Transport theory of granular swarms

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The transport of trace granular gas (swarm) in a carrier granular fluid is studied by means of the Boltzmann-Lorentz kinetic equation. Time-dependent perturbation theory is used to follow the evolution of the granular swarm from an arbitrary initial distribution. A nonhydrodynamic extension of the diffusion equation is derived, with transport coefficients that are time dependent and implicitly depend on the wave vector. Transport coefficients of any order are obtained as velocity moments of the solutions of the corresponding kinetic equations derived from the Boltzmann-Lorentz equation. For the special case of the initial distribution of swarm particles, transport coefficients are identified as time derivatives of the moments of the number density. Finally the granular particle transport theory is extended by the introduction of the concept of non-particle-conserving collisions.

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I. INTRODUCTION

During the past decade a lot of effort has been put into understanding of the behavior of granular media, due to their importance in various industrial and geological processes [1–4]. In modern technology, the blending of granular materials is an unavoidable step. Granular media are difficult to mix. At present, physicists do not know how to predict *a priori* whether two powders will mix or segregate when stirred together in a given blender. There are two dominant mixing mechanisms, both of them not yet completely understood. Convection is by far the faster and more efficient one for grains (as well as for fluids). Diffusion is much slower than convection, but it occurs in all directions. The relative importance of each mechanism is determined by the initial distribution of species in the mixer. In the absence of segregational tendencies between dissimilar particles, diffusion will eventually lead to a completely homogeneous mixture; when diffusion dominates, the mixing problem is reduced to that of finding relevant parameters that minimize the blending time. Better understanding of the transport process in granular mixtures should help in predicting whether a given flow will mix or segregate its constituents.

The case of polydisperse granular systems has been studied by several authors. Most of the previous theoretical work has been limited to the case of slightly inelastic hard spheres. Jenkins and Mancini [5] and Zamankhan [6] developed a kinetic theory, based on revised Enskog theory, in order to predict the transport properties of mixtures of smooth, slightly inelastic spheres. Recently, Garzó and Dufty [7] have provided a description of hydrodynamics in binary granular mixtures at low density, valid over the wide parameter range (mass ratio, diameters, concentrations, inelasticity parameters). References [8–10] examine the efficiency of diffusion as a mixing-segregating mechanism in granular materials.

In this paper we have developed the transport theory for a binary mixture in which one of the components (*swarm*) is present in the tracer concentration. For the purposes of this theory a granular swarm is defined as an ensemble of *independent* test particles moving in a background granular fluid. Swarm particle concentrations are assumed sufficiently low that both the mutual interactions between the swarm particles and the influence of the swarm on the background fluid can be neglected. The behavior of the swarm particles is therefore determined solely through collisions with the particles of carrier granular fluid and the external force field.

To develop the basic features of the theory of granular swarms and to simplify the analysis we consider the inelastic hard sphere model [11] only. The system of inelastic hard spheres (IHS's) represent an idealized model for *rapid granular flows*, where the dynamics of individual macroscopic particles is controlled by inelastic binary collisions, separated by ballistic propagation over a typical mean free path. Such flows do obey the conservation laws of mass and momentum, and can therefore be considered as fluids. However, energy is not conserved.

The analytical method we follow parallels our previous work on charged particle transport [12]. Using time-dependent perturbation theory generalized to non-Hermitian operators, we construct the transport theory of swarm particles as an initial value problem for the Boltzmann-Lorentz kinetic equation. Transport theory for granular swarms is somewhat more complicated than the analogous theory for charged particle swarms. The inherent time dependence of the reference state for carrier granular fluid introduces a new time scale [13–15] and requires some special attention in constructing the transport theory of granular swarms. Our definition of transport coefficients for swarm particles is applicable for an arbitrary nonstationary but homogeneous reference state of the surrounding granular fluid. The main result of this paper is a generalized diffusion equation (GDE) valid for all times, with an infinite set of transport coefficients which are expressed in terms of the solutions of a hierarchy of coupled linear integrodifferential equations.

The plan of the paper is as follows. In the remainder of the Introduction we review the elements of kinetic theory relevant for the subsequent discussions. In Sec. II the initial value problem for the Boltzmann-Lorentz kinetic equation is introduced and the corresponding transport theory is developed along lines which are a generalization of the perturbative method developed in [12]. In Sec. III we derive the nonhydrodynamic extension of the diffusion equation and

establish the connection between swarm particle flux and transport coefficients in the presence of the collision processes which do not conserve the number of swarm particles. Section IV is devoted to the analysis of the long-time behavior of transport coefficients. Finally, in the Conclusion, we summarize our main results. Some technical details of the calculations are given in the Appendixes.

Kinetic equation

We consider swarm particles of mass m in a granular fluid whose particles have mass m_0. All particles are smooth hard spheres ($d=3$) or disks ($d=2$). Collisions between swarm particles and fluid particles are characterized by a *constant* coefficient of normal restitution α, while the analogous quantity for collisions among fluid particles will be denoted by α_0. Both coefficients have values $0<\alpha,\alpha_0\leqslant 1$, with the value of unity corresponding to the elastic limit.

The revised Enskog kinetic equation for the one-particle distribution function $f(\vec{r},\vec{v},t)$ of the swarm particles is [16,17]

$$\left[\frac{\partial}{\partial t}+\vec{v}\cdot\frac{\partial}{\partial \vec{r}}+\vec{a}\cdot\frac{\partial}{\partial \vec{v}}\right]f(\vec{r},\vec{v},t)=J_E[f](\vec{r},\vec{v},t), \quad (1.1)$$

where J_E is the Enskog collision operator given by

$$J_E[f](\vec{r},\vec{v},t)=\overline{\sigma}^{d-1}\int d\vec{v}_1\int d\vec{e}(\vec{e}\cdot\vec{v}_r)\Theta(\vec{e}\cdot\vec{v}_r)[\alpha^{-2}\chi(\vec{r},\vec{r}-\overline{\sigma\vec{e}}|n,n_0)f(\vec{r},\vec{v}',t)f_0(\vec{r}-\overline{\sigma\vec{e}},\vec{v}_1',t)$$
$$-\chi(\vec{r},\vec{r}+\overline{\sigma\vec{e}}|n,n_0)f(\vec{r},\vec{v},t)f_0(\vec{r}+\overline{\sigma\vec{e}},\vec{v}_1,t)], \quad (1.2)$$

and $\vec{a}$ is the acceleration due to the external (gravitational) field. Here, $f_0(\vec{r},\vec{v},t)$ is the one-particle distribution function for the surrounding granular fluid, Θ is Heaviside step function, $\vec{e}$ is unit vector pointing from the center of the fluid particle to the center of the swarm particle at contact, and $\overline{\sigma}=(\sigma+\sigma_0)/2$, where σ and σ_0 are the diameters of swarm and fluid particles, respectively. The primes on the velocities denote the initial values $(\vec{v}',\vec{v}_1')$ that lead to $(\vec{v},\vec{v}_1)$ following a binary collision:

$$\vec{v}'=\vec{v}-\frac{(1+\alpha)\Delta}{\alpha(1+\Delta)}(\vec{e}\cdot\vec{v}_r)\vec{e}, \quad \vec{v}_1'=\vec{v}_1-\frac{(1+\alpha)}{\alpha(1+\Delta)}(\vec{e}\cdot\vec{v}_r)\vec{e}. \quad (1.3)$$

In the above expressions $\vec{v}_r=\vec{v}-\vec{v}_1$ and $\Delta=m_0/m$. Finally, $\chi[\vec{r},\vec{r}_1|n(t),n_0(t)]$ is the pair correlation function of elastic hard spheres in a spatially nonuniform equilibrium state.

