleons were published in [3–5]. They were followed by detailed studies of the electric radii as well as magnetic moments of all light and strange baryons [6]. Comparisons to corresponding predictions by other RCQM, such as the relativized one-gluon-exchange (OGE) RCQM of Bhaduri, Cohler, and Nogami, as parameterized in ref. [7], were given in [8]. In the latter paper also comparative studies of point-form and instant-form calculations of the nucleon electromagnetic form factors were made, in order to find out the essential differences between the spectator-model constructions in either the instant and point forms of Poincaré-invariant quantum mechanics [9]. More recently we have performed detailed investigations of the axial charges of the nucleon and N^* resonances [10]; this kind of studies have then also been extended to the axial charges of the whole octet and decuplet of light and strange baryons [11]. The axial charges are connected with the πNN coupling constant via the Goldberger-Treiman relation. Therefore it has been very interesting to study also the πNN as well as $\pi N\Delta$ interaction vertices [12]. With these investigations we have reached a microscopic description of the Q^2 dependences of the πNN and $\pi N\Delta$ form factors together with predictions for the corresponding coupling constants $f_{\pi NN}$ and $f_{\pi N\Delta}$, which were found in agreement with phenomenology.

Especially the point form results obtained from the GBE RCQM have been found to be everywhere in quite good an agreement with existing experimental data. Further fine-tuning of the description is probably only needed for such sensitive observables like the N electric radii, some baryon magnetic moments, and the N axial charge [5, 6, 10, 11]. The studies have recently been extended to the Δ

* Talk delivered by W. Plessas

and the lowest hyperon states [13], for which, of course, no experimental data exist. In some instances, however, comparisons to data from lattice QCD have been possible, showing again a reasonable agreement in most cases.

With regard to the N elastic electromagnetic form factors an interesting issue has come about by the recent publication of phenomenological data for the flavor contributions to these form factors [14]. We were immediately interested in the performance of the GBE RCQM with regard to the u- and d-flavor contributions to the proton and neutron electromagnetic form factors as well as the electric radii and magnetic moments. First results were already reported at the Bled Workshop in 2011 (see [15]) and subsequently published in [16]. For the flavor contributions to the Sachs electric and magnetic form factors of both the proton and the neutron surprisingly good agreement with experimental data published in [14] is achieved. Slight deviations occur close to zero momentum transfer, since the electric radii and magnetic moments are not perfectly reproduced by the GBE RCQM [6].

Driven by these successes we have extended the flavor analyses to all the other octet and decuplet baryons [17]. Again, no experimental data exist. However, in some cases we can compare to calculations of flavor components to electromagnetic baryon form factors from lattice QCD [18]. This applies specifically to Σ^-, Σ^0, Σ^+, Ξ^-, and Ξ^0 baryons. In all cases a remarkably good agreement is found. In Figs. 1 and 2 we show as typical examples the electric and magnetic form factors of Σ^+, for which also other lattice-QCD data exist.

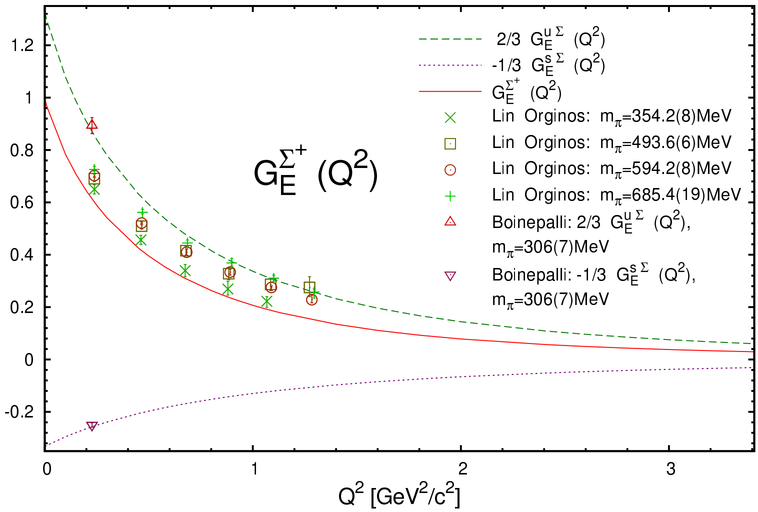

Fig. 1. Predictions of the GBE RCQM for the elastic electric form factor of Σ^+ (total: solid line, u-component: dashed line, s-component: dotted line) in comparison to data from lattice QCD for the total form factor [19] and for the u and s flavor contributions [18].

It should be emphasized that the covariant predictions of the GBE RCQM are parameter-free. No further parametrizations, such as meson-dressing effects nor constituent quark anomalous magnetic moments etc., have been included for the calculation of the electromagnetic current matrix elements. Still, a remarkably good agreement with the whole existing experimental data base and also with lattice-QCD data is generally achieved. It means that the RCQM is a reliable tool to treat at least the lowest-lying baryon states on reasonable grounds. Of course, refined wave functions such as the ones produced by the GBE RCQM must be employed and the framework must be fully relativistic.

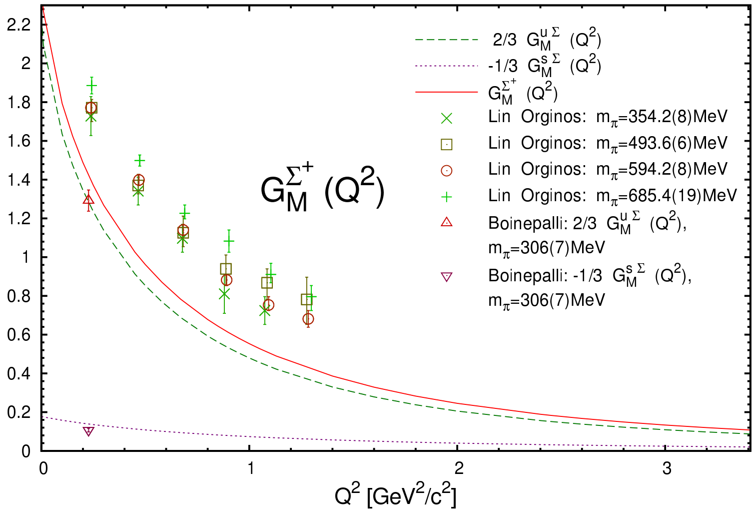

Fig. 2. Same as in Fig. 1 but for the elastic magnetic form factor of Σ^+.

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Pion- and photon-induced hadronic reactions in a combined coupled-channel analysis

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1 Introduction and Formalism

To gain insight into the non-perturbative sector of Quantum Chromodynamics the knowledge of the excited hadron spectrum is essential, providing the connection between experiment and QCD. Most resonances have been identified through elastic πN scattering in the past to present day. On the other hand, combining different reactions for resonance extraction allows to determine those states which couple only weakly to πN. The simultaneous analysis of different final states of pion- and photon-induced reactions is especially interesting regarding the new experimental window that has opened through the recent high-precision photon beam facilities, e.g., at ELSA, JLab and MAMI. Among other approaches, dynamical coupled-channel (DCC) models provide a sophisticated tool to analyze those data on excited baryons as they obey a maximum of theoretical requirements of the S-matrix such as analyticity to allow for a reliable extraction of resonances.

The DCC model developed and employed in this study (*Jülich model*) is based on an approach pursued over the years [1–9]. The scattering amplitude is obtained as the solution of a Lippmann-Schwinger equation (Eq. (1)) which guarantees two-body unitarity and approximates three-body unitarity,

$$\langle L'S'k'|T^{IJ}_{\mu\nu}|LSk\rangle = \langle L'S'k'|V^{IJ}_{\mu\nu}|LSk\rangle + \sum_{\gamma\, L''\, S''} \int_0^\infty k''^2\, dk'' \langle L'S'k'|V^{IJ}_{\mu\gamma}|L''S''k''\rangle \frac{1}{z - E_\gamma(k'') + i\epsilon} \langle L''S''k''|T^{IJ}_{\gamma\nu}|LSk\rangle \quad (1)$$

In Eq. 1 z is the scattering energy, $J\,(L)$ is the total angular (orbital angular) momentum, $S\,(I)$ is the total spin (isospin), $k(k', k'')$ are the incoming (outgoing, intermediate) momenta, and μ, ν, γ are channel indices. E_γ is the on-mass shell energy in channel γ [4]. The pseudo-potential V iterated in Eq. (1) is constructed from an effective interaction based on the Lagrangians of Wess and Zumino, supplemented by additional terms [2, 3] for including the Δ isobar, the ω, η, a_0 meson, and the σ. The channel space is given by $N\pi$, $N\eta$, $N\sigma$, $\Delta\pi$, $N\rho$, ΛK and ΣK. The non-resonant interactions are constructed of t- and u-channel exchanges of known mesons and baryons, while bare resonances can be considered as s-channel processes. The explicit treatment of the background in terms of t- and u-channel diagrams introduces strong correlations between the different partial waves and generates a non-trivial energy and angular dependence of the observables. Analyticity is respected in the sense that dispersive, real parts of intermediate states are included, as well as the correct structure of branch points, some of them being in the complex plane, and the correct off-shell behavior as dictated by the interaction Lagrangians. Thus, a reliable determination of resonance properties given in terms of pole positions and residues is possible. In the *Jülich model* SU(3) flavor symmetry is exploited to link the different reaction channels, while it is broken e.g. by physical masses and different cut-offs in the form factors of the vertices.

The extension of the model to photoproduction within a fully gauge-invariant approach has been accomplished recently [9].

In the following, the results of a simultaneous analysis of elastic πN-scattering and pion-induced K and η production within the framework of the *Jülich model* will be presented. In the present study, we perform a resonance analysis of the isospin $I = 1/2$ and $I = 3/2$ sector, considering the world data on the set of reactions $\pi^- p \to \eta n$, $K^0\Lambda$, $K^0\Sigma^0$, $K^+\Sigma^-$, and $\pi^+ p \to K^+\Sigma^+$, together with $\pi N \to \pi N$ scattering. Within the framework of DCC approaches, this is the first analysis of this type realized. The approach also includes the three effective $\pi\pi N$ channels $\pi\Delta$, σN and ρN. The considered energy range has been extended beyond 2 GeV and resonances up to $J = 9/2$ are included in this study.

The present study is the first step towards a global analysis of pion- and photon-induced production of πN, ηN, $K\Lambda$ and $K\Sigma$.

2 Results

While for the reaction $\pi N \to \pi N$ the partial waves from the GWU/SAID analysis [10] are used, for the inelastic channels, $\pi N \to \eta N$ and $\pi N \to KY$, we fit directly to total and differential cross sections as well as to polarization observables. The bulk of the existing data for the inelastic channels was obtained in the 1960's and 70's. Though many experiments have been carried out at different facilities, unfortunately, there are still energy ranges where the data situation is not ideal. All in all we include about 6000 data points in our analysis. The present solution was obtained in a fit procedure using MINUIT on the JUROPA supercomputer at the *Forschungszentrum Jülich*.

In the previous analysis [5], the reaction $\pi^+ p \to K^+\Sigma^+$ and πN scattering were considered and only resonance parameters, i.e. bare masses and couplings of the resonances to the different channels, were fitted. In this study, in addition the important T^{NP} parameters are varied. Those are the cut-offs of the form factors in t- and u-channel exchange diagrams.

Resonances with a total spin up to $J = 9/2$ are included, with the corresponding new parameters. One bare s-channel state is included in each of the $I = 1/2$ partial waves D13, D15, F15, P13, F17, H19 and G19, while we have two in S11 and P11. In the $I = 3/2$ sector, one bare s-channel state is included in the S31, D33, F35, P31, D35, F37, G37 and G39 partial waves and two are included in P33. These states couple to all channels πN, ρN, ηN, $\pi\Delta$, $K\Lambda$ and $K\Sigma$ if allowed by isospin. In total, we have 196 free parameters, of which 128 are resonance parameters and 68 belong to the T^{NP} part (t- and u-channel exchanges). The values of the parameters will be quoted elsewhere.

In Figs. 1, 2 and 3 we show a selection of our present results at typical energies.

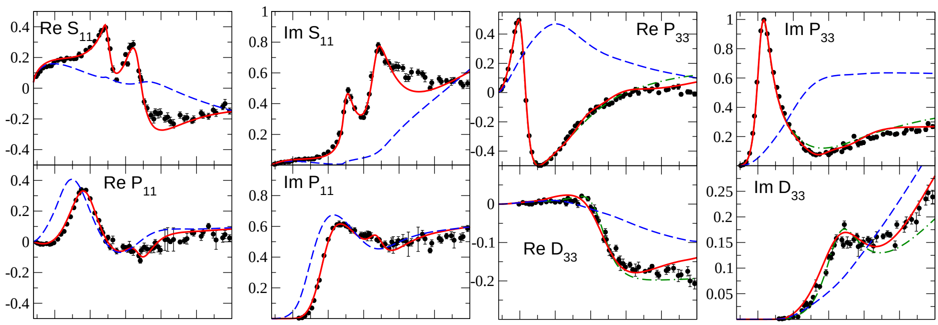

Fig. 1. Reaction $\pi N \to \pi N$, real and imaginary part of the S11, P11, P33 and D33 partial waves. (Red) solid lines: present solution. (Blue) dashed lines: only T^{NP}. (Green) dash-dotted lines: Jülich model, solution 2011 from Ref. [5]. Data points: GWU/SAID partial wave analysis (single energy solution) from Ref. [10]. (Preliminary)

In summary, a first combined analysis of the reactions $\pi N \to \pi N$, ηN, $K\Lambda$, and the three measured $K\Sigma$ final states $K^+\Sigma^+$, $K^0\Sigma^0$, and $K^+\Sigma^-$ within a dynamical coupled-channel framework has been performed. In the Lagrangian-based calculation, the full off-shell solution of the Lippman-Schwinger equation provides the correct analytic structure allowing for a reliable extrapolation into the complex plane to extract resonance pole positions and residues up to $J^P = 9/2^\pm$. The amplitude features also effective $\pi\pi N$ channels with branch points in the complex plane and a dispersive treatment of σ and ρ t-channel exchanges.

A publication of the full results together with a resonance analysis in terms of poles and residues is in progress.

The present results, in combination with the recent extension to pion photoproduction [9], will be used as input into a global study of pion- and photon-induced production of πN, ηN, $K\Lambda$ and $K\Sigma$. This means a major step towards the analysis of high-precision photoproduction data of ηN, $K\Lambda$, and $K\Sigma$ data produced, e.g., at ELSA, JLab, and MAMI.

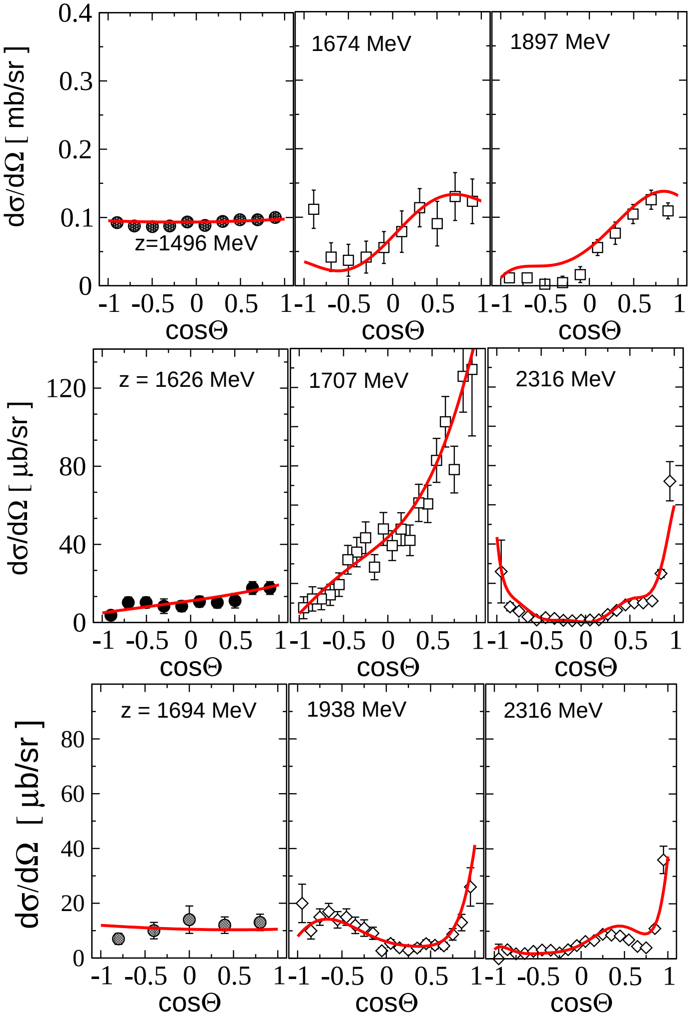

Fig. 2. Differential cross section for the reactions $\pi^- p \to \eta n$ (upper row), $\pi^- p \to K^0 \Lambda$ (middle) and $\pi^- p \to K^0 \Sigma^0$ (lower). (Red) solid lines: present solution. Selected results (Preliminary).

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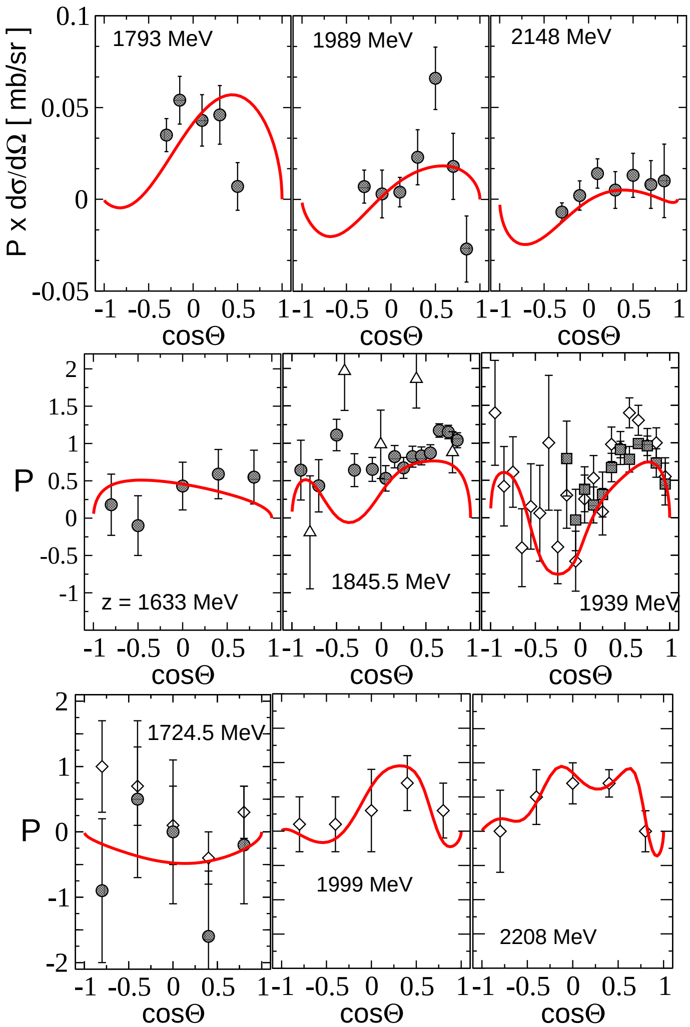

Fig. 3. Polarization for the reactions $\pi^- p \to \eta n$ (upper row), $\pi^- p \to K^0\Lambda$ (middle) and $\pi^- p \to K^0\Sigma^0$ (lower). (Red) solid lines: present solution. Selected results (Preliminary).

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The implications of the two solar mass neutron star for the strong interactions ⋆

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Abstract. The existence of a star with such a large mass means that the equation of state is stiff enough to provide a high enough pressure up to a fairly large density , about four times the nuclear density.

1 Introduction

Equations of state (EOS) that involve nonrelativistic constituents counteract gravitational infall of matter through a fermi pressure that is proportional to the density to the (5/3) power, unlike fermi pressures of relativistic constituents that go as density to the (4/3) power. Clearly the nonrelativistic nucleons are favoured over quarks for stiffer EOS's that can lead to larger mass for the stars.

However, a pure nonrelativistic fermi gas of neutrons is not sufficient to give large masses for neutron stars. Such a non interacting gas can give stars of maximum mass 0.7 solar mass - this a general relativistic effect coming from the Oppenheimer – Volkoff equation where the pressure needs to be proportional to density to a power greater than (5/3) . On the other hand, for white dwarfs fermi pressure of a nonrelativistic electron gas is all that is needed to counteract gravity and have stable stars. This enhanced pressure is provided by nuclear interactions like the hard core.

It is known that stars with soft, relativistic quark matter cores surrounded by a nonrelativistic n+p+e plasma in beta equilibrium can give maximum mass for neutron stars ~ 1.6 solar mass [1, 2].

It is also known that there are many nucleon based neutron stars models that have neutron stars with maximum mass above 2 solar masses, eg. the APR 98 EOS of Akmal, Pandharipande and Ravenhall [3].

If we can show that matter in neutron stars is entirely composed of nucleon degrees of freedom then we can have a simple resolution of this problem. *Can we?*

⋆ Talk delivered by V. Soni

2 The Maxwell construction between nuclear matter and quark matter

A simple way to look at whether nucleons can dissolve into quark matter is to plot E_B, the energy per baryon in the ground state of both phases versus $1/n_B$, where n_B is the baryon density. The slope of the common tangent between the two phases then gives the pressure and the intercept the common baryon chemical potential. For the quark matter equation of state see Fig.1.

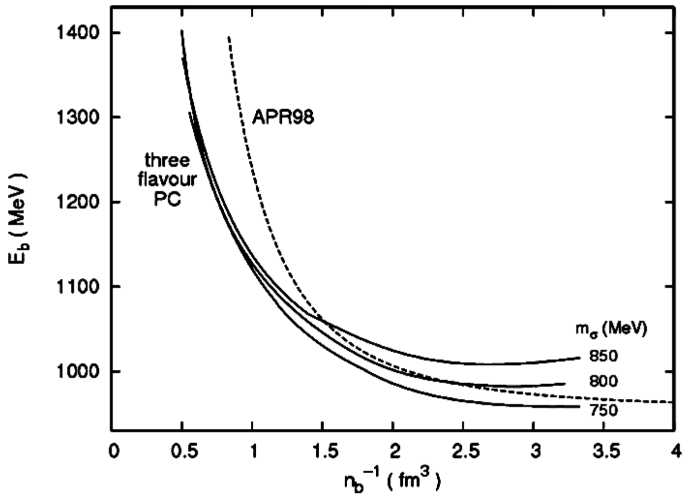

Fig. 1. The Maxwell construction: Energy per baryon plotted against the reciprocal of the baryon number density for APR98 equation of state (dashed line) and the 3-flavour pion-condensed phase (PC) for three different values of m_σ (solid lines). A common tangent between the PC phase and the APR98 phase in this diagram gives the phase transition between them. The slope of a tangent gives the negative of the pressure at that point, and its intercept gives the chemical potential. As this figure indicates, the transition pressure moves up with increasing m_σ, and at m_σ below ~750 MeV a common tangent between these two phases cannot be obtained. (From Fig. 2 of Soni and Bhattacharya [2] or Fig. 3 of the preprint [4])

This is based on an effective chiral symmetric theory that is QCD coupled to a chiral sigma model. The theory thus preserves the symmetries of QCD. In this effective theory chiral symmetry is spontaneously broken and the degrees of freedom are constituent quarks which couple to colour singlet, sigma and pion fields as well as gluons. The nucleon in such a theory is a colour singlet quark soliton with three valence quark bound states [5]. The quark meson couplings are set by matching mass of the nucleon to its experimental value and the meson self coupling which sets the tree level sigma particle mass is set from pi-pi scattering to be of order 800 MeV. Such an effective theory has a range of validity up to centre of mass energies (or quark chemical potentials) of ~ 800 MeV. For details we refer the reader to ref. [2].

This is the simplest effective chiral symmetric theory for the strong interactions at intermediate scale and we use this consistently to describe, both, the composite nucleon of quark boundstates and quark matter. We expect it to be valid till the intermediate scales quoted above. Of course inclusion of the higher mesonic degrees of freedom like the rho and A1 would make for a more complete description. We work at the mean field level the gluon interactions are subsumed in the colour singlet sigma and pion fields they generate. We could further add perturbative gluon mediated corrections but they do not make an appreciable difference.

As can be seen from Fig.1, it is the tree level value of the sigma mass that determines the intersection of the two phases; the higher the mass the higher the density at which the transition to quark matter will take place. In [2] it was found that above, $m_\sigma \sim 850$ MeV, stars with quark matter cores become unstable as their mass goes up beyond the allowed maximum mass. So, if we want purely nuclear stars we should, in this model, work at, $m_\sigma \geq 850$ MeV [2].

From Fig. 1, for the tree level value of the sigma mass ~850 MeV, the common tangent in the two phases starts at $1/n_B \sim 1.75$ fm^3 ($n_B \sim 0,57/\text{fm}^3$) in the nuclear phase of APR [A18 + dv +UIX] and ends up at $1/n_B \sim 1.25$ fm^3 ($n_B \sim 0.8/$ fm^3) in the quark matter phase.

At the above densities between the two phases there is a mixed phase at the pressure given by the slope of the common tangent and the at a baryon chemical potential given by the intercept of the common tangent on the vertical axis. If we are to stay in the nuclear phase the best way is to look at the central density of the nuclear (APR) stars and if it so happens that they are at lower density than that at which the above phase transition begins the we can safely say that the star remains in the nuclear phase.

Going Back to the APR phase in in fig 11 of APR [3] we find that for the APR [A18 + dv +UIX] the central density of a star of 1.8 solar mass is $n_B \sim 0.62$ /fm^3, very close to the initial density at which the phase transition begins.

The reason we are taking a static star mass of 1.8 solar mass from APR [3] is that for PSR-1614, the star is rotating fast at a period of 3 millisec and we expect a ~ 15% diminution of the central density from the rotation [6]. Equivalently, since the above paper reports results for static stars, the central density of a fast rotating 1.97 solar mass star ~ the central density of a static 1.8 solar mass star.

Now we have found that in above scenario the central density is of the same order as the density at which the above phase transition begins in the nuclear phase. Ideally we would like the central density to be a little less than the initial density at which the above phase transition begins in the nuclear phase.

3 Beyond the Maxwell tangent construction for the phase transition

How do we change the crossover and Maxwell tangent construction for the phase transition? There are 2 ways of moving the crossover between the 2 phases and also the initial density at which the above phase transition begins in the nuclear phase to higher density.

(i) By increasing the tree level mass of the sigma we can move the quark matter curve up (Fig. 1), thus moving the initial density at which the above phase transition begins in the nuclear phase to higher density. However we have to be careful. There is not much freedom here, as this is what also determines the $\pi - \pi$ scattering.

(ii) By softening the nuclear EOS at high density, e.g. by including hyperons or pi condensates. But this will increase the central density of the star and also reduce its maximum mass.

Of these the option (i) is a safer option as it does not disturb the central density or maximum mass of the nuclear star. However, the Maxwell construction is not the final word on the phase transition. The exact nature of the transition is not just given by the energy /baryon in the quark matter phase (which depends mainly on m_σ) but will depend on the quark binding inside the nucleon (which depends mainly o the quark meson coupling) and the nucleon nucleon repulsion as we squeeze them. This is not captured by the Maxwell construction.

The nucleon binding in this model is very high (Fig. 2) [5]

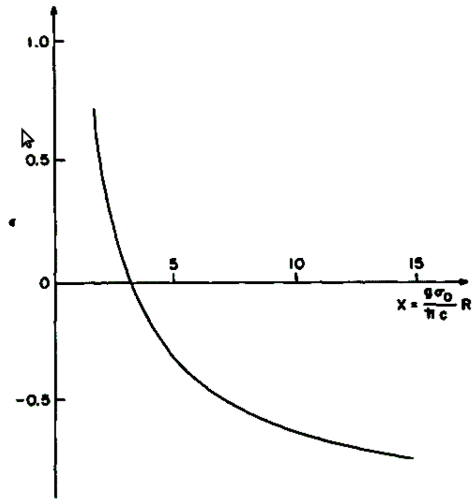

Fig. 2. Dependence of the quark energy on the soliton size X in the quark soliton model (From Fig. 2 of Kahana, Ripka and Soni [5])

The quark eigenfunctions are smaller than the radius of the nucleon; they spread over about 0.5 fermi. This yields a quark wave function size of ~1 fermi or kinetic energy of about 200 MeV. The unbound mass of the quark is given by gf_π ~ 500 MeV and effectively they must contribute 313 MeV to the mass of the nucleon , giving the quark binding energy of ~ 400 MeV.

We can see that the quarks will become unbound (go to the continuum) when the energy eigenvalue is larger than the unbound mass of the quark which is given by $m_{\text{free}} = g f_{\pi} \sim 500$ MeV. This happens when in the dimensionless units used in Fig. 2 $\epsilon \geq 1$ at $X = 3.12/1.94 = 1.6$. This translates into $R = (1.6/2.5)$ fm^{-1} $\sim$ 0.6 fm^{-1} .This is the effective radius of the squeezed nucleon at which the bound state quarks are liberated to the continuum. By inverting the volume occupied by the nucleon and assuming hexagonal close packing, this translates to nucleon density of $1/(6R^3) \sim 0.77$ fm^{-3}.

Thus the quark bound states in nucleon persist untill a much higher density $\sim 0.8/\text{fm}^3$. In other words, nucleons can survive well above the density at which the Maxwell phase transition begins and appreciably above the central density of the APR 2-solar-mass star.

Another feature is the the nucleon nucleon potential. It has been found for skyrmions and such quark-quark solitons with skyrmion configurations that there is a strong N-N repulsion that forces the lowest baryon number $N_B = 2$ configuration to become toroidal [7]. This is an indication that nucleon nucleon potential becomes strongly repulsive.

It thus follows that the phase transition from nuclear to quark matter will encounter a potential barrier before the quarks can go free. This effect cannot be seen by the coarse Maxwell construction which does not track their transition. This will modify the simple minded Maxwell construction which assumes only the energy and pressure that exist independently in the 2 phases. Here is where the internal structure of the nucleon will delay the transition.

All in all this produces a very plausible scenario of how the ~2 solar mass star can be achieved in a purely nuclear phase.

4 Consequences and discussion

A simple consequence of this unexpected scenario at high density is that the the phase diagram of QCD which plots temperature versus baryon chemical potential, the quark matter transition for finite density (in the range above) will be lifted up along the temperature axis.

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Highly excited states of baryons in large N_c QCD*

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Abstract. The masses of highly excited negative parity baryons belonging to the $N = 3$ band are calculated in the $1/N_c$ expansion method of QCD. We use a procedure which allows to write the mass formula by using a small number of linearly independent operators. The numerical fit of the dynamical coefficients in the mass formula show that the pure spin and pure flavor terms are dominant in the expansion, like for the $N = 1$ band. We present the trend of some important dynamical coefficients as a function of the band number N or alternatively of the excitation energy.

1 The status of the $1/N_c$ expansion method

The large N_c QCD, or alternatively the $1/N_c$ expansion method, proposed by 't Hooft [1] in 1974 and implemented by Witten in 1979 [2] became a valuable tool to study baryon properties in terms of the parameter $1/N_c$ where N_c is the number of colors. According to Witten's intuitive picture, a baryon containing N_c quarks is seen as a bound state in an average self-consistent potential of a Hartree type and the corrections to the Hartree approximation are of order $1/N_c$. These corrections capture the key phenomenological features of the baryon structure.

Ten years after 't Hooft's work, Gervais and Sakita [3] and independently Dashen and Manohar in 1993 [4] derived a set of consistency conditions for the pion-baryon coupling constants which imply that the large N_c limit of QCD has an exact contracted $SU(2N_f)_c$ symmetry when $N_c \to \infty$, N_f being the number of flavors. For ground state baryons the $SU(2N_f)$ symmetry is broken by corrections proportional to $1/N_c$ [5,6].

Analogous to s-wave baryons, consistency conditions which constrain the strong couplings of excited baryons to pions were derived in Ref. [7]. These consistency conditions predict the equality between pion couplings to excited states and pion couplings to s-wave baryons. These predictions are consistent with the nonrelativistic quark model.

A few years later, in the spirit of the Hartree approximation a procedure for constructing large N_c baryon wave functions with mixed symmetric spin-flavor parts has been proposed [8] and an operator analysis was performed for $\ell = 1$ baryons [9]. It was proven that, for such states, the $SU(2N_f)$ breaking occurs at

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order N_c^0, instead of $1/N_c$, as it is the case for ground and also for symmetric excited states $[56, \ell^+]$ (for the latter see Refs. [10,11]). This procedure has been extended to positive parity nonstrange baryons belonging to the $[70, \ell^+]$ multiplets with $\ell = 0$ and 2 [12]. In addition, in Ref. [12], the dependence of the contribution of the linear term in N_c, of the spin-orbit and of the spin-spin terms in the mass formula was presented as a function of the excitation energy or alternatively in terms of the band number N. Based on this analysis an impressive global compatibility between the $1/N_c$ expansion and the quark model results for $N = 0, 1, 2$ and 4 was found [13] (for a review see Ref. [14]). More recently the $[70, 1^-]$ multiplet was reanalyzed by using an exact wave function, instead of the Hartree-type wave function, which allowed to keep control of the Pauli principle at any stage of the calculations [21]. The novelty was that the isospin term, neglected previously [9] becomes as dominant in Δ resonances as the spin term in N^* resonances.

The purpose of this work is mainly to complete the analysis of the excited states by including the $N = 3$ band for which results were missing in the systematic analysis of Ref. [12]. An incentive for studying highly excited states with $\ell = 3$ has been given by a recent paper [15] where the compatibility between the two alternative pictures for baryon resonances namely the *quark-shell picture* and the *meson-nucleon scattering* picture defined in the framework of chiral soliton models [16,17] has been proven explicitly. This work was an extension of the analysis made independently by Cohen and Lebed [18, 19] and Pirjol and Schat [20] for low excited states with $\ell = 1$.

As explained below, we shall analyze the resonances thought to belong to the $N = 3$ band by using the procedure we have proposed in Ref. [21] for the $N = 1$ band. Details can be found in Ref. [22].

2 Mixed symmetric baryon states

If an excited baryon belongs to a symmetric SU(6) multiplet the N_c-quark system can be treated similarly to the ground state in the flavour-spin degrees of freedom, but one has to take into account the presence of an orbital excitation in the space part of the wave function [10, 11]. If the baryon state is described by a mixed symmetric representation of SU(6) , the [**70**] at $N_c = 3$, the treatment becomes more complicated. In particular, the resonances up to about 2 GeV are thought to belong to $[\mathbf{70}, 1^-]$, $[\mathbf{70}, 0^+]$ or $[\mathbf{70}, 2^+]$ multiplets and beyond to 2 GeV to $[\mathbf{70}, 3^-]$, $[\mathbf{70}, 5^-]$, etc.

There are two ways of studying mixed symmetric multiplets. The standard one is inspired by the Hartree approximation [8] where an excited baryon is described by a symmetric core plus an excited quark coupled to this core, see *e.g.* [9, 12, 23, 24]. The core is treated in a way similar to that of the ground state. In this method each $SU(2N_f) \times O(3)$ generator is separated into two parts

$$S^i = s^i + S^i_c; \quad T^a = t^a + T^a_c; \quad G^{ia} = g^{ia} + G^{ia}_c; \quad \ell^i = \ell^i_q + \ell^i_c, \tag{1}$$

where s^i, t^a, g^{ia} and ℓ^i_q are the excited quark operators and S^i_c, T^a_c, G^{ia}_c and ℓ^i_c the corresponding core operators.

As an alternative, we have proposed a method where all identical quarks are treated on the same footing and we have an exact wave function in the orbital-flavor-spin space. The procedure has been successfully applied to the $N = 1$ band [21, 25, 26]. In the following we shall adopt this procedure to analyze the $N = 3$ band.

3 The mass operator

When hyperons are included in the analysis, the SU(3) symmetry must be broken and the mass operator takes the following general form [27]

$$M = \sum_i c_i O_i + \sum_i d_i B_i. \tag{2}$$

The formula contains two types of operators. The first type are the operators O_i, which are invariant under $SU(N_f)$ and are defined as

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)}, \tag{3}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{SF}^{(k)}$ a k-rank tensor in SU(2)-spin. Thus O_i are rotational invariant. For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms. The rank $k = 2$ tensor operator of SO(3) is

$$L^{(2)ij} = \frac{1}{2}\left\{L^i, L^j\right\} - \frac{1}{3}\delta_{i,-j}\mathbf{L}\cdot\mathbf{L}, \tag{4}$$

which we choose to act on the orbital wave function $|\ell m_\ell\rangle$ of the whole system of N_c quarks (see Ref. [12] for the normalization of $L^{(2)ij}$). The second type are the operators B_i which are SU(3) breaking and are defined to have zero expectation values for non-strange baryons. Due to the scarcity of data in the $N = 3$ band hyperons, here we consider only one four-star hyperon $\Lambda(2100)7/2^-$ and accordingly include only one of these operators, namely $B_1 = -\mathcal{S}$ where $\mathcal{S}$ is the strangeness.

The values of the coefficients c_i and d_i which encode the QCD dynamics are determined from numerical fits to data. Table 1 gives the list of O_i and B_i operators together with their coefficients, which we believe to be the most relevant for the present study. The choice is based on our previous experience with the $N = 1$ band [26]. In this table the first nontrivial operator is the spin-orbit operator O_2. In the spirit of the Hartree picture [2] we identify the spin-orbit operator with the single-particle operator

$$\ell \cdot s = \sum_{i=1}^{N_c} \ell(i) \cdot s(i), \tag{5}$$

the matrix elements of which are of order N_c^0. For simplicity we ignore the two-body part of the spin-orbit operator, denoted by $1/N_c\,(\ell \cdot S_c)$ in Ref. [9], as being of a lower order (we remind that the lower case operators $\ell(i)$ act on the excited quark and S_c is the core spin operator).