The quantity $\chi[n,n_0]$ is a functional of the local partial densities $n(\vec{r},t)=\int d\vec{v}\, f(\vec{r},\vec{v},t)$ and $n_0(\vec{r},t)=\int d\vec{v}\, f_0(\vec{r},\vec{v},t)$. This implies that $\chi[n,n_0]$ and, consequently, $J_E[f]$ are highly nonlinear functionals of f through this density dependence. Since density-functional theories [18] are able to reliably reproduce the pair correlations in the fluid phase found in computer simulations, we will take $\chi[n,n_0]$ as a known functional.

The swarm particles diffuse in and are convected by the granular fluid. Their motion is also influenced by the gravitational field. The presence of a macroscopic flow of background fluid significantly modifies the transport properties, as swarm particles are now advected with the bulk motion of the fluid and generally mix by convective action at a much greater rate. In the kinetic theory of swarms a great simplification is possible because the influence of swarm particles on the background fluid is negligible. Therefore, the distribution function $f_0(\vec{r},\vec{v},t)$ obeys an *independent* nonlinear Enskog equation for inelastic particles. All properties of the surrounding fluid are assumed to be prescribed. For simplicity, here we will assume that the carrier granular fluid is in the homogeneous cooling state (HCS) described by a distribution of the form

$$f_0(\vec{r},\vec{v},t)\equiv f_H(\vec{v},t)=n_0 v_0^{-d}(t)\phi\left[\frac{v}{v_0(t)}\right], \quad (1.4)$$

where $v_0=[2k_BT_0(t)/m_0]^{1/2}$ is the thermal velocity of the fluid particles at time t, k_B is the Boltzmann constant, and ϕ is the scaling function. The granular temperature $T_0(t)$ is defined in the usual way:

$$\frac{d}{2}n_0k_BT_0(t)=\int d\vec{v}\frac{1}{2}m_0v^2f_H(\vec{v},t). \quad (1.5)$$

The HCS has been extensively studied [19–21] and used, because of its simplicity, as a reference state to build up theories for nonhomogeneous states. As indicated by Eq. (1.4) all the time dependence of the distribution f_H takes place through the granular temperature $T_0(t)$. This temperature decreases obeying Haff's law [22,23]

$$T_0(t)=T_0(0)\left[1+\frac{t}{t_c}\right]^{-2}. \quad (1.6)$$

The characteristic time of homogeneous cooling, t_c, can be determined from the second moment of the nonlinear Enskog-Boltzmann equation for a dense system of inelastic hard spheres [19]. The explicit form of the scaling function ϕ is only known in the first Sonine approximation [19].

The simplifying features of the swarm problem allow us to use the Boltzmann-Lorentz collision operator

$$J[f](\vec{r},\vec{v},t)=\chi(n_0)\overline{\sigma}^{d-1}\int \mathrm{d}\vec{v}_1\int \mathrm{d}\vec{e}(\vec{e}\cdot\vec{v}_r)\Theta(\vec{e}\cdot\vec{v}_r)$$
$$\times[\alpha^{-2}f(\vec{r},\vec{v}',t)f_H(\vec{v}_1',t)$$
$$-f(\vec{r},\vec{v},t)f_H(\vec{v}_1,t)], \tag{1.7}$$

instead of the Enskog collision operator (1.2). The frequency factor $\chi(n_0)$ is a constant that has been factored out of the collision integral. Indeed, the covolume effect and the screening effect responsible for modification of the binary-collision frequency are both due to the presence of background particles, which we assume to be at the HCS, with a *constant* density n_0.

The equilibrium radial distribution function at contact χ is independent of particle collisional properties and may be calculated from approximate formulas. The expression for the radial distribution at contact χ for mixtures of hard spheres $(d=3)$ that agrees best with numerical simulations is that of Mansoori *et al.* [24]. Using the assumption that swarm particles are very sparse, with density $n\ll n_0$, it can be written as

$$\chi(n_0)=\frac{1}{1-\xi_3}+\frac{3}{2}\frac{\xi_2}{(1-\xi_3)^2}\frac{\sigma\sigma_0}{\overline{\sigma}}+\frac{1}{2}\frac{\xi_2^2}{(1-\xi_3)^3}\left(\frac{\sigma\sigma_0}{\overline{\sigma}}\right)^2, \tag{1.8}$$

where $\xi_p=(\pi/6)n_0\sigma_0^p$, $p=2,3$.

The collision operator J depends on time through $f_H(\vec{v},t)$. The important feature is that operator J is the *linear* operator which acts on f only through its $\vec{v}$ dependence. Operator J is local in space and in time.

A freely evolving background fluid of IHS's, prepared initially in a HCS, is unstable against long-wavelength spatial fluctuations and eventually ends up in a state with inhomogeneities in the flow field $\vec{u}_0(\vec{r},t)$ (vortices) and in the density field $n_0(\vec{r},t)$ (clusters) [20,25–29]. In the present paper we are concerned only with the transport processes of granular swarms which precede the clustering of the background fluid.

II. INITIAL VALUE PROBLEM

In this section we introduce the initial value problem for the Boltzmann-Lorentz kinetic equation and develop its formal solution. We first write the kinetic equation (1.1) with the Boltzmann-Lorentz collision operator (1.7) in Fourier space. In this work we study the solution of this equation for a system of infinite volume, assuming that the one-particle distribution function $f(\vec{r},\vec{v},t)$ and its derivatives all vanish at large $\vec{r}$. This allows us to apply Fourier transform to Eqs. (1.1) and (1.7) to obtain

$$\frac{\partial}{\partial t}\Phi_{\vec{q}}(\vec{v},t)=\mathcal{L}_{\vec{q}}(t)\Phi_{\vec{q}}(\vec{v},t), \tag{2.1}$$

where $\Phi_{\vec{q}}(\vec{v},t)$ is the spatial Fourier transform of the one-particle distribution function

$$\Phi_{\vec{q}}(\vec{v},t)=\int \mathrm{d}\vec{r}\, e^{-i\vec{q}\cdot\vec{r}}f(\vec{r},\vec{v},t). \tag{2.2}$$

In Eq. (2.1) the operator $\mathcal{L}_{\vec{q}}(t)$ is

$$\mathcal{L}_{\vec{q}}(t)=\mathcal{M}(t)+\mathcal{P}_{\vec{q}}, \tag{2.3}$$

with

$$\mathcal{M}(t)=-\vec{a}\cdot\frac{\partial}{\partial\vec{v}}+J(t), \tag{2.4}$$

$$\mathcal{P}_{\vec{q}}=-i\vec{q}\cdot\vec{v}. \tag{2.5}$$

For the sake of compact notation, we introduce an abstract Hilbert space $\mathcal{H}$ to represent any function ψ of velocity $\vec{v}$. In other words, we consider $\psi(\vec{v})$ as the velocity-space representation of the vector $|\psi\rangle\in\mathcal{H}$, i.e., $\psi(\vec{v})=\langle\vec{v}|\psi\rangle$, which is standard notation borrowed from quantum mechanics [30]. In Hilbert space $\mathcal{H}$, the scalar (inner) product between two arbitrary vectors $|\varphi\rangle$ and $|\psi\rangle$ is defined as

$$\langle\varphi|\psi\rangle=\int \mathrm{d}\vec{v}\frac{1}{f_H(\vec{v},t)}\varphi^*(\vec{v})\psi(\vec{v}). \tag{2.6}$$

According to Eq. (2.6) we have

$$\hat{I}=\int \mathrm{d}\vec{v}\frac{1}{f_H(\vec{v},t)}|\vec{v}\rangle\langle\vec{v}| \tag{2.7}$$

and

$$\langle\vec{v}|\vec{v}'\rangle=f_H(\vec{v}',t)\,\delta(\vec{v}-\vec{v}'), \tag{2.8}$$

where $\hat{I}$ is the unit operator and δ is the delta function.

Likewise, a formal correspondence between operators $\mathcal{L}_{\vec{q}}(t)$, $\mathcal{M}(t)$, and $\mathcal{P}_{\vec{q}}$ and linear operators on Hilbert space $\mathcal{H}$ can be established:

$$\mathcal{M}(t)\to\hat{H}_0(t),\quad \mathcal{P}_{\vec{q}}\to\hat{H}'_{\vec{q}},$$
$$\mathcal{L}_{\vec{q}}(t)=\mathcal{M}(t)+\mathcal{P}_{\vec{q}}\to\hat{H}_{\vec{q}}(t)=\hat{H}_0(t)+\hat{H}'_{\vec{q}}. \tag{2.9}$$

For instance, the convective operator $\hat{H}'_{\vec{q}}=-i\vec{q}\cdot\hat{\vec{v}}$ acts on vector $|\psi\rangle\in\mathcal{H}$, according to

$$\hat{H}'_{\vec{q}}|\psi\rangle=\int \mathrm{d}\vec{v}\frac{1}{f_H(\vec{v},t)}\psi(\vec{v})(-i\vec{q}\cdot\hat{\vec{v}})|\vec{v}\rangle,\quad |\psi\rangle\in\mathcal{H}, \tag{2.10}$$

where $\hat{\vec{v}}$ is the vector operator defined by its components $\hat{v}_i$, $i=1,\ldots,d$, along d orthonormal axes, and $\hat{v}_i$, $i=1,\ldots,d$, are the usual multiplicative operators [30]. From

Eqs. (2.10) and (2.8) it follows that the velocity-space representation of the Hilbert space vector $\hat{H}'_{\vec{q}}|\psi\rangle$ is

$$\varphi(\vec{v})=\langle\vec{v}|\hat{H}'_{\vec{q}}|\psi\rangle=-i\vec{q}\cdot\vec{v}\,\psi(\vec{v}). \tag{2.11}$$

After these technical preliminaries, we proceed to consider an abstract initial value problem

$$\frac{\partial}{\partial t}|\Phi_{\vec{q}}(t)\rangle=\hat{H}_{\vec{q}}(t)|\Phi_{\vec{q}}(t)\rangle,\quad |\Phi_{\vec{q}}(t_0)\rangle=|\Phi^I_{\vec{q}}\rangle,\quad t\geqslant t_0. \tag{2.12}$$

Let the linear operator $\hat{U}_{\vec{q}}(t,s)$ map the solution $|\Phi_{\vec{q}}(s)\rangle$ at time s to the solution $|\Phi_{\vec{q}}(t)\rangle$ at time $t\geqslant s$:

$$|\Phi_{\vec{q}}(t)\rangle=\hat{U}_{\vec{q}}(t,s)|\Phi_{\vec{q}}(s)\rangle,\quad t\geqslant s\geqslant t_0. \tag{2.13}$$

The uniqueness of the solution of Eq. (2.12) implies that the family of evolution operators $\{\hat{U}_{\vec{q}}(t,s)|t\geqslant s\geqslant t_0\}$ satisfies

$$\hat{U}_{\vec{q}}(t,s)\hat{U}_{\vec{q}}(s,r)=\hat{U}_{\vec{q}}(t,r),\quad t\geqslant s\geqslant r\geqslant t_0,$$

$$\hat{U}_{\vec{q}}(t,t)=\hat{I},\quad t\geqslant t_0. \tag{2.14}$$

Furthermore, the evolution operator family $\{\hat{U}_{\vec{q}}(t,s),t\geqslant s\geqslant t_0\}$ governing Eq. (2.12) satisfies following differential equations [31]:

$$\frac{\partial}{\partial t}\hat{U}_{\vec{q}}(t,s)=\hat{H}_{\vec{q}}(t)\hat{U}_{\vec{q}}(t,s)=\hat{U}_{\vec{q}}(t,s)\hat{H}_{\vec{q}}(t),\quad \hat{U}_{\vec{q}}(s,s)=\hat{I},$$

$$\frac{\partial}{\partial s}\hat{U}_{\vec{q}}(t,s)=-\hat{U}_{\vec{q}}(t,s)\hat{H}_{\vec{q}}(s)=-\hat{H}_{\vec{q}}(s)\hat{U}_{\vec{q}}(t,s),$$

$$\hat{U}_{\vec{q}}(t,t)=\hat{I}. \tag{2.15}$$

Here, however, $\hat{H}_{\vec{q}}$ is *not* a Hermitian operator. Indeed, it is obvious that the convective operator $\hat{H}'_{\vec{q}}$ is anti-Hermitian,

$$\langle\varphi|\hat{H}'_{\vec{q}}|\psi\rangle=-\langle\psi|\hat{H}'_{\vec{q}}|\varphi\rangle^*,\quad |\psi\rangle,|\varphi\rangle\in\mathcal{H}, \tag{2.16}$$

where the asterisk represents complex conjugation.

Time-dependent perturbation method

In this section, we formulate the transport problem of swarm particles starting from the kinetic equation and use the time-dependent perturbation theory to determine the evolution operator $\hat{U}_{\vec{q}}(t,t_0)$,

$$|\Phi_{\vec{q}}(t)\rangle=\hat{U}_{\vec{q}}(t,t_0)|\Phi^I_{\vec{q}}\rangle,\quad t\geqslant t_0, \tag{2.17}$$

which describes the time evolution of the swarm particles in accordance with the kinetic equation (2.12). Since $\hat{H}_{\vec{q}}$ is not a Hermitian operator, it is obvious that the evolution operator $\hat{U}_{\vec{q}}(t,t_0)$ is not unitary.