Table 1. Operators and their coefficients in the mass formula obtained from numerical fits. The values of c_i and d_i are indicated under the heading Fit n $(n = 1, 2, 3, 4)$ from Ref. [22].

Operator	Fit 1 (MeV)	Fit 2 (MeV)	Fit 3 (MeV)	Fit 4 (MeV)
$O_1 = N_c \mathbb{1}$	$c_1 = 672 \pm 8$	$c_1 = 673 \pm 7$	$c_1 = 672 \pm 8$	$c_1 = 673 \pm 7$
$O_2 = \ell^i s^i$	$c_2 = 18 \pm 19$	$c_2 = 17 \pm 18$	$c_2 = 19 \pm 9$	$c_2 = 20 \pm 9$
$O_3 = \frac{1}{N_c} S^i S^i$	$c_3 = 121 \pm 59$	$c_3 = 115 \pm 46$	$c_3 = 120 \pm 58$	$c_3 = 112 \pm 42$
$O_4 = \frac{1}{N_c} [T^a T^a - \frac{1}{12} N_c (N_c + 6)]$	$c_4 = 202 \pm 41$	$c_4 = 200 \pm 40$	$c_4 = 205 \pm 27$	$c_4 = 205 \pm 27$
$O_5 = \frac{3}{N_c} L^i T^a G^{ia}$	$c_5 = 1 \pm 13$	$c_5 = 2 \pm 12$		
$O_6 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}$	$c_6 = 1 \pm 6$		$c_6 = 1 \pm 5$	
$B_1 = -\mathcal{S}$	$d_1 = 108 \pm 93$	$d_1 = 108 \pm 92$	$d_1 = 109 \pm 93$	$d_1 = 108 \pm 92$
χ^2_{dof}	1.23	0.93	0.93	0.75

The spin operator O_3 and the flavor operator O_4 are two-body and linearly independent. The expectation values of O_3 are simply equal to $\frac{1}{N_c} S(S+1)$ where S is the spin of the whole system. For nonstrange baryons the eigenvalue of O_4 is $\frac{1}{N_c} I(I+1)$ where I is the isospin. For the flavor singlet Λ the eigenvalue is $-(2N_c + 3)/4N_c$, favourably negative, as shown in Ref. [22].

Note that the definition of the operator O_4, indicated in Table 1, is such as to recover the matrix elements of the usual $1/N_c(T^a T^a)$ in SU(4), by subtracting $N_c(N_c + 6)/12$. This is understood by using Eq. (30) of Ref. [25] for the matrix elements of $1/N_c(T^a T^a)$ extended to SU(6). Then, it turns out that the expectation values of O_4 are positive for octets and decuplets and of order N_c^{-1}, as in SU(4), and negative and of order N_c^0 for flavor singlets.

The operators O_5 and O_6 are also two-body, which means that they carry a factor $1/N_c$ in the definition. However, as G^{ia} sums coherently, it introduces an extra factor N_c and makes all the matrix elements of O_6 of order N_c^0 [25]. These matrix elements are obtained from the formulas (B2) and (B4) of Ref. [26] where the multiplet $[70, 1^-]$ has been discussed. Interestingly, when $N_c = 3$, the contribution of O_5 cancels out for flavor singlets, like for $\ell = 1$ [26]. This property follows from the analytic form of the isoscalar factors given in Ref. [26].

We remind that the SU(6) generators S^i, T^a and G^{ia} and the $O(3)$ generators L^i of Eq. (4) act on the total wave function of the N_c system of quarks as proposed in Refs. [21], [25] and [26]. The advantage of this procedure over the standard one, where the system is separated into a ground state core + an excited quark, is that the number of relevant operators needed in the fit is usually smaller than the number of data and it allows a better understanding of their role in the mass formula, in particular the role of the isospin operator O_4 which has always been omitted in the symmetric core + excited quark procedure. We should also mention that in our approach the permutation symmetry is exact [21].

Among the operators containing angular momentum components, besides the spin-orbit, we have included the operators O_5 and O_6, to check whether or not they bring feeble contributions, as it was the case in the $N = 1$ band. From Table 1 one can see that their coefficients are indeed negligible either included together as in Fit 1 or separately as in Fit 2 and 3. Thus in the expansion series, besides O_1, proportional to N_c, the most dominant operators are the pure spin O_3 and the pure isospin O_4.

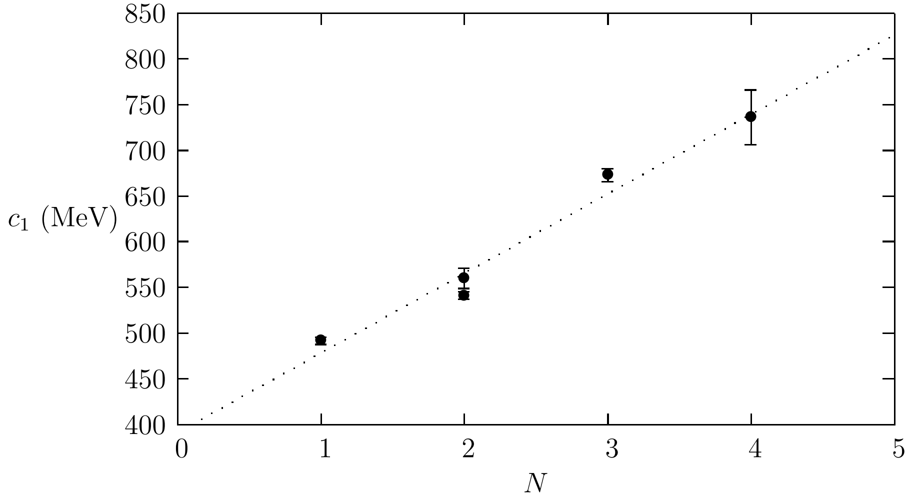

Fig. 1. The coefficient c_1 as a function of the band number N: $N = 1$ Ref. [26], N = 2 Ref. [10] for $[\mathbf{56}, 2^+]$ and Ref. [12] for $[\mathbf{70}, \ell^+]$, N = 3 Ref. [22], N = 4 Ref. [11]. The straight line is drawn to guide the eye.

4 Global results

The above analysis helps us to complete previous results for N = 1, 2 and 4 with the values of c_i obtained for N = 3. Therefore we can draw now a complete picture of the dependence of the coefficients c_1 and c_2 on N in analogy to Ref. [12] where results for N = 3 were missing. The new pictures are shown in Figs. 1 and 2. One can see that the values of c_1 follow nearly a straight line which can give rise to a Regge trajectory. Remember that c_1 describes the bulk content of the baryon mass, $c_1 N_c$ being the most dominant mass term. In a quark model language it represents the kinetic plus the confinement energy. As as discussed in Refs. [13, 14] the band number N also emerges from the spin independent part of a semi-relativistic quark model. If this part contributes to the total mass by a quantity denoted by M_0, then one can make the identification

$$c_1^2 = M_0^2/9 \tag{6}$$

when N_c = 3. In this way one can compare the Regge trajectory obtainable from the above results with that of a standard constituent quark model. It turns out

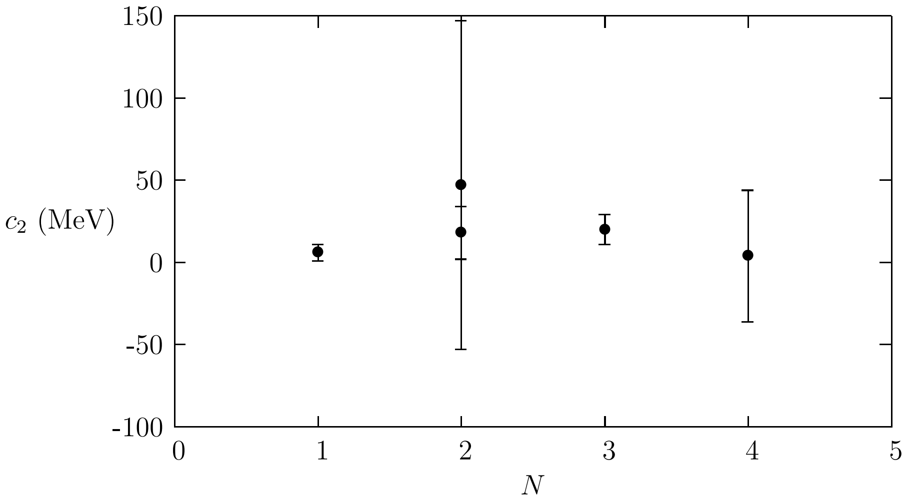

Fig. 2. Same as Figure 1 but for the coefficient c_2.

that they are close to each other [13,14]. and the value obtained here for c_1 at N = 3, missing in the previous work, is entirely compatible with the previous picture.

The behaviour of c_2 shows that the spin-orbit operator contributes very little to the mass, at all energies, in agreement to quark models, where it is usually neglected. Note that the behaviour of c_2 in Fig. 2 is slightly different from that of [12], because we presently take the value of c_2 at N = 1 from Ref. [26] (Fit 3 giving the lowest $\chi^2_{\rm dof}$) for consistency with our treatment, instead of that of Ref. [9], based on the ground state core + excited quark, the only available at the time the paper [12] was published.

We refrain ourselves from presenting the global picture of c_3, the spin term coefficient, because the results for positive parity mixed symmetric states are obtained on the one hand in the core + excited quark approach, where the isospin term is missing and on the other hand, for negative parity states where it is present, our approach is used. This term competes with the spin term. We plan to reanalyze the $[\mathbf{70}, \ell^+]$ multiplets before drawing a complete picture of c_3.

5 Conclusions

We have used a procedure which allows to write the mass formula by using a small number of linearly independent operators for spin-flavour mixed symmetric states of SU(6). The numerical fits of the dynamical coefficients in the mass formula for $N = 3$ band resonances show that the pure spin and pure flavor terms are dominant in the $1/N_c$ expansion, like for $N = 1$ resonances. This proves that the isospin term cannot be neglected, as it was the case in the ground state + excited quark procedure. We have shown the dependence of the dynamical coefficients c_1 and c_2 as a function of the band number N or alternatively of the excitation energy for $N = 1, 2, 3$ and 4 bands.

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Poles as a link between QCD and scattering theory (old and contemporary knowledge)

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An overview of existing knowledge about definition of a resonance, and quantification of resonance signals have been given. A special attention has paid to explaining why the definition of a resonance is in principle ill defined mathematical problem [1], and how it is overcame in physics reality [2]. A notion of scattering and resolvent resonances has been introduced, their interconnection and differences have been discussed, and reasons were presented why a pole as a resonance signal is the most acceptable solution [3]. The importance of multichannel analysis has been demonstrated for pole extraction giving the example of N(1710) P11 resonance where single channel πN elastic data are insufficient to establish its existence. Only inclusion of inelastic channels (η production and/or $K\Lambda$ channels) is needed [4]. The dangers when using Breit-Wigner parameters for quantifying resonance properties have been discussed, and use of phase-shift as a link between QCD and scattering theory has been mentioned by using Lüscher's theorem [5]. The present state of the art of baryon spectroscopy has been presented by showing the highlights form the Camogli Workshop [6].

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Complete Experiments for Pion Photoproduction

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Abstract. The possibilities of a model-independent partial wave analysis for pion, eta or kaon photoproduction are discussed in the context of 'complete experiments'. It is shown that the helicity amplitudes obtained from at least 8 polarization observables including beam, target and recoil polarization can not be used to analyze nucleon resonances. However, a truncated partial wave analysis, which requires only 5 observables will be possible with minimal model assumptions.

1 Introduction

Around the year 1970 people started to think about how to determine the four complex helicity amplitudes for pseudoscalar meson photoproduction from a complete set of experiments. In 1975 Barker, Donnachie and Storrow [1] published their classical paper on 'Complete Experiments'. After reconsiderations and careful studies of discrete ambiguities [2–4], in the 90s it became clear that such a model-independent amplitude analysis would require at least 8 polarization observables which have to be carefully chosen. There are plenty of possible combinations, but all of them would require a polarized beam and target and in addition also recoil polarization measurements. Technically this was not possible until very recently, when transverse polarized targets came into operation at Mainz, Bonn and JLab and furthermore recoil polarization measurements by nucleon rescattering has been shown to be doable. This was the start of new efforts in different groups in order to achieve the complete experimental information and a model-independent partial wave analysis [5–8].

2 Complete experiments

A complete experiment is a set of measurements which is sufficient to predict all other possible experiments, provided that the measurements are free of uncertainties. Therefore it is first of all an academic problem, which can be solved by mathematical algorithms. In practise, however, it will not work in the same way and either a very high statistical precision would be required, which is very unlikely, or further measurements of other polarization observables are necessary. Both problems, first the mathematical problem but also the problem for a physical experiment can be studied with the help of state-of-the-art models like MAID or partial wave analyses (PWA) like SAID. With high precision calculations the complete sets of observables can be checked and with pseudo-data, generated from models and PWA, real experiments can be simulated under realistic conditions.

2.1 Coordinate Frames

Experiments with three types of polarization can be performed in meson photoproduction: photon beam polarization, polarization of the target nucleon and polarization of the recoil nucleon. Target polarization will be described in the frame $\{x, y, z\}$, see Fig. 1, with the z-axis pointing into the direction of the photon momentum $\hat{k}$, the y-axis perpendicular to the reaction plane, $\hat{y} = \hat{k} \times \hat{q} / \sin\theta$, and the x-axis is given by $\hat{x} = \hat{y} \times \hat{z}$. For recoil polarization, traditionally the frame $\{x', y', z'\}$ is used, with the z'-axis defined by the momentum vector of the outgoing meson $\hat{q}$, the y'-axis is the same as for target polarization and the x'-axis given by $\widehat{x'} = \widehat{y'} \times \widehat{z'}$.

The photon polarization can be linear or circular. For a linear photon polarization ($P_T = 1$) in the reaction plane $(\hat{x}, \hat{z})$, $\varphi = 0$. Perpendicular, in direction $\hat{y}$, the polarization angle is $\varphi = \pi/2$. Finally, for right-handed circular polarization, $P_\odot = +1$.

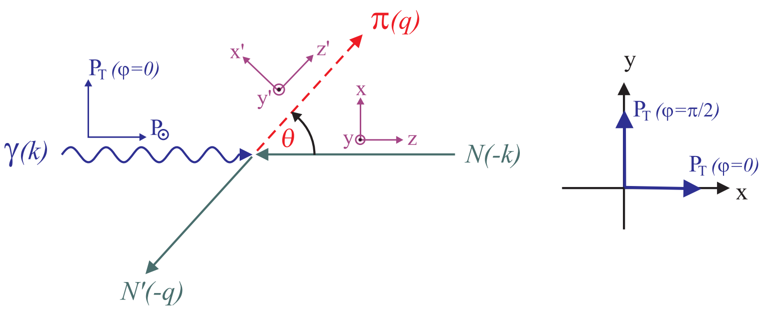

Fig. 1. Frames for polarization vectors in the CM.

The polarized differential cross section can be classified into three classes of double polarization experiments:
polarized photons and polarized target (types $(\mathcal{S}, \mathcal{BT})$)

$$\frac{d\sigma}{d\Omega} = \sigma_0\{1 - P_T\Sigma\cos 2\varphi + P_x(-P_T H \sin 2\varphi + P_\odot F) \\ +P_y(T - P_T P \cos 2\varphi) + P_z(P_T G \sin 2\varphi - P_\odot E)\}\,, \quad (1)$$

polarized photons and recoil polarization (types $(\mathcal{S}, \mathcal{BR})$)

$$\frac{d\sigma}{d\Omega} = \sigma_0\{1 - P_T\Sigma\cos 2\varphi + P_{x'}(-P_T O_{x'} \sin 2\varphi - P_\odot C_{x'}) \\ +P_{y'}(P - P_T T \cos 2\varphi) + P_{z'}(-P_T O_{z'} \sin 2\varphi - P_\odot C_{z'})\}\,, \quad (2)$$

polarized target and recoil polarization (types $(\mathcal{S}, \mathcal{TR})$)

$$\frac{d\sigma}{d\Omega} = \sigma_0\{1 + P_y T + P_{y'} P + P_{x'}(P_x T_{x'} - P_z L_{x'}) + P_{y'} P_y \Sigma + P_{z'}(P_x T_{z'} + P_z L_{z'})\}\,. \quad (3)$$

In these equations σ_0 denotes the unpolarized differential cross section, Σ, T, P are single-spin asymmetries $(\mathcal{S})$, E, F, G, H the beam-target asymmetries $(\mathcal{BT})$,

$O_{x'}$, $O_{z'}$, $C_{x'}$, $C_{z'}$ the beam-recoil asymmetries $(\mathcal{BR})$ and $T_{x'}, T_{z'}, L_{x'}, L_{z'}$ the target-recoil asymmetries $(\mathcal{TR})$. The polarization quantities are described in Fig. 1. The signs of the 16 polarization observables of Eq. (1,2,3) are in principle arbitrary, except for the cross section σ_0, which is naturally positive. For the 15 asymmetries we use the sign convention of Barker et al. [1], which is also used by the MAID and SAID partial wave analysis groups. For other sign conventions, see Ref. [9].

2.2 Amplitude analysis

Pseudoscalar meson photoproduction has 8 spin degrees of freedom, and due to parity conservation it can be described by 4 complex amplitudes of 2 kinematical variables. Possible sets of amplitudes are: Invariant amplitudes A_i, CGLN amplitudes F_i, helicity amplitudes H_i or transversity amplitudes b_i. All of them are linearly related to each other and further combinations are possible. Most often in the literature the helicity basis was chosen and the 16 possible polarization observables can be expressed in bilinear products

$$O_i(W, \theta) = \frac{q}{k} \sum_{k,\ell=1}^{4} \alpha_{k,\ell}\; H_k(W, \theta)\, H_l^*(W, \theta)\,, \tag{4}$$

where O_1 is the unpolarized differential cross section σ_0 and all other observables are products of asymmetries with σ_0, for details see Table 1.

From a complete set of 8 measurements $\{O_i(W, \theta)\}$ one can determine the moduli of the 4 amplitudes and 3 relative phases. But there is always an unknown overall phase, e.g. $\phi_1(W, \theta)$, which can not be determined by additional measurements. This is, however, not a principal problem as with the principally undetermined phase of a quantum mechanical wave function. Already in 1963 Goldberger et al. [10] discussed a method using the idea of a Hanbury-Brown and Twiss experiment, and very recently in 2012, Ivanov [11] discussed another method using vortex beams to measure the phase of a scattering amplitude. Both methods, however, are highly impractical for a meson photoproduction experiment.

Therefore, the complete information is contained in a set of 4 reduced amplitudes,

$$\tilde{H}_i(W, \theta) = H_i(W, \theta)\; e^{-i\,\phi_1(W, \theta)} \tag{5}$$

of which $\tilde{H}_1$ is a real function, the others are complex, resulting in a total of 7 real values for any given W and θ.