Let $\hat{U}_0(t,t_0)$ be the evolution operator corresponding to the unperturbed operator $\hat{H}_0(t)$; consequently, the operator $\hat{U}_0(t,t_0)$ satisfies the differential equation

$$\frac{\partial}{\partial t}\hat{U}_0(t,t_0)=\hat{H}_0(t)\hat{U}_0(t,t_0)=\hat{U}_0(t,t_0)\hat{H}_0(t),$$

$$\hat{U}_0(t_0,t_0)=\hat{I}. \tag{2.18}$$

The evolution operator $\hat{U}_{\vec{q}}(t,t_0)$ can be expressed in terms of the operator $\hat{S}_{\vec{q}}(t,t_0)$ defined as

$$\hat{U}_{\vec{q}}(t,t_0)=\hat{U}_0(t,t_0)\hat{S}_{\vec{q}}(t,t_0). \tag{2.19}$$

From Eqs. (2.15), (2.18), and (2.19) it follows that the time dependence of $\hat{S}_{\vec{q}}(t,t_0)$ is given by

$$\frac{\partial}{\partial t}\hat{S}_{\vec{q}}(t,t_0)=\hat{P}_{\vec{q}}(t)\hat{S}_{\vec{q}}(t,t_0),\quad \hat{S}_{\vec{q}}(t_0,t_0)=\hat{I}. \tag{2.20}$$

Here the operator $\hat{P}_{\vec{q}}(t)$ is a perturbation operator $\hat{H}'_{\vec{q}}$ in the ''interaction picture'':

$$\hat{P}_{\vec{q}}(t)=\hat{U}_0^{-1}(t,t_0)\hat{H}'_{\vec{q}}\hat{U}_0(t,t_0). \tag{2.21}$$

Equation (2.20) is equivalent to the integral equation

$$\hat{S}_{\vec{q}}(t,t_0)=\hat{I}+\int_{t_0}^{t}\mathrm{d}t_1\,\hat{P}_{\vec{q}}(t_1)\hat{S}_{\vec{q}}(t_1,t_0), \tag{2.22}$$

which can be solved by iteration (in powers of $\hat{P}_{\vec{q}}$), yielding

$$\hat{S}_{\vec{q}}(t,t_0)=\hat{I}+\sum_{p=1}^{\infty}\hat{S}^{(p)}_{q}(t,t_0), \tag{2.23}$$

where

$$\hat{S}^{(p)}_{q}(t,t_0)=\int_{t_0}^{t}\mathrm{d}t_1\,\hat{P}_{\vec{q}}(t_1)\int_{t_0}^{t_1}\mathrm{d}t_2\,\hat{P}_{\vec{q}}(t_2)\cdots\int_{t_0}^{t_{p-1}}\mathrm{d}t_p\,\hat{P}_{\vec{q}}(t_p),$$

$$p\geqslant 1. \tag{2.24}$$

From this result, with the aid of definitions (2.19) and 2.21, we get the following expansion for $\hat{U}_{\vec{q}}(t,t_0)$:

$$\hat{U}_{\vec{q}}(t,t_0)=\hat{U}^{(0)}(t,t_0)+\sum_{p=1}^{\infty}\hat{U}^{(p)}_{q}(t,t_0),$$

$$\hat{U}^{(0)}(t,t_0)=\hat{U}_0(t,t_0), \tag{2.25}$$

$$\hat{U}^{(p)}_{q}(t,t_0)=\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\cdots\int_{t_0}^{t_{p-1}}\mathrm{d}t_p\hat{U}^{(0)}(t,t_1)$$
$$\times\hat{H}'_{\vec{q}}\hat{U}^{(0)}(t_1,t_2)\hat{H}'_{\vec{q}}\cdots\hat{U}^{(0)}(t_{p-1},t_p)$$
$$\times\hat{H}'_{\vec{q}}\hat{U}^{(0)}(t_p,t_0),\quad p\geqslant 1. \tag{2.26}$$

Before finishing our formal manipulations, we still have to express the distribution function $|\Phi_{\vec{q}}(t)\rangle$ in terms of the convective operator $\hat{H}'_{\vec{q}}$. Using Eqs. (2.17) and (2.25), one obtains the expansion of $|\Phi_{\vec{q}}(t)\rangle$,

$$|\Phi_{\vec{q}}(t)\rangle = \sum_{p=0}^{\infty} |\Phi_{\vec{q}}^{(p)}(t)\rangle, \quad t \geq t_0, \tag{2.27}$$

where

$$|\Phi_{\vec{q}}^{(p)}(t)\rangle = \hat{U}_{\vec{q}}^{(p)}(t,t_0)|\Phi_{\vec{q}}^{I}\rangle, \quad t \geq t_0. \tag{2.28}$$

Inserting an explicit form of the convective operator $\hat{H}'_{\vec{q}} = -i\vec{q}\cdot\hat{\vec{v}}$ into Eq. (2.26) and by using Eqs. (2.27) and (2.28) we find that the vector $|\Phi_{\vec{q}}(t)\rangle$ can be expressed as

$$|\Phi_{\vec{q}}(t)\rangle = \sum_{p=0}^{\infty} (-i\vec{q})^p \odot_p \|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle, \qquad t \geq t_0, \tag{2.29}$$

where

$$\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle = \hat{U}^{(0)}(t,t_0)|\Phi_{\vec{q}}^{I}\rangle, \tag{2.30}$$

$$\begin{aligned}\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle = &\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{p-1}} dt_p \hat{U}^{(0)}(t,t_1)\hat{\vec{v}} \\ &\times \hat{U}^{(0)}(t_1,t_2)\hat{\vec{v}} \cdots \hat{U}^{(0)}(t_{p-1},t_p)\hat{\vec{v}} \\ &\times \hat{U}^{(0)}(t_p,t_0)|\Phi_{\vec{q}}^{I}\rangle, \quad p \geq 1. \end{aligned} \tag{2.31}$$

The quantities $(-i\vec{q})^p$ and $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ are tensors of rank p. The notation $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ signifies that such an object is a tensor of rank p whose components are not the usual $\mathbb{C}$ numbers, but rather are vectors in the Hilbert space $\mathcal{H}$. The symbol $\odot_p$ denotes the appropriate p-fold scalar product, i.e., $\hat{A}^{(p)} \odot_p \hat{B}^{(p)} = \Sigma_{\alpha_1 \cdots \alpha_p} A^{(p)}_{\alpha_1 \cdots \alpha_p} B^{(p)}_{\alpha_1 \cdots \alpha_p}$, for any tensors $\hat{A}^{(p)}$ and $\hat{B}^{(p)}$ of rank p. Cartesian components $\alpha_1, \ldots, \alpha_p = 1,2,3$, $p \geq 1$, of the tensor $(-i\vec{q})^p$ are $\mathbb{C}$ numbers given by

$$[(-i\vec{q})^p]_{\alpha_1 \cdots \alpha_p} = (-i)^p q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_p}, \tag{2.32}$$

while the components of tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ are vectors of Hilbert space $\mathcal{H}$ given by

$$\begin{aligned}&[\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle]_{\alpha_1 \cdots \alpha_p} \\ &= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{p-1}} dt_p \, \hat{U}^{(0)}(t,t_1)\hat{v}_{\alpha_1} \\ &\times \hat{U}^{(0)}(t_1,t_2)\hat{v}_{\alpha_2} \cdots \hat{U}^{(0)}(t_{p-1},t_p)\hat{v}_{\alpha_p} \\ &\times \hat{U}^{(0)}(t_p,t_0)|\Phi_{\vec{q}}^{I}\rangle \in \mathcal{H}. \end{aligned} \tag{2.33}$$

Equation (2.29) represents a formal solution of initial value problem (2.12). In a subsequent section we will use this formal solution to develop the hierarchy of kinetic equations and to introduce transport coefficients valid at *all* times, including the initial nonhydrodynamic stage of the evolution of swarm particles.

III. SHORT-TIME DEVELOPMENT OF GRANULAR SWARMS: TRANSPORT COEFFICIENTS

In this section we derive a hierarchy of kinetic equations for granular swarms and derive a general expression for transport coefficients. Changes needed to accommodate the possibility of nonconservative transport are considered in the subsection below.

Taking the time derivative of Eqs. (2.30) and (2.31), with the help of Eq. (2.18), we find that the tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ obey the following hierarchy of coupled differential equations:

$$\frac{\partial}{\partial t}\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle = \hat{H}_0(t)\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle, \quad \|\varkappa_{\vec{q}}^{(0)}(t_0)\rangle\rangle = |\Phi_{\vec{q}}^{I}\rangle, \quad t \geq t_0, \tag{3.1}$$

$$\frac{\partial}{\partial t}\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle = \hat{H}_0(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle + \hat{\vec{v}}\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle, \quad \|\varkappa_{\vec{q}}^{(p)}(t_0)\rangle\rangle = 0, \quad t \geq t_0, \quad p \geq 1. \tag{3.2}$$

The action of the vector operator $\hat{\vec{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ on the tensor $\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle$, $p \geq 1$, raises its rank by 1 and is defined as

$$[\hat{\vec{v}}\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle]_{\alpha_1} = \hat{v}_{\alpha_1}\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle,$$

$$[\hat{\vec{v}}\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle]_{\alpha_1 \cdots \alpha_p} = \hat{v}_{\alpha_1}[\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle]_{\alpha_2 \cdots \alpha_p}, \quad p \geq 2, \tag{3.3}$$

where, as before, $\alpha_1, \ldots, \alpha_p = 1,2,3$, $p \geq 1$.

It is convenient to introduce an infinite set of tensors

$$\hat{N}^{(p)}(\vec{q},t) = \langle f_H(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle, \quad p \geq 0. \tag{3.4}$$

Since $n_{\vec{q}}(t) = \langle f_H(t)|\Phi_{\vec{q}}(t)\rangle$, we get

$$n_{\vec{q}}(t) = \sum_{p=0}^{\infty} (-i\vec{q})^p \odot_p \hat{N}^{(p)}(\vec{q},t), \quad t \geq t_0. \tag{3.5}$$

Finally, to set up our transport theory we define transport coefficients by

$$\frac{\partial}{\partial t}\hat{N}^{(p)}(\vec{q},t) = \sum_{r=0}^{p} \hat{\omega}_{\vec{q}}^{(r)}(t) \otimes \hat{N}^{(p-r)}(\vec{q},t), \quad p \geq 0, \tag{3.6}$$

where $\hat{\omega}_{\vec{q}}^{(r)}(t)$ denote tensor transport coefficients of rank r, and the symbol $\otimes$ denotes the standard symmetrized outer tensor product defined as

$$[\hat{\omega}_{\vec{q}}^{(r)}(t)\otimes\hat{N}^{(p-r)}(\vec{q},t)]_{i_1\cdots i_p} = \frac{1}{p!}\sum_{(j_1,\ldots,j_p)\in P(i_1,\ldots,i_p)}[\hat{\omega}_{\vec{q}}^{(r)}(t)]_{j_1\cdots j_r}N^{(p-r)}_{j_{r+1}\cdots j_p}(\vec{q},t), \tag{3.7}$$

and the summation extends over all of indices $(j_1,\ldots,j_p)$ that are permutations $P(i_1,\ldots,i_p)$ of the indices on the left-hand side.

From definitions (3.6) and Eq. (3.5) it follows that

$$\frac{\partial}{\partial t}n_{\vec{q}}(t)-\sum_{p=0}^{\infty}(-i\vec{q})^p\odot_p\hat{\omega}_{\vec{q}}^{(p)}(t)n_{\vec{q}}(t)=0. \tag{3.8}$$

This last equation is often called the generalized diffusion equation. It describes the temporal evolution of the $n_{\vec{q}}(t)$ in terms of an infinite set $\{\hat{\omega}_{\vec{q}}^{(p)}|p\geqslant 0\}$ of transport coefficients. It should be stressed that it is valid for *all* times and for *arbitrary* initial conditions. Since the $\vec{q}$ dependence of transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}$, $p\geqslant 0$, has its origin in the $\vec{q}$ dependence of the initial vector $|\Phi_{\vec{q}}^I\rangle$ [see Eqs. (2.30) and (2.31)], we conclude that they can be related to the corresponding Fourier component of the initial distribution. In other words, transport coefficients are time-dependent functionals of the initial conditions.

The derivative with respect to time occurring in Eq. (3.6) can be eliminated with the help of Eqs. (3.1) and (3.2). Combining Eqs. (3.4),(3.6) and (3.1),(3.2) we obtain

$$\hat{\omega}_{\vec{q}}^{(0)}(t)=\frac{1}{\langle\langle f_H(t)\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle}\langle f_H(t)|\hat{H}_0(t)\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle, \tag{3.9}$$

$$\hat{\omega}_{\vec{q}}^{(p)}(t)=\frac{1}{\langle\langle f_H(t)\|\varkappa_{\vec{q}}^{(0)}(t)\rangle\rangle}\Big[\langle f_H(t)|\hat{\vec{v}}\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle + \langle f_H(t)|\hat{H}_0(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle-\sum_{r=0}^{p-1}\hat{\omega}_{\vec{q}}^{(r)}(t) \otimes\langle f_H(t)\|\varkappa_{\vec{q}}^{(p-r)}(t)\rangle\rangle\Big], \quad p\geqslant 1. \tag{3.10}$$

For a given initial condition $|\Phi_{\vec{q}}^I\rangle$, kinetic equations (3.1) and (3.2) and expressions (3.9) and (3.10) determine both the time-dependent tensors $\|\varkappa_{\vec{q}}^{(r)}(t)\rangle\rangle$ and the transport coefficients $\hat{\omega}_{\vec{q}}^{(r)}$ for all $r\leqslant p$ and $p\geqslant 0$.

For the purpose of analysis in subsequent sections it is useful to write down the first three kinetic equations of hierarchy (3.1) and (3.2) in the velocity-space representation. Using the correspondences (2.9) and $\langle\langle\vec{v}\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle \equiv\hat{f}_{\vec{q}}^{(p)}(\vec{v},t)$, $p\geqslant 0$, we immediately get, for $p=0$,

$$\frac{\partial}{\partial t}f_{\vec{q}}^{(0)}(\vec{v},t)+\vec{a}\cdot\frac{\partial}{\partial\vec{v}}f_{\vec{q}}^{(0)}(\vec{v},t)=J[f_{\vec{q}}^{(0)}](\vec{v},t),$$

$$f_{\vec{q}}^{(0)}(\vec{v},t_0)=\langle\vec{v}|\Phi_{\vec{q}}^I\rangle\equiv f_{\vec{q}}^I(\vec{v}). \tag{3.11}$$

Swarm particles may freely exchange momentum and energy with the surrounding granular fluid and, therefore, these are not invariants of the collision operator J [Eq. (1.7)]. There is only one collision invariant, corresponding to the number of swarm particles. This is analogous to the situation in the Brownian motion theory of a granular gas [13,14] or to the situation in the self-diffusion in freely evolving granular gases [15,32]. The presence of this invariant implies that

$$\langle f_H(t)|\hat{H}_0(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle=\int d\vec{v}\,J[\hat{f}_{\vec{q}}^{(p)}](\vec{v},t)=0, \quad p\geqslant 0. \tag{3.12}$$