Figure 2 shows two of such amplitude analyses with a complete set of 8 observables and an overcomplete set of 10 observables. The data used for this analysis has been generated as pseudo-data from Monte-Carlo events according to the Maid2007 solution, see Sect. 3. The figure shows the real parts of two out of four reduced helicity amplitudes, $\mathrm{Re}\tilde{H}_1$ and $\mathrm{Re}\tilde{H}_4$. While the solution with the complete set of 8 observables results in a rather bad description of the true amplitudes, the solution of the overcomplete set gives a satisfactory result.

Table 1. Spin observables for pseudoscalar meson photoproduction involving beam, target and recoil polarization in 4 groups, $\mathcal{S}, \mathcal{B}\mathcal{T}, \mathcal{B}\mathcal{R}, \mathcal{T}\mathcal{R}$. A phase space factor q/k has been omitted in all expressions and the asymmetries are given by $A = \hat{A}/\sigma_0$. In column 2 the observables are expressed in terms of the Walker helicity amplitudes [12] and in column 3 in $\sin\theta$ and $x = \cos\theta$ with the leading terms for an S, P wave truncation.

Spin Obs	Helicity Representation	Partial Wave Expansion
σ_0	$\frac{1}{2}(\lvert H_1\rvert^2 + \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$A_0^\sigma + A_1^\sigma x + A_2^\sigma x^2 + \cdots$
$\hat{\Sigma}$	$\mathrm{Re}(H_1 H_4^* - H_2 H_3^*)$	$\sin^2\theta(A_0^\Sigma + \cdots)$
$\hat{T}$	$\mathrm{Im}(H_1 H_2^* + H_3 H_4^*)$	$\sin\theta(A_0^T + A_1^T x + \cdots)$
$\hat{P}$	$-\mathrm{Im}(H_1 H_3^* + H_2 H_4^*)$	$\sin\theta(A_0^P + A_1^P x + \cdots)$
$\hat{G}$	$-\mathrm{Im}(H_1 H_4^* + H_2 H_3^*)$	$\sin^2\theta(A_0^G + \cdots)$
$\hat{H}$	$-\mathrm{Im}(H_1 H_3^* - H_2 H_4^*)$	$\sin\theta(A_0^H + A_1^H x + \cdots)$
$\hat{E}$	$\frac{1}{2}(-\lvert H_1\rvert^2 + \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$A_0^E + A_1^E x + A_2^E x^2 + \cdots$
$\hat{F}$	$\mathrm{Re}(H_1 H_2^* + H_3 H_4^*)$	$\sin\theta(A_0^F + A_1^F x + \cdots)$
$\widehat{O_{x'}}$	$-\mathrm{Im}(H_1 H_2^* - H_3 H_4^*)$	$\sin\theta(A_0^{O_{x'}} + A_1^{O_{x'}} x + A_2^{O_{x'}} x^2 + \cdots)$
$\widehat{O_{z'}}$	$\mathrm{Im}(H_1 H_4^* - H_2 H_3^*)$	$\sin^2\theta(A_0^{O_{z'}} + A_1^{O_{z'}} x + \cdots)$
$\widehat{C_{x'}}$	$-\mathrm{Re}(H_1 H_3^* + H_2 H_4^*)$	$\sin\theta(A_0^{C_{x'}} + A_1^{C_{x'}} x + A_2^{C_{x'}} x^2 + \cdots)$
$\widehat{C_{z'}}$	$\frac{1}{2}(-\lvert H_1\rvert^2 - \lvert H_2\rvert^2 + \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$A_0^{C_{z'}} + A_1^{C_{z'}} x + A_2^{C_{z'}} x^2 + A_3^{C_{z'}} x^3 + \cdots$
$\widehat{T_{x'}}$	$\mathrm{Re}(H_1 H_4^* + H_2 H_3^*)$	$\sin^2\theta(A_0^{T_{x'}} + A_1^{T_{x'}} x + \cdots)$
$\widehat{T_{z'}}$	$\mathrm{Re}(H_1 H_2^* - H_3 H_4^*)$	$\sin\theta(A_0^{T_{z'}} + A_1^{T_{z'}} x + A_1^{T_{z'}} x^2 + \cdots)$
$\widehat{L_{x'}}$	$-\mathrm{Re}(H_1 H_3^* - H_2 H_4^*)$	$\sin\theta(A_0^{L_{x'}} + A_1^{L_{x'}} x + A_2^{L_{x'}} x^2 + \cdots)$
$\widehat{L_{z'}}$	$\frac{1}{2}(\lvert H_1\rvert^2 - \lvert H_2\rvert^2 - \lvert H_3\rvert^2 + \lvert H_4\rvert^2)$	$A_0^{L_{z'}} + A_1^{L_{z'}} x + A_2^{L_{z'}} x^2 + A_3^{L_{z'}} x^3 + \cdots$

2.3 Truncated partial wave analysis

Even with the help of unitarity in form of Watson's theorem, the angle-dependent phase $\phi_1(W, \theta)$ cannot be provided. This has very strong consequences, namely a partial wave decomposition would lead to wrong partial waves, which would be useless for nucleon resonance analysis. It becomes obvious in the following schematic formula

$$f_\ell(W) = \frac{2}{2\ell+1} \int \tilde{H}(W, \theta) e^{i\phi(W,\theta)} P_\ell(\cos\theta)\, d\cos\theta\,, \tag{6}$$

where the desired partial wave $f_\ell(W)$ cannot be obtained from the reduced helicity amplitudes $\tilde{H}(W, \theta)$ alone, as long as the angle dependent phase $\phi(W, \theta)$ is unknown.

Our main goal in the data analysis of photoproduction is the search for nucleon resonances and their properties. To better reach this goal, one can directly perform a partial wave analysis from the observables without going through the underlying helicity amplitudes. Such an analysis would be a truncated partial wave analysis (TPWA) with a minimal model dependence (i) from the truncation of the series at a maximal angular momentum ℓ_{max} and (ii) from an overall unknown phase as in the case of the amplitude analysis in the previous paragraph. However, in the TPWA the overall phase would be only a function of energy and with additional theoretical help it can be constrained without strong model as-

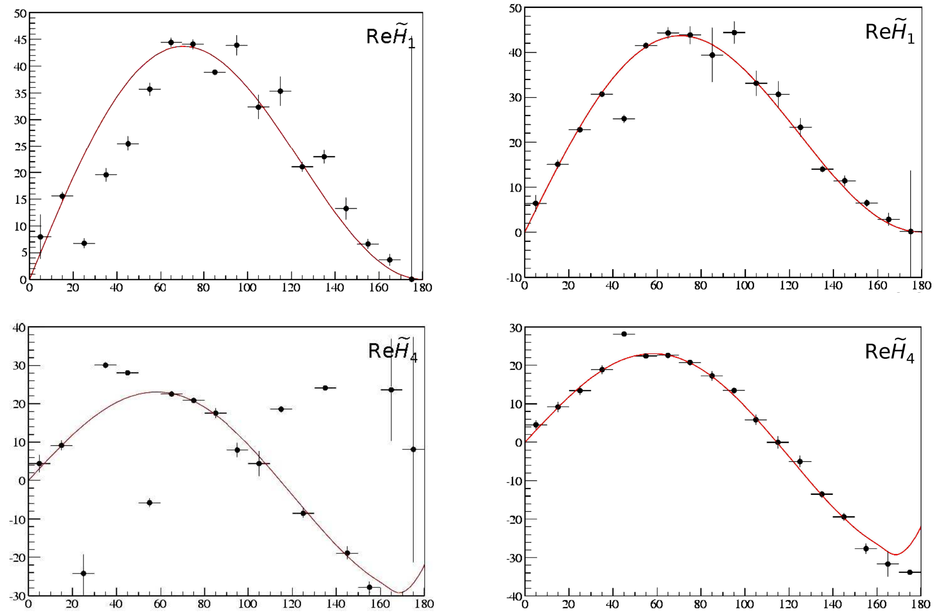

Fig. 2. Comparison of the reduced helicity amplitudes Re$\tilde{H}_1$ and Re$\tilde{H}_4$ between a pseudo-data analysis with a complete dataset of 8 observables: $\sigma_0, \Sigma, T, P, E, G, O_{x'}, C_{x'}$ (left 2 panels) and with an overcomplete dataset of 10 observables with additional F, H (right 2 panels) for $\gamma p \to \pi^0 p$ at $E = 320$ MeV as a function of the c.m. angle θ. The solid red curves show the MAID2007 solutions. Amplitudes are in units of $10^{-3}/m_{\pi^+}$.

sumptions. Such a concept was already discussed and applied for γ, π in the 80s by Grushin [13] for a PWA in the region of the $\Delta(1232)$ resonance.

Formally, the truncated partial wave analysis can be performed in the following way. All observables can be expanded either in a Legendre series or in a $\cos\theta$ series

$$O_i(W, \theta) = \frac{q}{k} \sin^{\alpha_i}\theta \sum_{k=0}^{2\ell_{max}+\beta_i} A_k^i(W) \cos^k\theta \,, \tag{7}$$

$$A_k^i(W) = \sum_{\ell,\ell'=0}^{\ell_{max}} \sum_{k,k'=1}^{4} \alpha_{\ell,\ell'}^{k,k'} \mathcal{M}_{\ell,k}(W) \mathcal{M}^*_{\ell',k'}(W) \,, \tag{8}$$

where k, k' denote the 4 possible electric and magnetic multipoles for each πN angular momentum $\ell \geq 2$, namely $\mathcal{M}_{\ell,k} = \{E_{\ell+}, E_{\ell-}, M_{\ell+}, M_{\ell-}\}$. For an S, P truncation ($\ell_{max} = 1$) there are 4 complex multipoles $E_{0+}, E_{1+}, M_{1+}, M_{1-}$ leading to 7 free real parameters and an arbitrary phase, which can be put to zero for the beginning. In Table 1 we list the expansion coefficients for all observables that appear in an S, P wave expansion. Already from the 8 observables of the first two groups ($\mathcal{S}, \mathcal{B}T$) one can measure a set of 16 coefficients, from which we only need 8 well selected ones for a unique mathematical solution. This can be achieved by a measurement of the angular distributions of only 5 observables, e.g. $\sigma_0, \Sigma, T, P, F$ or $\sigma_0, \Sigma, T, F, G$. In the first example one gets even 10 coefficients, from which e.g.

A_1^P and A_0^F can be omitted. In the second case, there are 9 coefficients, of which A_0^F can be omitted. In practise one can select those coefficients, which have the smallest statistical errors, and therefore, the biggest impact for the analysis by keeping in mind that all discrete ambiguities are resolved.

As has been shown by Omelaenko [14] the same is true for any PWA with truncation at ℓ_{max}. For the determination of the $8\ell_{max} - 1$ free parameters one has the possibility to measure $(8\ell_{max}, 8\ell_{max}, 8\ell_{max} + 4, 8\ell_{max} + 4)$ coefficients for types $(\mathcal{S}, \mathcal{BT}, \mathcal{BR}, \mathcal{TR})$, respectively.

3 Partial wave analysis with pseudo-data

In a first numerical attempt towards a model-independent partial wave analysis, a procedure similar to the second method, the TPWA, described above, has been applied [6], and pseudo-data, generated for γ, π^0 and γ, π^+ have been analyzed.

Events were generated over an energy range from $E_{lab} = 200 - 1200$ MeV and a full angular range of $\theta = 0 - 180°$ for beam energy bins of $\Delta E_\gamma = 10$ MeV and angular bins of $\Delta\theta = 10°$, based on the MAID2007 model predictions [15]. For each observable, typically $5 \cdot 10^6$ events have been generated over the full energy range. For each energy bin a single-energy (SE) analysis has been performed using the SAID PWA tools [16].

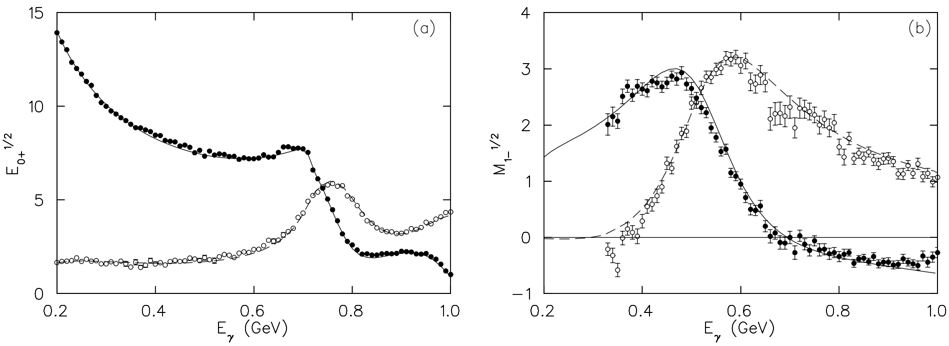

Fig. 3. Real and imaginary parts of (a) the S_{11} partial wave amplitude $E_{0+}^{1/2}$ and (b) the P_{11} partial wave amplitude $M_{1-}^{1/2}$. The solid (dashed) line shows the real (imaginary) part of the MAID2007 solution, used for the pseudo-data generation. Solid (open) circles display real (imaginary) single-energy fits (SE6p) to the following 6 observables without any recoil polarization measurement: $d\sigma/d\Omega$, two single-spin observables Σ, T and three beam-target double polarization observables E, F, G. Multipoles are in millifermi units.

A series of fits, SE4p, SE6p and SE8p have been performed [6] using 4, 6 and 8 observables, respectively. Here the example using 6 observables $(\sigma_0, \Sigma, T, E, F, G)$ is demonstrated, where no recoil polarization has been used. As explained before, such an experiment would be incomplete in the sense of an 'amplitude analysis', but complete for a truncated partial wave analysis. In Fig. 3 two multipoles $E_{0+}^{1/2}$ and $M_{1-}^{1/2}$ for the S_{11} and P_{11} channels are shown and the SE6p fits are compared to the MAID2007 solution. The fitted SE solutions are very close to the MAID solution with very small uncertainties for the S_{11} partial wave. For the P_{11} partial wave we obtain a larger statistical spread of the SE solutions. This is typical for the $M_{1-}^{1/2}$ multipole, which is generally much more difficult to obtain with good accuracy [15], because of the weaker sensitivity of the observables to this magnetic multipole. But also this multipole can be considerably improved in an analysis with 8 observables [6].

4 Summary and conclusions

It is shown that for an analysis of N^* resonances, the amplitude analysis of a complete experiment is not very useful, because of an unknown energy and angle dependent phase that can not be determined by experiment and can not be provided by theory without a strong model dependence. However, the same measurements or even less will be very useful for a truncated partial wave analysis with minimal model dependence due to truncations and extrapolations of Watson's theorem in the inelastic energy region. A further big advantage of such a PWA is a different counting of the necessary polarization observables, resulting in very different sets of observables. While it is certainly helpful to have polarization observables from 3 or 4 different types, for a mathematical solution of the bilinear equations one can find minimal sets of only 5 observables from only 2 types, where either a polarized target or recoil polarization measurements can be completely avoided.

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News from Belle: Recent Spectroscopy Results

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Abstract. This paper reports on some of the latest spectroscopic measurements performed with the experimental data collected by the Belle spectrometer, which has been operating at the KEKB asymmetric-energy e^+e^- collider in the KEK laboratory in Tsukuba, Japan.

1 Introduction

The Belle detector [1] at the asymmetric-energy e^+e^- collider KEKB [2] has accumulated about 1 ab^{-1} of data by the end of its operation in June 2010. The KEKB collider, called a *B-factory*, most of the time operated near the $\Upsilon(4S)$ resonance, but it has accumulated substantial data samples also at other Υ resonances, like $\Upsilon(1S)$, $\Upsilon(2S)$ and $\Upsilon(5S)$, as well as in the nearby continuum. In particular, the data samples at the $\Upsilon(4S)$ and $\Upsilon(5S)$ resonances are by far the largest available in the world, corresponding to integrated luminosities of 798 fb^{-1} and 123 fb^{-1}, respectively. Large amount of collected experimental data and excellent detector performance enabled many interesting spectroscopic results, including discoveries of new hadronic states and studies of their properties. This report covers most recent and interesting spectroscopic measurements—performed with either charmonium(-like) and bottomonium(-like) states.

2 Bottomonium and Bottomonium-like States

The Belle collaboration used a data sample at the CM energy around the $\Upsilon(5S)$ mass 10.89 GeV, and found large signals for decays into $\pi^+\pi^-\Upsilon(1S)$, $\pi^+\pi^-\Upsilon(2S)$ and $\pi^+\pi^-\Upsilon(3S)$ final states [3]. If these transitions are only from the $\Upsilon(5S)$ resonance, then the corresponding partial widths are more than two orders of magnitude larger than the corresponding partial widths for $\Upsilon(4S)$, $\Upsilon(3S)$ and $\Upsilon(2S)$ decays to $\pi^+\pi^-\Upsilon(1S)$. These results motivate a search for the $h_b(mP)$ resonances in the $\Upsilon(5S)$ data. $h_b(1P)$ and $h_b(2P)$ states are observed in the missing mass spectrum of $\pi^+\pi^-$ pairs for the $\Upsilon(5S)$ decays, with significances of 5.5σ and 11.2σ, respectively [4]. This is the first observation of the $h_b(1P)$ and $h_b(2P)$ spin-singlet bottomonium states in the reaction $e^+e^- \to h_b(mP)\pi^+\pi^-$ at the $\Upsilon(5S)$ energy. Later $h_b(1P)$ and $h_b(2P)$ were studied in the $\Upsilon(5S) \to h_b\pi^+\pi^- \to \gamma\eta_b(1S)\pi^+\pi^-$

* Representing the Belle Collaboration.

Decay mode	Branching fraction in %
$h_b(1P) \to \gamma\eta_b(1S)$	$49.2 \pm 5.7^{+5.6}_{-3.3}$
$h_b(2P) \to \gamma\eta_b(1S)$	$22.3 \pm 3.8^{+3.1}_{-3.3}$
$h_b(2P) \to \gamma\eta_b(2S)$	$47.5 \pm 10.5^{+6.8}_{-7.7}$

Table 1. The branching fractions for $h_b \to \gamma\eta_b$ decays, as measured by Belle.

decay [5]. In the same final state, Belle observes [5] also the first evidence for a $\eta_b(2S)$ in $\Upsilon(5S) \to h_b(2P)\pi^+\pi^- \to \gamma\eta_b(2S)\pi^+\pi^-$ decay. The width of $\eta_b(2S)$ is small, with $\Gamma = (4 \pm 8)$ MeV. Branching fractions for observed radiative h_b decays are summarized in Table 1.