From Eqs. (3.9) and (3.12), we get

$$\hat{\omega}_{\vec{q}}^{(0)}(t)=0, \quad t\geqslant t_0. \tag{3.13}$$

Further, for $p=1$ we get

$$\frac{\partial}{\partial t}\vec{f}_{\vec{q}}^{(1)}(\vec{v},t)+\vec{a}\cdot\frac{\partial}{\partial\vec{v}}\vec{f}_{\vec{q}}^{(1)}(\vec{v},t)=J[\vec{f}_{\vec{q}}^{(1)}](\vec{v},t)+\vec{v}f_{\vec{q}}^{(0)}(\vec{v},t),$$

$$\vec{f}_{\vec{q}}^{(1)}(\vec{v},t_0)=0. \tag{3.14}$$

Using Eq. (3.10) and $\hat{\omega}_{\vec{q}}^{(0)}(t)=0$, we can establish the following expression for the drift velocity:

$$\vec{W}_{\vec{q}}(t)\equiv\hat{\omega}_{\vec{q}}^{(1)}(t)=\frac{1}{\int d\vec{v}\,f_{\vec{q}}^{(0)}(\vec{v},t)}\int d\vec{v}\,\vec{v}f_{\vec{q}}^{(0)}(\vec{v},t). \tag{3.15}$$

Finally, we also rewrite Eqs. (3.2) and (3.10) for $p=2$:

$$\frac{\partial}{\partial t}\hat{f}_{\vec{q}}^{(2)}(\vec{v},t)+\vec{a}\cdot\frac{\partial}{\partial\vec{v}}\hat{f}_{\vec{q}}^{(2)}(\vec{v},t)=J[\hat{f}_{\vec{q}}^{(2)}](\vec{v},t)+\vec{v}\otimes\vec{f}_{\vec{q}}^{(1)}(\vec{v},t),$$

$$\hat{f}_{\vec{q}}^{(2)}(\vec{v},t_0)=0, \tag{3.16}$$

$$\hat{D}_{\vec{q}}(t)\equiv\hat{\omega}_{\vec{q}}^{(2)}(t)=\frac{1}{\int d\vec{v}f_{\vec{q}}^{(0)}(\vec{v},t)}\Big[\int d\vec{v}\,\vec{v}\otimes\vec{f}_{\vec{q}}^{(1)}(\vec{v},t) -\vec{W}_{\vec{q}}(t)\otimes\int d\vec{v}\,\vec{f}_{\vec{q}}^{(1)}(\vec{v},t)\Big], \tag{3.17}$$

where $\hat{D}_{\vec{q}}(t)$ denotes the diffusion tensor. It can be seen that the diffusion tensor $\hat{D}_{\vec{q}}(t)$ is anisotropic, as expected [33]. It is straightforward to obtain expressions for the third- and higher-order transport coefficients.

Equations (3.11)–(3.17) give the general expressions for the first three transport coefficients. Using further simplify-

ing assumptions regarding the initial conditions which are discussed in Sec. IV, it is possible to reduce these general results to the well-known expressions found in Chapman-Enskog theory. The details of this reduction for a particular case of self-diffusion are given in Appendix C.

Bulk and flux transport coefficients

In the development of transport theory in previous sections it was assumed that the number and properties of swarm particles were not changed in collisions with particles of background granular fluid. In this section and in the remaining part of this paper, we remove this restriction and allow nonconservative processes to take place. The term ''nonconservative processes'' is to be interpreted in a wide sense; it includes all processes which do not conserve the number of swarm particles and/or their other physical properties. Non-particle-conserving collisions are usual for real granular systems. Examples of such processes are the permanent deformation or cracking of swarm particles. Particles created in such collisions develop a *new* granular swarm, with different transport properties. In this paper we have considered only the equation for a one-component granular swarm. This should be sufficient for a general understanding. In an actual ''reacting'' system several granular species are usually involved. We characterize all these possible ''nonconservative'' processes by the respective collision operator J^R. The kinetic equation (1.1) for the one-particle distribution function of granular swarms in the presence of nonconservative processes is corrected with a ''reactive'' term J^R. We suppose that the collision operator J^R is *linear* and depends functionally on the background fluid distribution $f_H(\vec{v},t)$. It is therefore a time-dependent operator. For the present purposes, however, where the aim is a formal structure of the transport theory of granular swarms, further details of this operator are not needed.

Now notice that the operator $\hat{H}_0(t)$ contains two terms: the particle conserving term $\mathcal{M}(t)$ [Eq. (2.4)] and the ''reactive'' collision term $J^R(t)$ which, under the correspondence (2.9), become operators $\hat{H}_0^{PC}(t)$ and $\hat{H}_0^R(t)$, respectively. Hence the non-particle-conserving terms will survive in Eqs. (3.9) and (3.10):

$$\langle f_H(t)|\hat{H}_0(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle=\langle f_H(t)|\hat{H}_0^R(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle\neq 0,\quad p\geqslant 0. \tag{3.18}$$

Inserting Eq. (3.18) into Eqs. (3.9) and (3.10), we see that the presence of nonconservative processes alters the transport coefficients in two ways. First, there are explicitly $\hat{H}_0^R(t)$-dependent terms in Eqs. (3.9) and (3.10), and second, there is an implicit change in tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ since the kinetic equations (3.1) and (3.2) are now different.

Note that when nonconservative processes are present the calculation of a transport coefficient of rank p requires solutions of the kinetic equations (3.1) and (3.2) up to order p. In the absence of nonconservative processes, solutions of kinetic equations to the order $p-1$ suffice for the same purpose.

After these generalizations, we want to establish the connection between the swarm particle flux $\vec{\Gamma}(\vec{r},t)=\int d\vec{v}\,\vec{v}f(\vec{r},\vec{v},t)$ and transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}(t)$, $p\geqslant 0$. From the definition (2.2) we have that the Fourier transform of $\vec{\Gamma}(\vec{r},t)$ is given by

$$\vec{\Gamma}_{\vec{q}}(t)=\langle f_H(t)|\hat{\vec{v}}|\Phi_{\vec{q}}(t)\rangle. \tag{3.19}$$

Inserting Eq. (2.29) into Eq. (3.19), we arrive at

$$\vec{\Gamma}_{\vec{q}}(t)=\sum_{p=0}^{\infty}(-i\vec{q})^p\odot_p\langle f_H(t)|\hat{\vec{v}}\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle. \tag{3.20}$$

After some algebra, we obtain

$$\vec{\Gamma}_{\vec{q}}(t)=\sum_{p=0}^{\infty}(-i\vec{q})^p\odot_p\hat{\Omega}_{\vec{q}}^{(p+1)}(t)n_{\vec{q}}(t), \tag{3.21}$$

where

$$\hat{\Omega}_{\vec{q}}^{(p)}(t)=\hat{\omega}_{\vec{q}}^{(p)}(t)-\hat{R}_{\vec{q}}^{(p)}(t),\quad p\geqslant 1, \tag{3.22}$$

$$\hat{R}_{\vec{q}}^{(p)}(t)=-\frac{1}{n_{\vec{q}}(t)}\langle f_H(t)|[\hat{\omega}_{\vec{q}}^{(0)}(t)\hat{I}-\hat{H}_0^R(t)]\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle,\quad p\geqslant 1. \tag{3.23}$$

For completeness we put $\hat{\Omega}_{\vec{q}}^{(0)}(t)\equiv\hat{\omega}_{\vec{q}}^{(0)}(t)$. The details of this calculation are given in Appendix A.

In classical near-equilibrium theories the flux $\vec{\Gamma}(\vec{r},t)$ is usually expressed in the form $\vec{\Gamma}(\vec{r},t)=n(\vec{r},t)\mathcal{K}\vec{E}^{\rm ext}-\mathcal{D}[\partial n(\vec{r},t)/\partial\vec{r}]$ where $\mathcal{K}$ and $\mathcal{D}$ are classical mobility and diffusion coefficients, respectively, and $\vec{E}^{\rm ext}$ is the external field strength. By analogy with this classical prescription, we could define ''flux'' transport coefficients $\hat{\Omega}_{\vec{q}}^{(p)}(t)$, $p\geqslant 0$, in accordance with Eqs. (3.21)–(3.23). Reaction-corrected transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}(t)$ are often called ''bulk'' transport coefficients. In the absence of reactive processes $\hat{R}_{\vec{q}}^{(p)}(t)$ vanishes for any $p\geqslant 1$, and the ''bulk'' and ''flux'' transport coefficients become *identical*. Thus, in the latter case, with $\hat{\omega}_{\vec{q}}^{(0)}\equiv 0$, the drift velocity $\vec{W}_{\vec{q}}(t)$ is determined by Eqs. (3.11) and (3.15) and the diffusion tensor $\hat{D}_{\vec{q}}(t)$ by Eqs. (3.14) and (3.17).

IV. SPACE-TIME EVOLUTION OF GRANULAR SWARMS

We base our transport theory on the revised Enskog kinetic theory (RET) for the hard sphere fluid [34,35]. The RET is exact for times much shorter than the mean free time between collisions. On this time scale it describes the one-particle distribution function $f(\vec{r},\vec{v},t)$ for arbitrary spatial variations. Outside this time regime, the Enskog equation is not exact, because it does not take into account the velocity correlations built up by sequences of correlated binary collisions. However, RET takes into account static short-range

correlations caused by excluded volume effects. The kinetic equations (3.1) and (3.2) and corresponding transport coefficients (3.9) and (3.10) obtained from these equations are correct within the framework of assumptions in which Enskog theory is considered to be applicable. This remark is essential; the validity of the Enskog description is an underlying assumption of this work.

This section is devoted to the analysis of the long-time behavior of transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}(t)$, $p \geqslant 0$. It is a difficult problem which cannot be solved in full generality at the present time. Recently, we have analyzed the foundations of the transport theory of charged particle swarms in *rarefied* neutral gases in the presence of *static* and *uniform* external electric field [12]. Except for minor technical details, our strategy was the same as the one we followed in Secs. II and III to establish the generalized diffusion equation [Eq. (3.8)] from the Enskog equation. There is, however, a very important difference between the transport theory of charged particle and granular swarms. The transport theory of charged particle swarms is based on the Boltzmann equation, with the equilibrium state of a neutral gas as the reference state. As a consequence, the corresponding unperturbed collision operator $\hat{H}_0$ [Eq. (2.9)] is time independent. For granular swarms, the energy dissipation of the surrounding granular fluid plays a very crucial role, and it is responsible for the time dependence of the unperturbed collision operator $\hat{H}_0(t)$. To extract any information about the long-time behavior of either transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}(t)$, $p \geqslant 0$, or one-particle distribution function $f(\vec{r},\vec{v},t)$, we must analyze the asymptotic behavior of the tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$, $p \geqslant 0$. This is, of course, a difficult problem, because the tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$ involve the evolution operator $\hat{U}_0(t,t_0)$, which is very complicated and cannot be evaluated in closed form.

In the case of charged particle swarms we have performed an analysis of the long-time behavior of tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$, $p \geqslant 0$. The remarkable theorem has been proved that a sufficient condition for the existence of a hydrodynamic regime is the existence of an isolated eigenvalue $\hat{\omega}_*^{(0)}$ of the operator $\hat{H}_0$ which is separated from the rest of the spectrum by a gap along the real axis [12,36]. Such an assumption implies the separation of the relaxation time scale $\tau_0 \propto (d_0)^{-1}$ (d_0 is the length of gap in the spectrum) and the hydrodynamic time scale $\tau_h \propto [q(k_B T)^{1/2}]^{-1}$ [37] (τ_h is the time a swarm particle needs to travel the length of macroscopic gradients; $k_B T$ is the mean random energy of a swarm particle). This means that in the long-time limit ($t \gg \tau_0$) all $\hat{\omega}_{\vec{q}}^{(p)}(t)$ become time *and* $\vec{q}$ independent in the *same* characteristic time and achieve their hydrodynamic values

$$\hat{\omega}_{\vec{q}}^{(p)}(t) \simeq \hat{\omega}_*^{(p)}, \quad t \gg \tau_0, \quad p \geqslant 0. \tag{4.1}$$

The transport coefficients $\hat{\omega}_*^{(p)}$ as well as the one-particle distribution function $f(\vec{r},\vec{v},t)$ can be evaluated in non-Hermitian perturbation theory, as demonstrated in Ref. [12]. These results show that hydrodynamic behavior is always linked to the forgetting of the initial conditions through the relaxation.

As discussed above, the theory of granular swarms is more complicated than the corresponding theory of charged particle swarms due to the presence of an additional time scale characterizing cooling processes. For this reason it is difficult to analyze the case of arbitrary initial conditions, and we limit ourselves to the special case where the initial distribution is given by

$$|\Phi_{\vec{q}}^{I}\rangle = |f_0\rangle n_{\vec{q}}(t_0). \tag{4.2}$$

Inserting this initial value into Eqs. (2.30) and (2.31) we get

$$\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle = \|\varkappa^{(p)}(t)\rangle\rangle n_{\vec{q}}(t_0), \quad p \geqslant 0, \quad t \geqslant t_0, \tag{4.3}$$

where the tensors $\|\varkappa^{(p)}(t)\rangle\rangle$, $p \geqslant 0$, are $\vec{q}$ independent. From Eqs. (3.9) and (3.10) and Eq. (4.3) we conclude that all transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}(t)$, $p \geqslant 0$, also become $\vec{q}$ independent, i.e.,

$$\hat{\omega}_{\vec{q}}^{(p)}(t) \equiv \hat{\omega}^{(p)}(t), \quad p \geqslant 0, \quad t \geqslant t_0. \tag{4.4}$$

Fourier inversion $\mathcal{F}^{-1}$ of Eq. (4.4) gives

$$\mathcal{F}^{-1}[\hat{\omega}_{\vec{q}}^{(p)}(t)] \equiv \hat{\omega}^{(p)}(\vec{r},t) = \hat{\omega}^{(p)}(t)\,\delta(\vec{r}), \quad p \geqslant 0, \quad t \geqslant t_0, \tag{4.5}$$

and we find that, for a class of initial conditions (4.2), the transport coefficients are $\vec{q}$ independent for all times.