Comparable rates of $h_b(1P)$ and $h_b(2P)$ production indicate a possible exotic process that violates heavy quark spin-flip and this motivates a further study of the resonant structure in $\Upsilon(5S) \to h_b(mP)\pi^+\pi^-$ and $\Upsilon(5S) \to \Upsilon(nS)\pi^+\pi^-$ decays [6]. Due to the limited statistics, only the study of $M(h_b(mP)\pi)$ distribution is possible for $h_b(mP)\pi^+\pi^-$, while in the case of $\Upsilon(nS)\pi^+\pi^-$ decay modes the Dalitz plot analysis can be performed. As a result, two charged bottomonium-like resonances, $Z_b(10610)$ and $Z_b(10650)$, are observed with signals in five different decay channels, $\Upsilon(nS)\pi^\pm$ $(n = 1, 2, 3)$ and $h_b(mP)\pi^\pm$ $(m = 1, 2)$. The averaged values for the mass and widths of the two states are calculated to be: $M(Z_b(10610)) = (10607.2 \pm 2.0)$ MeV, $\Gamma(Z_b(10610)) = (18.4 \pm 2.4)$ MeV and $M(Z_b(10650)) = (10652.2 \pm 1.5)$ MeV, $\Gamma(Z_b(10650)) = (11.5 \pm 2.2)$ MeV. The measured masses are only a few MeV above the thresholds for the open beauty channels $B^*\overline{B}$ (10604.6 MeV) and $B^*\overline{B}^*$ (10650.2 MeV) [9], which could indicate a molecular nature of the two observed states. Angular analysis of charged pion distributions favours the $J^P = 1^+$ spin-parity assignment for both $Z_b(10610)$ and $Z_b(10650)$.

3 Charmonium and Charmonium-like States

There has been a renewed interest in charmonium spectroscopy since 2002. The attention to this field was drawn by the discovery of the two missing $c\bar{c}$ states below the open-charm threshold, $\eta_c(2S)$ and $h_c(1P)$ [7,8] with J^{PC}=0^{-+} and 1^{+-}, respectively, but even with the discoveries of new new charmonium-like states (so called “XYZ” states).

3.1 The $X(3872)$ news

The storyabout the so called “XYZ” states began in 2003, when Belle reported on $B^+ \to K^+J/\psi\pi^+\pi^-$ analysis, where a new state decaying to $J/\psi\pi^+\pi^-$ was discovered [10]. The new state, called X(3872), was soon confirmed and also intensively studied by the CDF, DØ and *BABAR* collaborations [11–19]. So far it has been established that this narrow state ($\Gamma = (3.0^{+1.9}_{-1.4} \pm 0.9)$ MeV) has a mass of

(3872.2 ± 0.8) MeV, which is very close to the $D^0\overline{D}^{*0}$ threshold [9]. The intensive studies of several $X(3872)$ production and decay modes suggest two possible J^{PC} assignments, 1^{++} and 2^{-+}, and establish the $X(3872)$ as a candidate for a loosely bound $D^0\overline{D}^{*0}$ molecular state. However, results provided substantial evidence that the $X(3872)$ state must contain a significant $c\overline{c}$ component as well.

Recently, Belle performed a study of $B \to (c\overline{c}\gamma)K$ using the final data sample with 772 million of $B\overline{B}$ pairs collected at the $\Upsilon(4S)$ resonance [20]. Pure $D^0\overline{D}^{*0}$ molecular model [21] predicts $\mathcal{B}(X(3872) \to \psi'\gamma)$ to be less than $\mathcal{B}(X(3872) \to J/\psi\gamma)$. Results by the *BABAR* collaboration [19] show that $\mathcal{B}(X(3872) \to \psi'\gamma)$ is almost three times that of $\mathcal{B}(X(3872) \to J/\psi\gamma)$, which is inconsistent with the pure molecular model, and can be interpreted as a large $c\overline{c} - D^0\overline{D}^{*0}$ admixture. We observe $X(3872) \to J/\psi\gamma$ together with an evidence for $\chi_{c2} \to J/\psi\gamma$ in $B^\pm \to J/\psi\,\gamma K^\pm$ decays, while in our search for $X(3872) \to \psi'\gamma$ no significant signal is found. We also observe $B \to \chi_{c1}K$ decays in both, charged as well as neutral B decays. The obtained results suggest that the $c\overline{c}$-$D^0\overline{D}^{*0}$ admixture in $X(3872)$ may not be as large as discussed above.

New results for the $X(3872) \to J/\psi\pi^+\pi^-$ decay modes in $B^+\to K^+X(3872)$ and $B^0\to K^0\ (\to\pi^+\pi^-)X(3872)$ decays are obtained with the complete Belle data set of 772 million $B\overline{B}$ pairs collected at the $\Upsilon(4S)$ resonance [22]. The results for the $X(3872)$ mass and width are obtained by a 3-dimensional fit to distributions of the three variables: beam-constrained-mass $M_{bc}= \sqrt{(E^{cms}_{beam})^2 - (p^{cms}_B)^2}$ (with the beam energy E^{cms}_{beam} and the B-meson momentum p^{cms}_B both measured in the centre-of-mass system), the invariant mass $M_{inv}(J/\psi\pi^+\pi^-)$ and the energy difference $\Delta E = E^{cms}_B - E^{cms}_{beam}$ (where E^{cms}_B is the B-meson energy in the centre-of-mass system). As a first step, the fit is performed for the reference channel $\psi'\to J/\psi\pi^+\pi^-$, and the resolution parameters are then fixed for the fit of the $X(3872)$. The mass, determined by the fit, is $(3871.84\pm0.27\pm0.19)$ MeV. Including the new Belle result, the updated world-average mass of the $X(3872)$ is $m_X=(3871.67\pm0.17)$ MeV. If the $X(3872)$ is an S-wave $D^{*0}\overline{D}^0$ molecular state, the binding energy E_b would be given by the mass difference $m(X)-m(D^{*0})-m(D^0)$. With the current value of $m(D^0)+m(D^{*0})=(3871.79 \pm 0.30)$ MeV [9], a binding energy of $E_b=(-0.12\pm0.35)$ MeV can be calculated, which is surprisingly small and would indicate a very large radius of the molecular state.

The best upper limit for the $X(3872)$ width was 2.3 MeV (with 90% C.L.), obtained by previous Belle measurement [10]. The 3-dimensional fits are more sensitive to the natural width, which is smaller than the detector resolution ($\sigma \sim 4$ MeV). Due to the fit sensitivity and the calibration performed on the reference channel $\psi'\to J/\psi\pi^+\pi^-$, the updated upper limit for the $X(3872)$ width is about 1/2 of the previous value: $\Gamma(X(3872)) < 1.2$ MeV at 90% C.L.

Previous studies performed by several experiments suggested two possible J^{PC} assignments for the $X(3872)$, 1^{++} and 2^{-+}. In the recent Belle analysis [20], the $X(3872)$ quantum numbers were also studied with the full available data sample collected at the $\Upsilon(4S)$ resonance. At the current level of statistical sensitivity it is not possible to distinguish completely between the two possible quantum number assignments, so both hypotheses are still allowed. Possible C-odd neutral partners of X(3872) are also searched, but no signal is found for this type of states.

4 Summary and Conclusions

Many new particles have already been discovered during the operation of the Belle experiment at the KEKB collider, and some of them are mentioned in this report. Some recent Belle results also indicate that analogs to exotic charmonium-like states can be found in $b\overline{b}$ systems. Although the operation of the experiment has finished, data analyses are still ongoing and therefore more interesting results on charmonium(-like) and bottomonium(-like) spectroscopy can still be expected from Belle in the near future.

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Meson electro-production in the region of the Delta(1700) D33 resonance

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Abstract. We apply a coupled channel formalism incorporating quasi-bound quark-model states to calculate the D13, D33 and D15 scattering and electro-production amplitudes. The meson-baryon vertices for πN, $\pi\Delta$ (s- and d-waves), ρN, $\pi N(1440)$, $\pi N(1535)$, $\pi\Delta(1600)$ and $\sigma\Delta(1600)$ channels are determined in the Cloudy Bag Model. We use the same values for the model parameters as in the case of the P11, P33 and S11 partial waves except for the strength of the coupling of the d-wave mesons to quarks which has to be increased in order to reproduce the width of the observed D-wave resonances. The electro-production amplitudes exhibit a consistent behavior in all channels but are too weak in the resonance region.

1 Introduction

This work is a continuation of a joint project on the description of baryon resonances performed by the Coimbra group (Manuel Fiolhais and Pedro Alberto) and the Ljubljana group (Simon Širca and B. G.) [1–9]. In our previous works [5–7] we have successfully applied our method which incorporates excited baryons represented as quasi-bound quark-model states into a coupled channel formalism using the K-matrix approach [5] to calculate the scattering and the electro-production amplitudes in the P11, P33 and S11 partial waves. In the present work we extend of the approach to low lying negative parity D-wave resonances.

In the next section we give a short review of the method and in the following sections we discuss in more detail scattering and electro-production in the D13 and D33 and D15 partial waves.

2 The method

We limit ourselves to a class of chiral quark models in which mesons couple linearly to the quark core. In such cases the elements of the K matrix in the basis with good total angular momentum J and isospin T can be cast in the form [5]:

$$K^{JT}_{M'B'\,MB} = -\pi\mathcal{N}_{M'B'}\langle\Psi^{MB}_{JT}\|V_{M'}(k)\|\widetilde{\Psi}_{B'}\rangle\,, \qquad \mathcal{N}_{MB} = \sqrt{\frac{\omega_M E_B}{k_M W}}\,. \tag{1}$$

Here ω_M and k_M are the energy and momentum of the incoming (outgoing) meson, $|\widetilde{\Psi}_B\rangle$ is properly normalized baryon state and E_B is its energy, W is the invariant energy of the meson-baryon system, and $|\Psi^{MB}\rangle$ is the principal value state

$$|\Psi^{MB}_{JT}\rangle = \mathcal{N}_{MB}\left\{[a^\dagger(k_M)|\widetilde{\Psi}_B\rangle]^{JT} + \sum_{\mathcal{R}} c^{MB}_{\mathcal{R}}|\Phi_{\mathcal{R}}\rangle \right.$$
$$\left. + \sum_{M'B'}\int \frac{dk\,\chi^{M'B'\,MB}(k,k_M)}{\omega_k + E_{B'}(k) - W}\,[a^\dagger(k)|\widetilde{\Psi}_{B'}\rangle]^{JT}\right\}. \qquad (2)$$

The first term represents the free meson (π, η, ρ, K, ...) and the baryon (N, Δ, Λ, ...) and defines the channel, the next term is the sum over *bare* tree-quark states $\Phi_{\mathcal{R}}$ involving different excitation of the quark core, the third term introduces meson clouds around different isobars, $E(k)$ is the energy of the recoiled baryon. We assume that the two pion decay proceeds either through an unstable meson (ρ-meson, σ-meson, ...) or through a baryon resonance ($\Delta(1232)$, $N^*(1440)$ ...). The meson amplitudes $\chi^{M'B'\,MB}(k,k_M)$ are proportional to the (half) off-shell matrix elements of the K-matrix and are determine by solving a Lippmann-Schwinger type of equation. The resulting matrix elements of the K-matrix take the form

$$K_{M'B'\,MB}(k,k_M) = -\sum_{\mathcal{R}} \frac{\mathcal{V}^{M}_{B\mathcal{R}}(k_M)\mathcal{V}^{M'}_{B'\mathcal{R}}(k)}{Z_{\mathcal{R}}(W)(W-W_{\mathcal{R}})} + K^{\mathrm{bkg}}_{M'B'\,MB}(k,k_M)\,, \qquad (3)$$

where the first term represents the contribution of various resonances while $K^{\mathrm{bkg}}_{M'B'\,MB}(k,k_M)$ originates in the non-resonant background processes. Here $\mathcal{V}^{M}_{B\mathcal{R}}$ is the dressed matrix element of the quark-meson interaction between the resonant state and the baryon state in the channel MB, and $Z_{\mathcal{R}}$ is the wave-function normalization. The physical resonant state $\mathcal{R}$ is a superposition of the dressed states built around the bare 3-quark states $\Phi_{\mathcal{R}'}$. The T matrix is finally obtained by solving the Heitler's equation

$$T_{MB\,M'B'} = K_{MB\,M'B'} + i\sum_{M''B''} T_{MB\,M''B''}K_{M''B''\,M'B'}\,. \qquad (4)$$

Considering meson electro-production, the T matrix for $\gamma N \to MB$ satisfies

$$T_{MB\,\gamma N} = K_{MB\,\gamma N} + i\sum_{M'B'} T_{MB\,M'B'}K_{M'B'\,\gamma N}\,. \qquad (5)$$

In the vicinity of a chosen resonance ($\mathcal{R}$) we write (see (3)):

$$K_{MB\,\gamma N} = -\frac{\mathcal{V}^{M}_{B\mathcal{R}}\mathcal{V}^{\gamma}_{N\mathcal{R}}}{Z_{\mathcal{R}}(W)(W-W_{\mathcal{R}})} - \sum_{\mathcal{R}'\neq\mathcal{R}} \frac{\mathcal{V}^{M}_{B\mathcal{R}'}\mathcal{V}^{\gamma}_{N\mathcal{R}'}}{Z_{\mathcal{R}'}(W)(W-W_{\mathcal{R}'})} + B^{\mathrm{bkg}}_{MB\,\gamma N}\,. \qquad (6)$$

We manipulate the first term:

$$\frac{\mathcal{V}^{M}_{B\mathcal{R}}\mathcal{V}^{\gamma}_{N\mathcal{R}}}{Z_{\mathcal{R}}(W)(W-W_{\mathcal{R}})} = \frac{{\mathcal{V}^{M}_{B\mathcal{R}}}^{2}}{Z_{\mathcal{R}}(W)(W-W_{\mathcal{R}})}\,\frac{\mathcal{V}^{\gamma}_{N\mathcal{R}}}{\mathcal{V}^{M}_{B\mathcal{R}}} = \left(K_{MB\,MB} - K^{\mathrm{bkg}}_{MB\,MB}\right)\frac{\mathcal{V}^{\gamma}_{N\mathcal{R}}}{\mathcal{V}^{M}_{B\mathcal{R}}}$$

so that (5) takes the form

$$\begin{aligned}
T_{MB\,\gamma N} &= \left(K_{MB\,MB} + i \sum_{M'B'} T_{MB\,M'B'} K_{M'B'\,MB} \right) \frac{\mathcal{V}^{\gamma}_{N\mathcal{R}}}{\mathcal{V}^{M}_{B\mathcal{R}}} \\
&\quad + K^{\mathrm{bkg}}_{MB\,\gamma N} + i \sum_{M'B'} T_{MB\,M'B'} K^{\mathrm{bkg}}_{M'B'\,\gamma N} \\
&= \frac{\mathcal{V}^{\gamma}_{N\mathcal{R}}}{\mathcal{V}^{M}_{B\mathcal{R}}}\, T_{MB\,MB} + T^{\mathrm{bkg}}_{MB\,MB} \equiv T^{\mathrm{res}}_{MB\,\gamma N} + T^{\mathrm{bkg}}_{MB\,\gamma N}\,, \qquad (7)
\end{aligned}$$

which means that the T matrix for elektro-production can be split into the resonant part and the background part; the latter is the solution of the Heitler equation with the "background" K-matrix defined as

$$K^{\mathrm{bkg}}_{MB\,\gamma N} = -K^{\mathrm{bkg}}_{MB\,MB} \frac{\mathcal{V}^{\gamma}_{N\mathcal{R}}}{\mathcal{V}^{M}_{B\mathcal{R}}} - \sum_{\mathcal{R}'\neq\mathcal{R}} \frac{\mathcal{V}^{M}_{B\mathcal{R}'} \mathcal{V}^{\gamma}_{N\mathcal{R}'}}{Z_{\mathcal{R}'}(W)(W - W_{\mathcal{R}'})} + B^{\mathrm{bkg}}_{MB\,\gamma N}\,.$$

Note that $\mathcal{V}^{\gamma}_{N\mathcal{R}}(k_\gamma)$ is proportional to the helicity amplitudes while the strong amplitude $\mathcal{V}^{M}_{B\mathcal{R}}(k_M)$ to $\sqrt{\Gamma_{MB}}$ and to ζ, the sign of the phase of the meson decay.

3 The D-wave resonances in the Cloudy Bag Model

In the quark model, the negative parity D-wave resonances are described by a single quark $l = 1$ orbital excitation. The two D13 (flavor octet, $J = \frac{3}{2}$) resonances are the superposition of the $S = \frac{1}{2}$ and $S = \frac{3}{2}$ configurations, the D33 resonance (flavour decouplet) has $S = \frac{1}{2}$, while the D15 resonance (octet, $J = \frac{5}{2}$) has $S = \frac{3}{2}$. We use the j–j coupling scheme in which the resonances take the following forms:

$$\begin{aligned}
N(1520)D13 &= -\sin\vartheta_d |{}^4\mathbf{8}_{3/2}\rangle + \cos\vartheta_d |{}^2\mathbf{8}_{3/2}\rangle \\
&= c^1_S |(1s)^2 1p_{3/2}\rangle_{MS} + c^1_A |(1s)^2 1p_{3/2}\rangle_{MA} + c^1_P |(1s)^2 1p_{1/2}\rangle\,, \qquad (8) \\
N(1700)D13 &= \cos\vartheta_d |{}^4 8_{3/2}\rangle + \sin\vartheta_d |{}^2 8_{3/2}\rangle \\
&= c^2_S |(1s)^2 1p_{3/2}\rangle_{MS} + c^2_A |(1s)^2 1p_{3/2}\rangle_{MA} + c^2_P |(1s)^2 1p_{1/2}\rangle\,, \qquad (9) \\
\Delta(1700)D33 &= |{}^2\mathbf{10}_{3/2}\rangle = \frac{\sqrt{5}}{3} |(1s)^2 1p_{3/2}\rangle - \frac{2}{3} |(1s)^2 1p_{1/2}\rangle\,, \qquad (10) \\
N(1675)D15 &= |{}^4\mathbf{8}_{5/2}\rangle = |(1s)^2 1p_{3/2}\rangle\,. \qquad (11)
\end{aligned}$$

Here MS and MA denote the mixed symmetric and the mixed antisymmetric representation, and

$$c^1_S = \frac{2}{3}\sin\vartheta_d + \sqrt{\frac{5}{18}}\cos\vartheta_d, \quad c^1_A = -\frac{\sqrt{2}}{2}\cos\vartheta_d, \quad c^1_P = -\frac{\sqrt{5}}{3}\sin\vartheta_d + \frac{\sqrt{2}}{3}\cos\vartheta_d\,. \qquad (12)$$

The $l = 2$ pions couple only to $j = 3/2$ quarks; the corresponding interaction in the Cloudy Bag Model takes the form

$$V^{\pi}_{2mt}(k) = \frac{1}{2f_\pi} \sqrt{\frac{\omega_{p_{3/2}}\omega_s}{(\omega_{p_{3/2}} - 2)(\omega_s - 1)}}\, \frac{\sqrt{2}}{2\pi} \frac{k^2}{\sqrt{\omega_k}} \frac{j_2(kR)}{kR} \sum_{i=1}^{3} \tau_t(i)\, \Sigma^{[\frac{1}{2}\frac{3}{2}]}_{2m}(i)\,, \qquad (13)$$

where

$$\Sigma_{1m}^{[\frac{1}{2}\frac{3}{2}]} = \sum_{m_s m_j} C_{\frac{3}{2}m_j 1m}^{\frac{1}{2}m_s} |s m_s\rangle\langle p_{3/2} m_j| \,, \qquad \omega_s = 2.043\,, \quad \omega_{p_{3/2}} = 3.204\,.$$

In the case of P11, P33 and S11 waves we have used the bag radius $R = 0.83$ fm which determines the range of quark-pion interaction corresponding to the cut-off $\Lambda \sim 550$ MeV/c, and the value for $f_\pi = 76$ MeV which reproduces the experimental value of the πNN coupling constant. For the d-wave pions it turns out that the range predicted by (13) is too large while the resulting coupling strength is too weak. We have therefore modified the interaction in such a way as to correspond to $\Lambda \sim 550$ MeV/c, while the coupling strength has been increased by a factor 1.7 – 2.75 (depending on the considered resonance).

4 Scattering amplitudes

The effect of the form factor and the strength of quark-meson coupling discussed in the previous section is most clearly seen in the case of the D15 where the background effects as well as the influence of other resonances are almost negligible. Using our standard value for the cut-off parameter we have to increase the quark model coupling constant by a factor of 2.75 in order to obtain an almost perfect fit to the data in the region of the resonance.

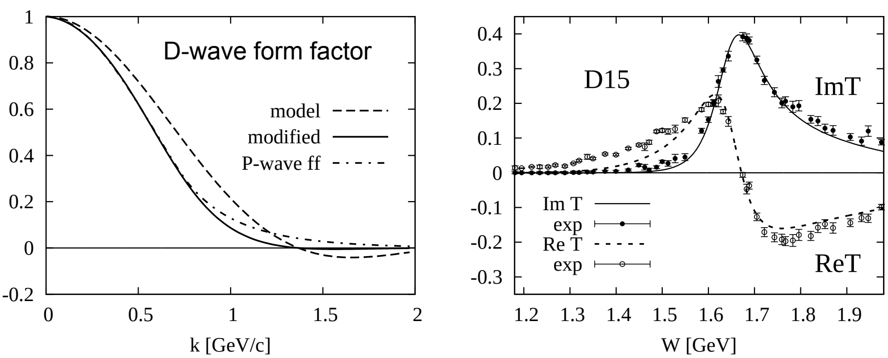

Fig. 1. The form factor for the D-wave pions (left panel), and the real and the imaginary part of the D15 scattering amplitude (right). The data points are from [10].

The data for elastic scattering in the D13 partial wave show almost no sign of the second resonance N(1700). Since the $l = 2$ pions most strongly couple to the $|(1s)^2 1p_{3/2}\rangle_{MA}$ configuration, the absence of the second resonance can be most easily explained by the vanishing of the c_A^2 coefficient in (9), $c_A^2 = -\sin\theta_d/\sqrt{2}$. This suggests $\theta_d = 0$. In our model the resonances are mixed through the pion interaction which changes slightly the above conclusion leading to the choice $\theta_d \approx 10°$ for the optimal mixing. At this energy range the effect of the cut-off is less pronounced; the quark-model prediction for the πNR coupling constant has to be increased by a factor of 1.7, while that to the Δ decreased by a factor of one half.

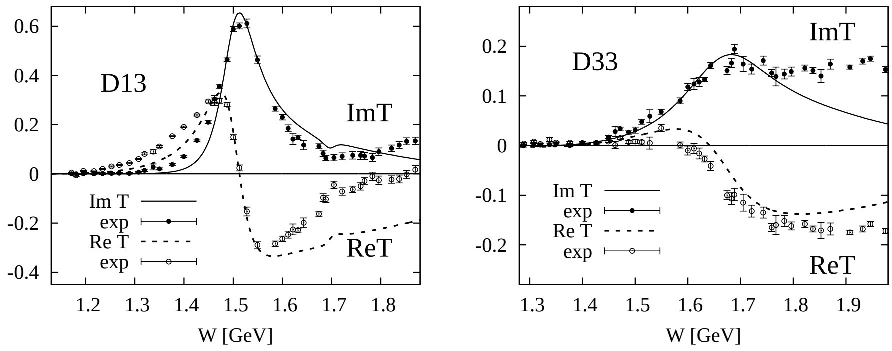

Fig. 2. The real and the imaginary part of the D13 wave scattering amplitude (left), and for the D33 wave (right). The data points are from [10].

In the vicinity of the D33 resonance the elastic amplitude is dominated by the coupling of the elastic channel to the $\pi\Delta(1232)$ channel. The d-wave pion coupling to the nucleon is increased by a factor of 2.5 with respect to the quark model value, while the model value for s-wave coupling to the $\Delta(1232)$ is not modified. Increasing the latter coupling brings the real part of the amplitude closer to the data, however the behavior of the photo-production amplitudes, presented in the next section, is deteriorated.

5 Electro-production

The electro-production amplitudes are obtained by evaluating the EM current consisting of the quark and the pion part between the nucleon ground state and the resonant state. The corresponding helicity amplitude $\mathcal{V}^{\gamma}_{N\mathcal{R}}$ in (7) reads

$$\mathcal{V}^{\gamma}_{N\mathcal{R}}(k_\gamma) = \frac{e}{\sqrt{2\omega_\gamma}}\langle\mathcal{R}|j_{EM}(k_\gamma)|N\rangle,$$

where the resonant state stemming from the second and the third term in (2) consists of the bare-quark part and the meson cloud

$$|\mathcal{R}\rangle = \frac{1}{\sqrt{Z_\mathcal{R}}}\left\{|\Phi_\mathcal{R}\rangle - \sum_{MB}\int \frac{dk \quad \mathcal{V}^{M}_{B\mathcal{R}}(k)}{\omega_k + E_B - W}\,[a^\dagger(k)|\widetilde{\Psi}_B\rangle]^{JT}\right\}. \qquad (14)$$

The background term entering (7) is dominated by the pion-pole term and the u-channel process which originate from the first term in (2).

In Figs. 3 – 6 the transverse photo-production amplitudes for the partial D13, D33 and D15 partial waves calculated in our model are compared to the data as well as to the analysis of the MAID group [11]. While our calculation correctly reproduce the behavior of the amplitudes at the energies close to the threshold where they are dominated by the pion-pole term, their strength in the resonance region is typically a factor 0.5 to 0.7 weaker compared to the value of the electric transverse amplitude as deduced from the experiment, and even weaker in the case of the magnetic amplitude. The pertinent multipoles are sensitive to the nucleon's periphery which is apparently not adequately reproduced in the bag model, as we have already noticed when analyzing the coupling of the resonance to the d-wave pions. Here the pion cloud effect are relatively weak as a consequence of cancellations of different terms, and contribute at the level of 10 % to 20 % to the amplitudes.

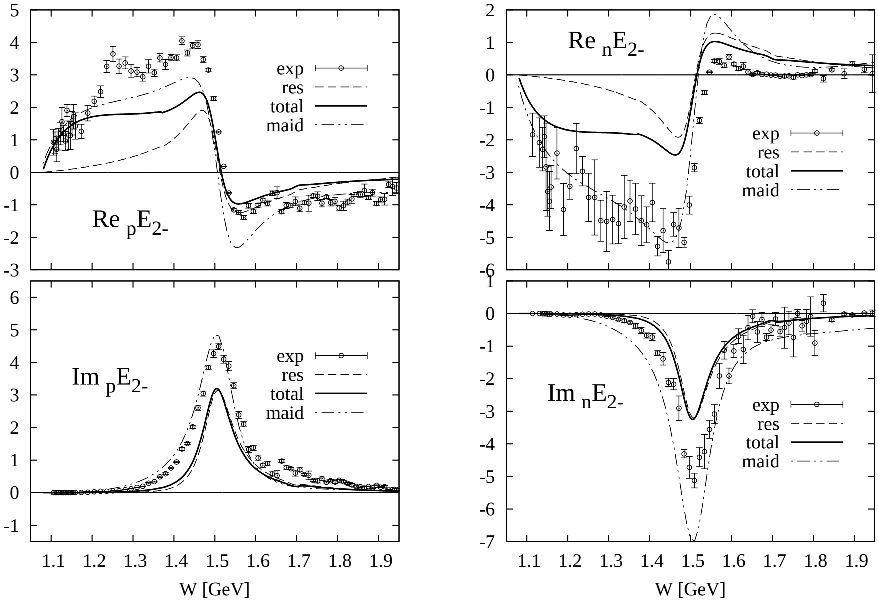

Fig. 3. The real and the imaginary part of the proton and neutron multipoles E_{2-} for the D13 wave in units $10^{-3}/m_\pi$ (preliminary). The data points are from [10], "maid" corresponds to the partial wave analysis from [11].

Nonetheless, we should stress that the amplitudes exhibit a consistent behavior in all considered partial waves. In particular, our model correctly predicts that in the D13 partial wave the ${}_nE^{1/2}_{2-}$ multipole amplitude is weaker than the corresponding ${}_nE^{1/2}_{2-}$ amplitude, and that the ${}_nM^{1/2}_{2-}$ amplitude almost vanishes. Similarly, for the D15 partial wave the quark model predicts that the quark contribution to the ${}_pM^{1/2}_{2-}$ multipole vanishes and only the pion cloud contributes to the resonant part of the amplitude. The non-zero quark contribution in the case of the neutron multipole is however too weak to reproduce the data.

6 Discussion

Comparing the present results with the results for other partial waves obtained in chiral quark models we notice a general trend that the quark core alone does not provide sufficient strength to reproduced the observed resonance excitation amplitudes. The best known example is the P33 partial wave in which case the quark contribution to the electric dipole excitation of the $\Delta(1232)$ is estimated

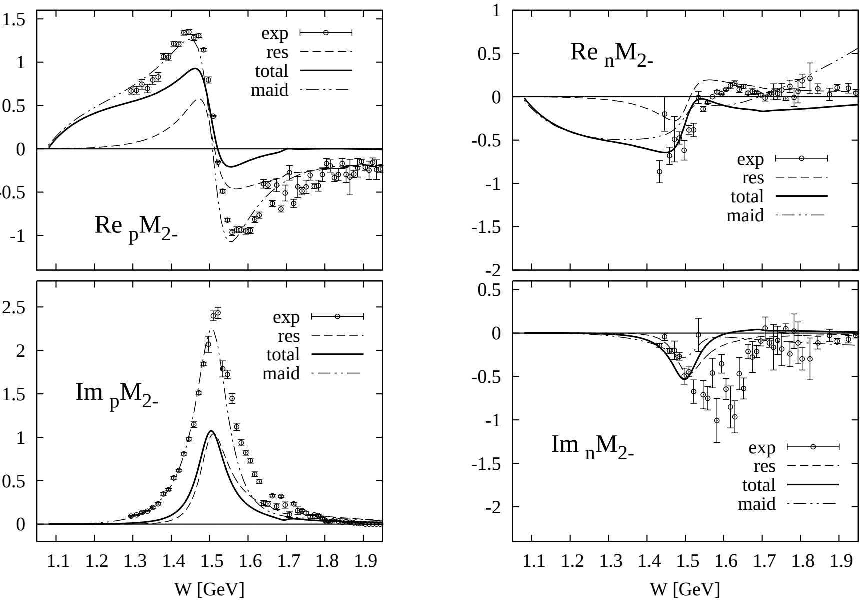

Fig. 4. The M_{2-} multipole, notation as in Fig. 3.

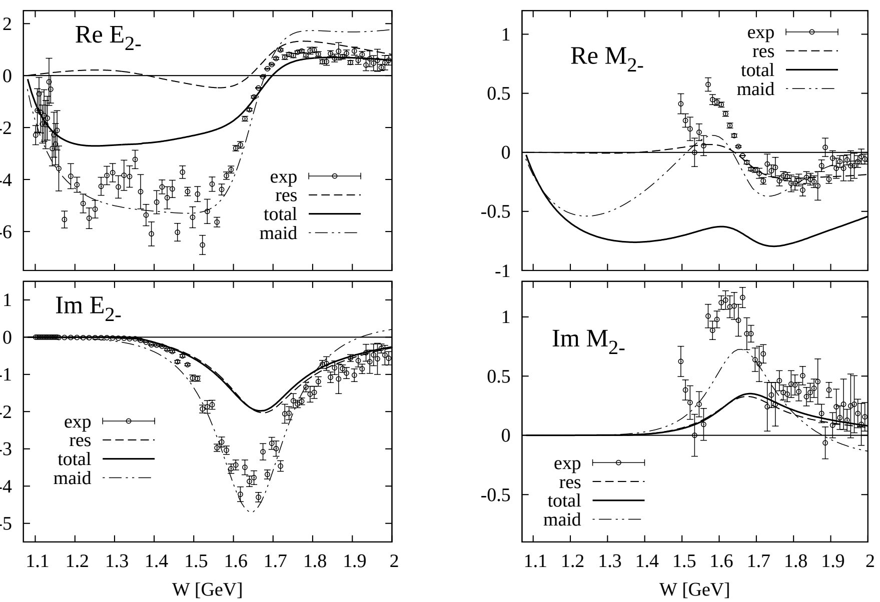

Fig. 5. E_{2-} and M_{2-} amplitudes for the D33 wave, notation as in Fig. 3.

by only 60 % while the rest is attributed to the pion cloud [1]. In the present calculation the pion cloud effects turn out not to be that important. In fact, we have noticed a considerable cancellation of different contributions of the meson cloud, e.g. the vertex correction due to pion loops and the genuine contribution

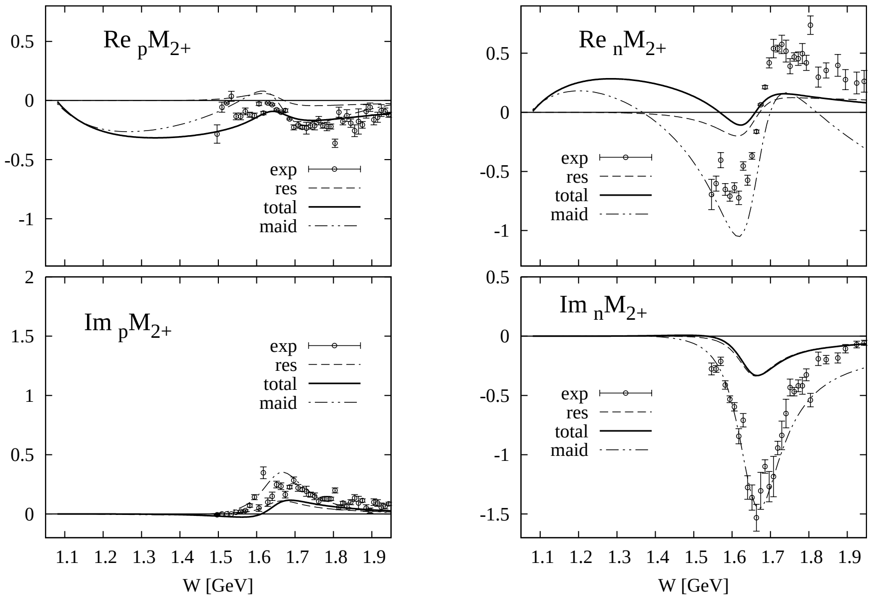

Fig. 6. The M_{2+} amplitudes for the D15 wave, notation as in Fig. 3.

of the pion cloud to the EM current. It is therefore possible that a calculation in a more elaborate chiral quark model could provide a better agreement with the data. To conclude, the overall qualitative agreement with the multipole analysis in the D13, D33 and D15 partial waves prove that the quark-model explanation of the D-wave resonance as the p-wave excitation of the quark core supplemented by the meson cloud is sensible and that no further degrees of freedom are needed.

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Scattering phase shifts and resonances from lattice QCD

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Most of hadrons are hadronic resonances - they decay quickly via the strong interactions. Among all the resonances, only the ρ meson has been properly simulated as a resonance within lattice QCD up to know. This involved the simulation of the $\pi\pi$ scattering in p-wave, extraction of the scattering phase shift and determination of m_R and Γ via the Breit-Wigner like fit of the phase shift.

In the past year, we performed first exploratory simulations of $D\pi$, $D^*\pi$ and $K\pi$ scattering in the resonant scattering channels [1,2]. Our simulations are done in lattice QCD with two-dynamical light quarks at a mass corresponding to $m_\pi \simeq$ 266 MeV and the lattice spacing $a = 0.124$ fm.

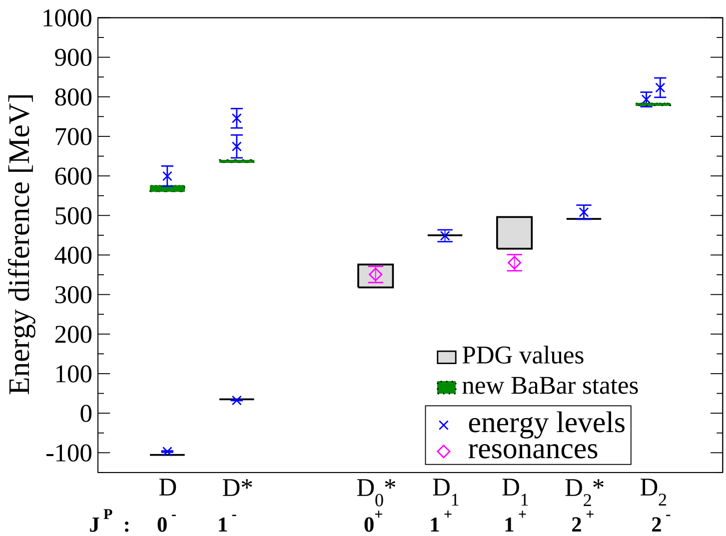

Fig. 1. Energy differences $\Delta E = E - \frac{1}{4}(M_D + 3M_{D^*})$ for D meson states in our simulation [1] and in experiment; the reference spin-averaged mass is $\frac{1}{4}(M_D + 3M_{D^*}) \approx 1971$ MeV in experiment. Magenta diamonds give resonance masses for states treated properly as resonances, while those extracted naively assuming $m_n = E_n$ are displayed as blue crosses [1].

The masses and widths of the broad scalar $D_0^*(2400)$ and the axial $D_1(2430)$ charmed-light resonances are extracted by simulating the corresponding $D\pi$ and $D^*\pi$ scattering on the lattice [1]. The resonance parameters are obtained using a Breit-Wigner fit of the elastic phase shifts. The resulting $D_0^*(2400)$ mass is 351 ± 21 MeV above the spin-average $\frac{1}{4}(m_D + 3m_{D^*})$, in agreement with the experimental value of 347 ± 29 MeV above. The resulting $D_0^* \to D\pi$ coupling $g^{lat} = 2.55 \pm 0.21$ GeV is close to the experimental value $g^{exp} = 1.92 \pm 0.14$ GeV, where g parametrizes the width $\Gamma \equiv g^2 p^\star/s$. The resonance parameters for the broad $D_1(2430)$ are also found close to the experimental values; these are obtained by appealing to the heavy quark limit, where the neighboring resonance $D_1(2420)$ is narrow. The simulation of the scattering in these channels incorporates quark-antiquark as well as $D^{(*)}\pi$ interpolators, and we use distillation method for contractions. The resulting D-meson spectrum is compared to the experimental one in Fig. 1.

In addition, the ground and several excited charm-light and charmonium states with various J^P are calculated using standard quark-antiquark interpolators. The lattice results for the charmonium are compared to the experimental levels in Fig. 2.

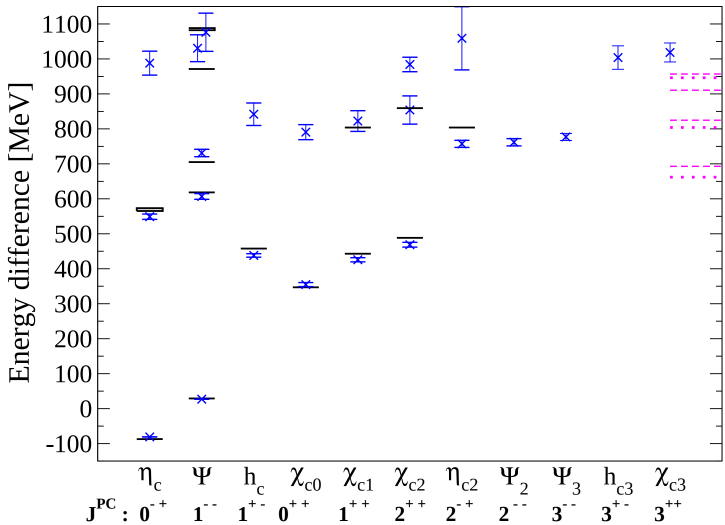

Fig. 2. Energy differences $\Delta E = E - \frac{1}{4}(M_{\eta_c} + 3M_{J\psi})$ for charmonium states in our simulation [1] and in experiment; reference spin-averaged mass is $\frac{1}{4}(M_{\eta_c} + 3M_{J\psi}) \approx 3068$ MeV in experiment. The magenta lines on the right denote relevant lattice and continuum $\bar{D}^{(*)}D^{(*)}$ thresholds.

We also simulated $K\pi$ scattering in s-wave and p-wave for both isospins $I = 1/2,\ 3/2$ using quark-antiquark and meson-meson interpolating fields [2]. Fig. 3 shows the resulting energy levels of $K\pi$ in a box. In all four channels we observe the expected $K(n)\pi(-n)$ scattering states, which are shifted due to the interaction. In both attractive $I = 1/2$ channels we observe additional states that are related

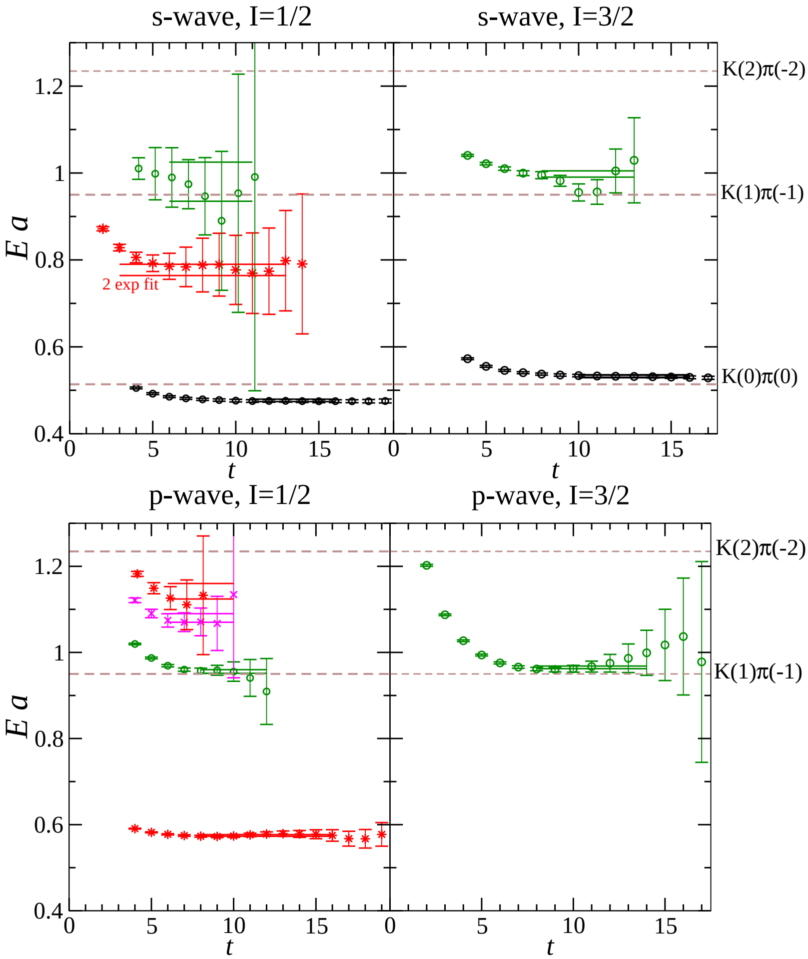

Fig. 3. The energy levels $E(t)a$ of the $K\pi$ in the box for all four channels (multiply by $a^{-1} = 1.59$ GeV to get the result in GeV). The horizontal broken lines show the energies $E = E_K + E_\pi$ of the non-interacting scattering states $K(n)\pi(-n)$ as measured on our lattice; $K(n)\pi(-n)$ corresponds to the scattering state with $p^* = \sqrt{n}\frac{2\pi}{L}$. Note that there is no $K(0)\pi(0)$ scattering state for p-wave. Black and green circles correspond to the shifted scattering states, while the red stars and pink crosses correspond to additional states related with resonances.

to resonances; we attribute them to $K_0^*(1430)$ in s-wave and $K^*(892)$, $K^*(1410)$ and $K^*(1680)$ in p-wave. We extract the elastic phase shifts δ at several values of the $K\pi$ relative momenta. The resulting phases exhibit qualitative agreement with the experimental phases in all four channels, as shown in Fig. 4. In addition to the values of the phase shifts shown in Fig. 4, we also extract the values of the phase shift close to the threshold, which are expressed in terms of the scattering lengths in [2].

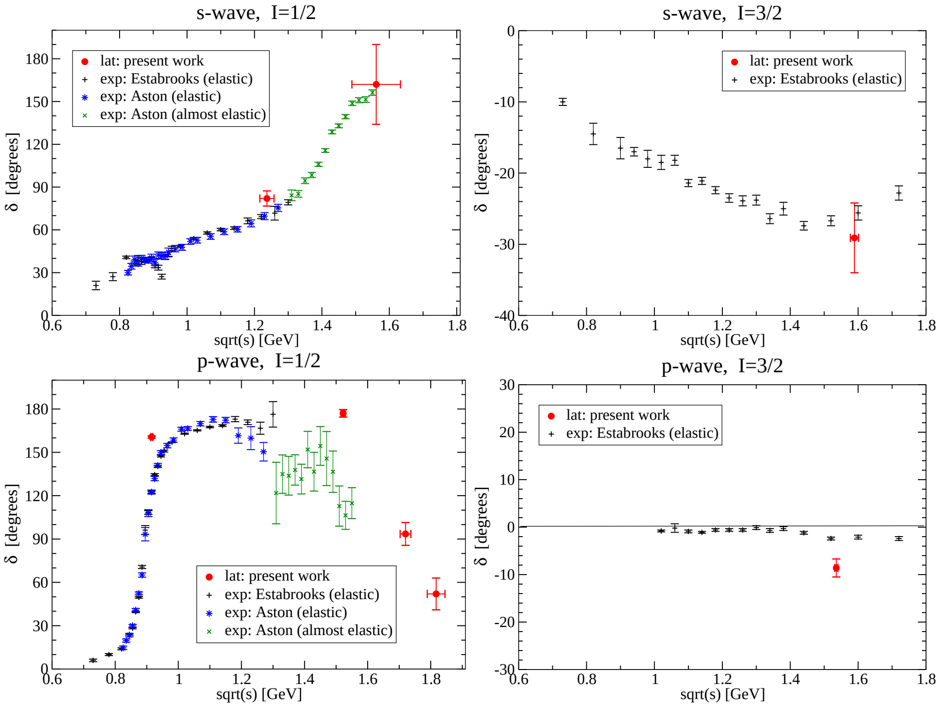

Fig. 4. The extracted $K\pi$ scattering phase shifts δ_ℓ^I in all four channels $l = 0, 1$ and $I = 1/2,\ 3/2$. The phase shifts are shown as a function of the $K\pi$ invariant mass $\sqrt{s} = M_{K\pi} = \sqrt{(p_\pi + p_K)^2}$. Our results (red circles) apply for $m_\pi \simeq 266$ MeV and $m_K \simeq 552$ MeV in our lattice simulation. In addition to the phases provided in four plots, we also extract the values of $\delta_0^{1/2,\ 3/2}$ near threshold $\sqrt{s} = m_\pi + m_K$, but these are provided in the form of the scattering length in the main text (as they are particularly sensitive to $m_{\pi,K}$). Our lattice results are compared to the experimental elastic phase shifts (both are determined up to multiples of 180 degrees).

We believe that these simulations of the $D\pi$, $D^*\pi$ and $K\pi$ scattering in the resonant channels represent encouraging step to simulate resonances properly from first principle QCD. There are many other exciting resonances waiting to be simulated along the similar lines.

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Scattering of nucleon on a superheavy neutron *

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Abstract. The scattering cross section of a superheavy baryon on a nucleon is estimated. The possibility that such a superheavy baryon (from a higher quark family) might be a viable candidate for the dark matter, is discussed.

1 Introduction

The purpose of this talk is twofold.

(i) Scattering of a light cluster on a superheavy cluster is a challenging few-body problem. The energy scales and consequently the sizes of both clusters differ by 5-6 orders of magnitude. Due to colour neutrality of unperturbed clusters, the strong interaction acts only at a very short distance via the virtual colour-octet colour-octet Van der Waals excitation. The novel feature is the van der Waals interaction at contact separation. Moreover, due to the small size of the superheavy cluster the effective quark-quark interaction is expected to be coulomb-like and this feature might be tested even in bottomium collisions.

(ii) We want to show that clusters of strongly interacting particles are viable candidates for dark matter provided their masses are large enough. Then both the number density of dark matter particles is small and their cross section is small due to their small size.

We require that the number of collisions of dark matter particles against the detector is either consistent with the DAMA experiment [1] (if confirmed) or lower (if DAMA is not confirmed). It turns out that superheavy quarks must have a mass of about 100 TeV or more in order to have a low enough collision rate by weak interaction. Surprisingly, at this mass the strong cross section is much smaller than the weak cross section and can be neglected.

As an example we take the superheavy quarks from the unified *Spin-Charge-Family* theory [2–6] which has been developed by one of the authors (SNMB) in the recent two decades. For a short review, we invite the reader to read the Bled 2010 Proceedings [7]. In this theory eight families of quarks and leptons are predicted, with the fifth family decoupled from the lower ones and therefore rather stable. The most promising candidates for dark matter are the superheavy neutrons (the $n_5 = u_5 d_5 d_5$ clusters) of the fifth family.

* Talk delivered by M. Rosina

There is a danger in this proposal. Either the charged baryon $u_5u_5u_5$ or the charged baryon $d_5d_5d_5$ could be the lightest, depending on whether u_5 or d_5 is lighter. Charged clusters cannot, of course, constitute dark matter. Forming the atoms with the first family electrons they would have far too large scattering amplitude to be consistent with the properties of dark matter. However, if one takes into account also the electro-weak interaction between quarks, then the neutral baryon $n_5 = u_5d_5d_5$ can be the lightest, provided the u-d mass difference is not too large. We have put limits on the u-d quark mass differences in ref. [7] and we briefly repeat the result (choosing $\alpha_{EM} = 1/128,\ \alpha_W = 1/32,\ \alpha_Z = 1/24$).

For superheavy quarks, the colour interaction is assumed to be coulombic and we solve the Hamiltonian for the three-quark system

$$H = 3m_5 + \sum_i \frac{\mathbf{p}_i^2}{2m_5} - \frac{(\sum_i \mathbf{p}_i)^2}{6m_5} - \sum_{i<j} \frac{2}{3}\frac{\alpha_s}{r_{ij}}.$$

For the choice of the average quark mass $m_5 = 100$ TeV and $\alpha_s = 1/13$ the binding energy is $E_0 = -\eta\alpha_s^2 m_5 = -0.39$ TeV and the average quark momentum $p = \sqrt{2m_5\, E_{kin}/3} = 5.1$ TeV . (The coefficient η has been obtained variationally).

The electroweak interaction prefers the neutral $u_5d_5d_5$ and it cannot decay into $d_5d_5d_5$ or $u_5u_5d_5$ provided

$$-0.026\,\mathrm{TeV}\ < m_{u5} - m_{d5} <\ 0.39\,\mathrm{TeV}.$$

This limits are not very narrow, but they are narrow compared to the mass scale of $m_5 = 100$ TeV.

2 The weak $(u_5d_5d_5)$ – $(u_1d_1d_1)$ cross section

It is easy to calculate the scattering amplitude since the superheavy neutron is a point particle compared to the range of the weak interaction and its quark structure is not important. Only Z-exchange matters since there is not enough energy to excite $u_5d_5d_5$ into $d_5d_5d_5$ or $u_5u_5d_5$ via W-exchange. We consider only the scattering on neutron (the “charge” of proton almost happens to cancel!). Also, we consider only the Fermi (vector) matrix element, since it adds coherently in heavy nuclei, while the Gamov-Teller (axial) has many cancellations in spin coupling.

$$\mathcal{M} = [\frac{1}{2}t_0(1) - \sin^2\vartheta_W e(1)]\,\frac{g_Z^2}{m_Z^2}\,[\frac{1}{2}t_0(5) - \sin^2\vartheta_W e(5)] = \frac{G_F}{2\sqrt{2}}$$

$$\sigma_n = 2\pi|\mathcal{M}|^2\,\frac{4\pi p_1^2}{(2\pi)^3 v^2} = \frac{m_{n1}^2}{\pi}\,|\mathcal{M}|^2 = \frac{G_F^2 m_{n1}^2}{8\pi} = 1.9\times 10^{-13}\mathrm{fm}^2\,.$$

We should note that the cross section does not depend on the mass m_{n5} provided it is much larger than m_{n1} of the first family. For a heavy target

$$\sigma_A = \sigma_n\,(A-Z)^2A^2$$

The rate at a detector of ${}^{23}_{11}$Na ${}^{127}_{53}$I per kilogram of detector is

$$R_{1kg} = \sigma_A \, N_A \, \frac{\rho_{n5} \times v}{m_{n5}}$$

$$R_{1kg} = \sigma_n \, [(A_{Na} - Z_{Na})^2 A_{Na}^2 + (A_I - Z_I)^2 A_I^2] \, \frac{N_{avogadro}}{A_{Na} + A_I} \, \frac{\rho_{n5} \times v}{m_{n5}} = 1.3/\text{day}$$

We used the data $\rho_{n5} = 0.3\,\text{GeV}\,\text{cm}^{-3}$, $m_{n5} = 300\,\text{TeV}$, $v = 230\,\text{km/s}$.

This can be compared to the rate claimed by the DAMA collaboration:

$$\Delta R_{1kg}(\text{DAMA}) = 0.02/\text{day}, \quad R_{1kg}(\text{DAMA}) \sim (0.1 \leftrightarrow 1)/\text{day}.$$

This comparison was used to decide about the choice of m_5 in our example. If DAMA results are not confirmed, m_5 should be even larger.

3 The strong MESON – meson cross section

This Section is a **lesson** for a future realistic calculation of the $(u_5 d_5 d_5) - (u_1 d_1 d_1)$ scattering . We want to show that for superheavy quarks the strong cross section is much smaller than the weak cross section and can be neglected. For this purpose we need only an estimate and not a detailed calculation. Meson-meson scattering offers a good estimate since the baryon in a quark-diquark approximation resembles a meson. However, this lesson is very relevant for botomium scattering and for future heavy baryons in the 10-100 GeV region.

Here we present the trial functions of the light and heavy meson, together with relevant quantities such as the chromomagnetic dipole moment D of the heavy meson sitting in the dipole field G of the light meson. Note that m and M are quark masses and $\alpha = \frac{4}{3}\alpha_s$.

$$\mathbf{r} = \mathbf{r_q} - \mathbf{r_{\bar{q}}}, \quad b = 1/(\tfrac{1}{2}m)\alpha \qquad\qquad \mathbf{R} = \mathbf{R_Q} - \mathbf{R_{\bar{Q}}}, \quad B = 1/(\tfrac{1}{2}M)\alpha \ll b$$

$$\psi_0 = (2/\sqrt{4\pi b^3}) \, \exp(-r/b) \qquad\qquad \Psi_0 = (2/\sqrt{4\pi B^3}) \, \exp(-R/B)$$

$$\psi_z = \frac{2^{-3/2}}{\sqrt{4\pi f^3}} \, (r/f) \cos\vartheta \, \exp(-r/f) \qquad\qquad \Psi_z = \frac{2^{-3/2}}{\sqrt{4\pi B^3}} \, (R/B) \cos\Theta \, \exp(-R/B)$$

$$\epsilon_0 = -(1/2)(\tfrac{1}{2}m)\alpha^2 \qquad\qquad E_0 = -(1/2)(\tfrac{1}{2}M)\alpha^2$$

$$\epsilon_{z,kin} = +(1/8)(\tfrac{1}{2}m)\alpha^2 \, (b/f)^2 \qquad\qquad E_z = -(1/8)(\tfrac{1}{2}M)\alpha^2$$

$$G_z = \langle \psi_z | z/(r/2)^3 | \psi_0 \rangle = \gamma/\sqrt{fb^3} \qquad\qquad D = \langle \Psi_z | Z | \Psi_0 \rangle = \beta B$$

$$\gamma = 16\sqrt{2}/3 = 7.542 \qquad\qquad \beta = 2^{15/2}/3^5 = 0.745$$

The meson wavefunctions get "decorated" with colour factors

$$\phi_0 = \psi_0 \, \frac{(r[gb] + g[br] + b[rg])}{\sqrt{3}}, \quad \phi_{z3} = \psi_z \, \frac{(r[gb] - g[br])}{\sqrt{2}}$$

$$\Phi_0 = \Psi_0 \frac{(r[gb] + g[br] + b[rg])}{\sqrt{3}}, \quad \Phi_{z3} = \Psi_z \frac{(r[gb] - g[br])}{\sqrt{2}}$$

We write explicitly only the spatial excitation in the z-direction and colour excitation in the "third colour" $\omega = 3$. Others behave similarly.

We shall need the colour matrix element

$$\left\langle \frac{r[gb] - g[br]}{\sqrt{2}} \right| \frac{\lambda^3_q}{2} - \frac{\lambda^3_{\bar{q}}}{2} \left| \frac{r[gb] + g[br] + b[rg]}{\sqrt{3}} \right\rangle = \sqrt{\frac{2}{3}}$$

For color neutral hadrons, the dominant term in the expansion yields the effective dipole–dipole, colour-octet – colour-octet potential

$$\hat{V}_{\text{dipole}} = \alpha_s \left(\mathbf{R_Q} \frac{\overrightarrow{\lambda_Q}}{2} + \mathbf{R_{\bar{Q}}} \frac{\overrightarrow{\lambda_{\bar{Q}}}}{2} \right) \left(\frac{\mathbf{r_q}}{r_q^3} \frac{\overrightarrow{\lambda_q}}{2} + \frac{\mathbf{r_{\bar{q}}}}{r_{\bar{q}}^3} \frac{\overrightarrow{\lambda_{\bar{q}}}}{2} \right),$$

The perturbation term between the unperturbed ground state and the virtual excitation is then

$$V'_{z,3} = \alpha_s \langle \Psi_z \psi_z | \left\{ \frac{Z}{2} \right\} \sqrt{\frac{2}{3}} \left\{ \frac{z/2}{(r/2)^3} \right\} \sqrt{\frac{2}{3}} | \Psi_0 \psi_0 \rangle = \frac{\alpha_s D_z G_z}{6}$$

$$V'_{x,\omega} = V'_{y,\omega} = V'_{z,\omega} \equiv V' \quad \text{equal for all } \omega\,.$$

The second order perturbation theory then gives the effective potential between the two clusters

$$V_{eff} = -24 \frac{V'^2}{(E_z - E_0) + \epsilon_{z,kin}}$$

We have neglected $\epsilon_{z,pot}$ and ϵ_0. The factor 24 comes from 3 spacial and 8 colour degrees of freedom.

$$V_{eff} = -\frac{2}{3} \frac{(\alpha_s D_z G_z)^2}{(3/8)(\frac{1}{2}M)(4\alpha_s/3)^2 + (1/8)(\frac{1}{2}m)(4\alpha_s/3)^2(b/f)^2}$$

$$V_{eff} = -\frac{2(\beta\gamma B)^2}{fb^3(M + (1/3)m(b/f)^2)}$$

Note that α_s has canceled. Minimization with respect to f gives $f/b = \sqrt{m/3M} << 1$. Finally, we get

$$V_{eff} = -\frac{\sqrt{3}\beta^2\gamma^2}{b^3}\left(\frac{m}{M}\right)^{3/2}\frac{B}{m}$$

Here we took the distance between the two clusters $U = 0$. We assume

$$V_{eff}(U) = V_{eff}(U = 0)\ \exp(-2U/b).$$

In Born approximation (with the mass of the lighter cluster $m_q + m_{\bar{q}} = 2m$) we get

$$a = \frac{(2m)}{2\pi}\int V_{eff}(U)d^3U = \sqrt{3}\beta^2\gamma^2\left(\frac{m}{M}\right)^{3/2} B.$$

Let us give a numerical example with the choice

$$m = 300\,\mathrm{MeV}, \quad M = \tfrac{1}{2}m_Q = 100\,\mathrm{TeV}, \quad m/M = 3\cdot 10^{-6}, \quad \alpha_s = 1/13$$

$$a = \sqrt{3}\beta^2\gamma^2(\frac{m}{M})^{3/2}\,B = 1.1\cdot 10^{-11}\ \mathrm{fm}$$

$$\sigma = 4\pi a^2 = 1.5\cdot 10^{-21}\ \mathrm{fm}^2$$

4 Conclusion

Regarding the weak interaction, the scattering rate of superheavy clusters is inversely proportional to their mass because (i) their weak cross section is independent of the heavy mass if it is large enough and (ii) because their number density is inversely proportional to their mass for the known dark matter density. This argument requires the superheavy quark mass to be about 100 TeV (if DAMA experiment is confirmed) or more.

For such a heavy mass, **the strong cross section is MUCH SMALLER than the weak cross section.** The reason is (i) the small size of the heavy hadron, $B = 3.8\cdot 10^{-5}$ fm and moreover, (ii) the suppression factor $(m/M)^3$ which is a consequence of colour neutrality of both clusters so that they interact only by induced color dipoles (“van der Waals interaction”).

The lesson from the heavy hadron – light hadron scattering will be useful also for not-so-exotic processes such as botomium and bbb scattering.

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Approaching the spin structure of ^{3}He by polarization observables

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Abstract. The E05-102 experiment at Jefferson Laboratory (TJNAF) was devised to study the double-polarization asymmetries in electron-induced deuteron, proton, and neutron knockout from polarized ^{3}He at low momentum transfers, in a wide range of missing momenta. With this advanced experimental technique, we strive to obtain a much clearer insight into the ground-state structure of ^{3}He, the cornerstone nucleus widely used as the effective neutron target. An order of magnitude improvement in the statistical uncertainties with respect to existing measurements is anticipated. We report on the status of the ongoing data analysis.

1 Physics motivation

The primary motivation to study electron-induced processes involving the ^{3}He nucleus (see [1] and references therein) is to understand the ground-state structure of this nucleus. This structure is not only interesting by itself; it is also important to study it in order to be able to interpret all data “on the neutron” for which ^{3}He acts as an effective target to a very good approximation. Contrary to common belief, there is no widely adopted consensus about the exact level at which this approximation can be treated as “good” or “good enough”.

A precise understanding of the transition between the experimental data acquired on ^{3}He targets and the observables corresponding to the neutron has become a burning issue since the statistical precision of recently performed (or future) experiments is so large that the systematical uncertainties of this computational transition procedure have become comparable to it. Some of the most interesting observables fall into this category, like e.g. the neutron elastic form-factors

$$G_E^n\,, \quad G_M^n\,,$$

and the polarized quark structure functions corresponding to the neutron,

$$G_E^n\,, \quad G_M^n\,, \quad A_1^n\,, \quad g_1^n\,, \quad g_2^n\,,$$

as well as the studies of the GDH sum rule.

One of the main complications, of course, is that the protons in ^{3}He partly polarized due to the presence of S'- and D-state components of the ground-state wave-function. (The ground-state configuration of ^{3}He is intimately connected to another open question of differences in RMS-radii, $\langle r^2 \rangle^{1/2}$, of ^{3}H as opposed to ^{3}He, a matter largely unresolved due to an almost complete lack of measurements on tritium.) The manifestations of the distribution of spin, orbital angular momentum and isospin within ^{3}He appear to be most prominent and unambiguous in double-polarization asymmetries for electron-induced deuteron, proton, and neutron knockout from polarized ^{3}He. Numerous discrepancies among the state-of-the-art theories persist for these observables.

In short, understanding the role of D and S' states as the two most relevant sub-leading components of the ^{3}He wave-function, and of the spin- and isospin-dependence of reaction mechanisms on ^{3}He is one of the key issues in the "Standard Model" of few-body theory.

2 The measurements

The exclusive cross-section for electron-induced deuteron knockout (with both the beam and the target polarized) has the form

$$\frac{d\sigma(h, \mathbf{S})}{d\Omega_e \, dE_e \, d\Omega_d \, dp_d} = \frac{d\sigma_0}{d\Omega_e \, dE_e \, d\Omega_d \, dp_d} \left[1 + \mathbf{S} \cdot \boldsymbol{A}^0 + h(A_e + \mathbf{S} \cdot \boldsymbol{A}) \right] \, .$$

In the experiment described in this contribution, we measured two components of $\boldsymbol{A}$ (or linear combinations thereof), which correspond to the transverse and longitudinal double-polarization asymmetries

$$A_{x,z} = \frac{[d\sigma_{++} + d\sigma_{--}] - [d\sigma_{+-} + d\sigma_{-+}]}{[d\sigma_{++} + d\sigma_{--}] + [d\sigma_{+-} + d\sigma_{-+}]} \, ,$$

where the subscript signs denote the helicities of the electron beam and the orientation of the target spin. The target was polarized along the beam-line and perpendicular to it (in both sideways directions). Similarly, the asymmetries for exclusive processes in which the proton and the neutron were knocked out (with obvious modifications to the above formulas) have been measured.

Since the transverse and longitudinal asymmetries in each channel have very distinct sensitivities to the dominant S and the sub-dominant D and S' component of the ^{3}He as functions of missing momentum, our experiment carries an immense resolving power for testing theories mentioned below. The fact that several exclusive channels were measured at the same time at approximately the same four-momentum transfer of about 0.2 to 0.3 $(\mathrm{GeV}/c)^2$, in a large range of missing momenta, and with excellent statistical and systematical uncertainties, is another landmark feature of this experiment.

The resulting asymmetries will be compared to state-of-the-art theories of the ^{3}He nucleus. We exploit the calculations of the Bochum/Krakow group [4] that apply a full Faddeev approach with the AV18 NN-potential and the Urbana IX three-nucleon force, together with a complete treatment of final-state interactions (FSI) and meson-exchange currents (MEC).

Also available to use are the calculations of the Hannover/Lisbon group [5] that are also full Faddeev, but with a coupled-channel extension and refit of the CD-Bonn NN-potential. They also incorporate FSI and MEC, while the effective three-nucleon force and two-body currents are provided by inclusion of the Δ as an active degree-of-freedom. Coulomb interaction for outgoing charged baryons is also included.

The group from Pisa has also provided us with their calculations based on the AV18 potential and the Urbana IX force in which the FSI are included by means of the variational pair-hyperspherical harmonics expansion, and MEC are also accounted for. This is not a Faddeev-type calcuation, but its accuracy is assumed to be completely equivalent to it [6]. All three predictions (full calculations only) are presented in comparison to the anticipated experimental uncertainties in Fig. 1.

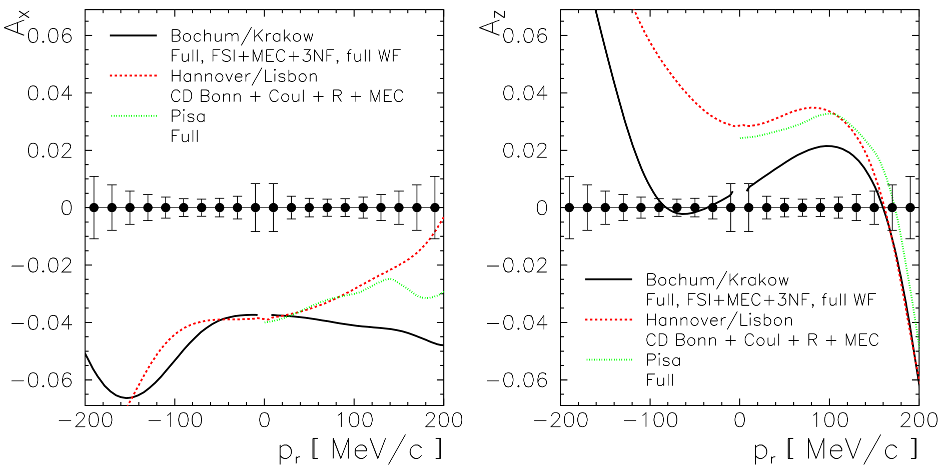

Fig. 1. The predictions for the asymmetries A_x and A_y in the quasi-elastic $^3\mathbf{He}(\mathbf{e}, e'd)$ process. The anticipated experimental uncertainties and three calculations by the Bochum/Krakow, Hannover/Lisbon and Pisa groups are shown.

3 Status of data analysis

The polarizations of the electron beam and the target have been established, and the beam and target monitoring apparatus have been calibrated. The magneto-optical properties of the BigBite spectrometer that was used to detect the charged hadrons have been determined [3], and the tracking and PID detectors have been calibrated, along with the neutron detector and the spectrometer used to detect the electrons. Presently the analysis work is focused on the correct averaging of the theoretical asymmetries over the relatively large experimental acceptance. To this purpose, we have obtained the calculations of the asymmetries on a rather dense grid of points in the (E_e, θ_e) plane that covers the majority of our acceptance, as shown in Fig. 2. The additional dimension in which averaging is performed is the deuteron (or proton) emission angle with respect to the virtual photon.

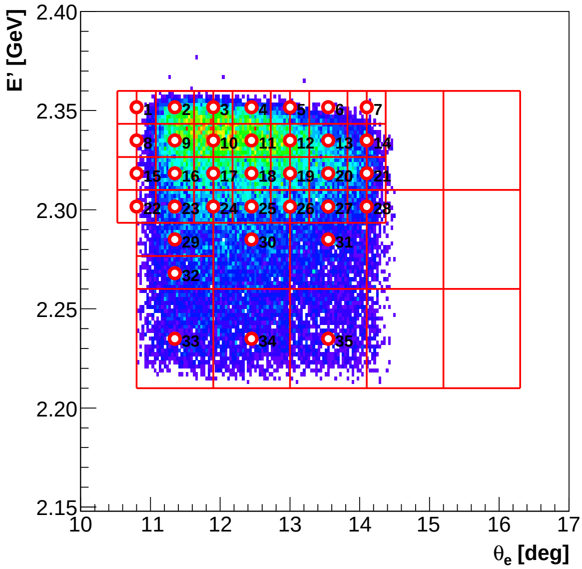

Fig. 2. The grid in (E_e, θ_e) plane on which the theoretical calculations will be performed, thus covering the most relevant parts of the experimental acceptance in the $^3\mathbf{He}(\mathbf{e}, e'd)$ channel. The high density of points is needed for reliable acceptance averaging because the asymmetries have a strong dependence on the energy transfer $E_e - E_e'$ (vertical axis).

The statistics of the data is sufficient to achieve a precision better than 2% on the asymmetries in each 20 MeV/c bin in missing momentum, ranging to about 200 MeV/c in the deuteron channel and about 300 MeV/c in the proton channels. Similar accuracy will be achieved in the neutron channel, and an even better one in the inclusive channels, which are a "bonus" of our experiment.

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DREENA framework: high pt predictions and proposal of a new observable

Friday, 7 September 2018 16:05 (20)

I will present our newly developed DREENA framework, which allows predicting energy loss of high pt partons traversing quark gluon plasma (QGP). The framework is based on dynamical energy loss formalism, and is applied to both the medium with constant temperature (DREENA-C) and to evolving medium modeled by Bjorken 1+1D expansion (DREENA-B). The formalism allows to generate predictions for both light and heavy flavor observables, for different centralities and collision energies, as well as different experiments and collision systems. Accordingly, I will first show that our post-dictions agree well with a wide range of data at different centralities. Furthermore, I will show that the predictions, which were published well before the data became available, agree very well with these data, again explaining some of the experimentally observed, but intuitively unexpected, suppression patterns. I will also propose a new observable, which allows clearly distinguishing between different energy loss mechanisms, as well as numerical predictions and simple scaling arguments that support this proposal. The first steps in our work towards the application of this model as a novel high-precision tomographic tool of QGP medium, will also be discussed.

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Session Classification : Open Heavy Flavour

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Contribution ID: **328** Type: **3a) Heavy-flavours and quarkonia (TALK)**

DREENA framework: predictions, comparison with experimental data, and proposal of a new observable

Thursday, 4 October 2018 11:05 (20 minutes)

We will present our newly developed DREENA framework, which allows predicting energy loss of high $p_\perp$ partons traversing quark gluon plasma (QGP). The framework is based on dynamical energy loss formalism, and is applied to both the medium with constant temperature (DREENA-C) [1] and to evolving medium modeled by Bjorken 1+1D expansion (DREENA-B) [2]. The formalism allows making numerical predictions for a wide number of observables, centralities and collision energies, and for different experiments and collision systems. Accordingly, we will first show that our postdictions agree well with a wide range of data at different centralities. Furthermore, we will show that the predictions, which were published well before the data became available, agree very well with these data, again explaining some of the experimentally observed, but intuitively unexpected, suppression patterns. We will also propose a new observable [3], which allows clearly distinguishing between different energy loss mechanisms, as well as numerical predictions and simple scaling arguments that support this proposal. The first steps in our work towards the application of this model as a novel high-precision tomographic tool of QGP medium, will also be discussed.

[1] D. Zigic, I. Salom, J. Auvinen, M. Djordjevic and M. Djordjevic, arXiv:1805.03494 [nucl-th]

[2] D. Zigic, I. Salom, M. Djordjevic and M. Djordjevic, arXiv:1805.04786 [nucl-th].

[3] M. Djordjevic, D. Zigic, M. Djordjevic and J. Auvinen, arXiv:1805.04030 [nucl-th].

Summary

Primary authors: DJORDJEVIC, Magdalena (Institute of Physics Belgrade); AUVINEN, Jussi (Institute of Physics Belgrade); DJORDJEVIC, Marko; ZIGIC, Dusan (Institute of Physics Belgrade); Dr SALOM, Igor (Institute of Physics Belgrade)

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Numerical predictions of DREENA-C and DREENA-B frameworks

We here present two frameworks that allow generating a wide range of predictions from the dynamical energy loss formalism [1,2]. In distinction to majority of other methods, the dynamical energy loss formalism takes into account a realistic medium composed of dynamical scattering centers. The first framework (DREENA-C) [1] applies to the medium with constant temperature, while the second framework applies to evolving medium modeled by Bjorken 1+1D expansion (DREENA-B) [2]. We here present joint R_{AA} and v_2 predictions for light and heavy flavor, for different systems, centralities and collision energies. DREENA-C (constant temperature) predictions overestimate v_2. For DREENA-B (Bjorken expansion) we obtain a good agreement with both R_{AA} and v_2 data. Introducing medium evolution has a larger effect on v_2, but for precision predictions it also has to be taken into account for R_{AA}. These results argue the dynamical energy loss formalism can provide a basis for a state of the art QGP tomography tool.

[1] D. Zigic, I. Salom, J. Auvinen, M. Djordjevic and M. Djordjevic, arXiv:1805.03494 [nucl-th]
[2] D. Zigic, I. Salom, M. Djordjevic and M. Djordjevic, arXiv:1805.04786 [nucl-th]

Summary

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Institute of Physics
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P R E F A C E

This volume contains some reviews and original research contributions, which are presented at the **8th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics**, held in Belgrade (Serbia), August 24–31, 2014 (http://www.mphys8.ipb.ac.rs). The programme of this meeting was mainly oriented towards some recent developments in gravity and cosmology, string and quantum field theory, and some relevant mathematical methods. We hope that articles presented here will be valuable literature not only for the participants of this meeting but also for many other PhD students and researchers in modern mathematical and theoretical physics. We are grateful to all authors for writing their contributions for these proceedings.

The previous seven meetings in this series of summer schools and conferences on modern mathematical physics were also held in Serbia: Sokobanja, 13–25 August 2001; Kopaonik, 1–12 September 2002; Zlatibor, 20–31 August 2004; Belgrade, 3–14 September 2006; Belgrade, 6–17 July 2008; Belgrade, 14–23 September 2010; and Belgrade, 9–19 September 2012. The corresponding proceedings of all these meetings were published by the Institute of Physics, Belgrade, and are available in the printed form as well as online at the meeting websites.

This eighth meeting took place at the "Milutin Milankovic Society" located in Belgrade downtown. There was conference dinner in Little Bay restaurant and excursion to Sremski Karlovci and Novi Sad. We hope that all attending this meeting will recall it as a useful and pleasant event, and will wish to participate again in the future.

We wish to thank all lecturers and other speakers for their interesting and valuable talks. We also thank all participants for their active participation. Financial support of our sponsors: *Ministry of Education, Science and Technological Development of the Republic of Serbia, Belgrade; ICTP – SEENET–MTP grant RRJ-09 "Cosmology and Strings", Niš, Serbia; Project 174012 (Geometry, Education and Visualization with Applications) Belgrade,* was very significant for realization of this activity.

June 2015

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