Although the previous formulation in Fourier space is very convenient for mathematical analysis, it is not very useful for discussing the physical meaning of the results. For this latter purpose, let us go back to configuration space. Applying the well-known convolution theorem for Fourier transforms on the GDE [Eq. (3.8)], we get immediately

$$\frac{\partial}{\partial t} n(\vec{r},t) - \sum_{p=0}^{\infty} \left(-\frac{\partial}{\partial \vec{r}} \right)^p \odot_p \int d\vec{r}_1\, \hat{\omega}^{(p)}(\vec{r}-\vec{r}_1,t)\, n(\vec{r}_1,t) = 0. \tag{4.6}$$

We see that the left-hand side of Eq. (4.6) involves a nonlocal dependence on the number density $n(\vec{r},t)$. The transport coefficients $\hat{\omega}^{(p)}(\vec{r},t)$, $p \geqslant 0$, connect the time evolution of $n(\vec{r},t)$ in an arbitrary point $\vec{r}$ to its value in other points. This is in accordance with the fact that granular materials are intrinsically nonlocal [3].

Greater insight into the physical interpretation of the transport coefficients $\hat{\omega}^{(p)}(\vec{r},t)$, $p \geqslant 0$, can be obtained by taking spatial moments of the number density $n(\vec{r},t)$. Let $\psi(\vec{r})$ be any function of $\vec{r}$ and let us define

$$\langle \psi(\vec{r}) \rangle_n \equiv \frac{1}{N} \int d\vec{r}\, \psi(\vec{r}) n(\vec{r},t), \quad N \equiv N(t) = \int d\vec{r}\, n(\vec{r},t). \tag{4.7}$$

Assuming that $n(\vec{r},t)$, together with its derivatives, vanish at $|\vec{r}|\to\infty$, we obtain from Eq. (4.6) the following equation for the time development of the averages $\langle\psi(\vec{r})\rangle_n$:

$$\frac{\partial}{\partial t}\langle\psi(\vec{r})\rangle_n+\frac{1}{N(t)}\frac{\mathrm{d}N(t)}{\mathrm{d}t}\langle\psi(\vec{r})\rangle_n - \frac{1}{N(t)}\sum_{p=0}^{\infty}\int \mathrm{d}\vec{r}_1\,\hat{\omega}^{(p)}(\vec{r}_1,t) \odot_p \int \mathrm{d}\vec{r}\,n(\vec{r}-\vec{r}_1,t)\left(\frac{\partial}{\partial\vec{r}}\right)^p\psi(\vec{r})=0. \quad (4.8)$$

The details of the derivation of Eq. (4.8) are given in Appendix B.

If $\psi(\vec{r})$ is a polynomial of order s in $\vec{r}$, then in Eq. (4.8) only the transport coefficients of order $p\leqslant s$ occur. Taking successive moments $(\psi(\vec{r})=1,\vec{r},\vec{r}\otimes\vec{r},\ldots)$, after some algebra we have

$$\frac{1}{N(t)}\frac{\mathrm{d}N(t)}{\mathrm{d}t}=\int \mathrm{d}\vec{r}\,\hat{\omega}^{(0)}(\vec{r},t), \quad (4.9)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{r}\rangle_n=\int \mathrm{d}\vec{r}\,\hat{\omega}^{(1)}(\vec{r},t)-\int \mathrm{d}\vec{r}\,\hat{\omega}^{(0)}(\vec{r},t)\vec{r}, \quad (4.10)$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\langle\vec{r}\otimes\vec{r}\rangle_n-\langle\vec{r}\rangle_n\otimes\langle\vec{r}\rangle_n) = \int \mathrm{d}\vec{r}\,\hat{\omega}^{(2)}(\vec{r},t)-\frac{1}{2}\int \mathrm{d}\vec{r}[\hat{\omega}^{(1)}(\vec{r},t)\otimes\vec{r}+\vec{r}\otimes\hat{\omega}^{(1)}(\vec{r},t)] + \frac{1}{2}\int \mathrm{d}\vec{r}\,\hat{\omega}^{(0)}(\vec{r},t)\vec{r}\otimes\vec{r}, \quad (4.11)$$

where the operation $\otimes$ has its usual meaning as defined in Eq. (3.7).

In general, the quantities on the left-hand sides (LHS's) of Eqs. (4.9)–(4.11) are time dependent. As such they can be used in analysis of computer experiments. Their form is independent of whether ''reactive'' processes (Sec. III) are present or not. Physically, the time derivative on the LHS of Eq. (4.10) can be interpreted as the time-dependent velocity of the center of mass of the granular swarm. The tensor quantity on the LHS of Eq. (4.11) represents the time-dependent rate of change of the mean-square width of the granular swarm or equivalently the time-dependent rate of the spreading of the granular swarm.

Thus far our discussion of the real-space formulation was completely general. Let us consider again the initial state $f(\vec{r},\vec{v},t_0)=f_0(\vec{v})n(\vec{r},t_0)$ which separates the velocity and space-time dependences [Eq. (4.2)]. According to Eq. (4.5), from Eq. (4.6) and Eqs. (4.9)–(4.11) we get immediately generalized diffusion equation

$$\frac{\partial}{\partial t}n(\vec{r},t)-\sum_{p=0}^{\infty}\hat{\omega}^{(p)}(t)\odot_p\left(-\frac{\partial}{\partial\vec{r}}\right)^p n(\vec{r},t)=0 \quad (4.12)$$

and the following expressions for the transport coefficients:

$$R(t)\equiv-\hat{\omega}^{(0)}(t)=-\frac{1}{N(t)}\frac{\mathrm{d}N(t)}{\mathrm{d}t}, \quad (4.13)$$

$$\vec{W}(t)\equiv\hat{\omega}^{(1)}(t)=\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{r}\rangle_n, \quad (4.14)$$

$$\hat{D}(t)\equiv\hat{\omega}^{(2)}(t)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\langle\vec{r}\otimes\vec{r}\rangle_n-\langle\vec{r}\rangle_n\otimes\langle\vec{r}\rangle_n) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle(\vec{r}-\langle\vec{r}\rangle_n)\otimes(\vec{r}-\langle\vec{r}\rangle_n)\rangle_n. \quad (4.15)$$

These equations are exact for all times if the initial condition has the assumed form.

Recently, Brey and co-workers [14] have proposed a theory of Brownian motion in a granular gas in the absence of an external field based on the Fokker-Planck equation. For the initial condition defined by Eq. (4.2) these authors have shown that the number density of heavy particle granular swarms in a granular gas of much lighter particles obeys Eq. (4.12) truncated at $p=2$. Our theory yields Eq. (4.12) without restrictions on the ratio of gas to swarm particle masses. Furthermore, the results presented here are not restricted to small gradients in the density of swarm particles.

To establish further the contacts of our formalism with previously studied particular cases we consider the self-diffusion in freely evolving granular gas. In Appendix A we show that our general results yield the well-known kinetic equations and the transport coefficients obtained in the standard Chapman-Enskog treatment [15].

Turning to the flux transport coefficients $\hat{\Omega}_q^{(p)}$, $p\geqslant 1$, from Eqs. (3.22), (3.23), and (4.3) we obtain that they also become $\vec{q}$ independent when the velocity and space-time dependences of the initial conditions separate:

$$\hat{\Omega}_q^{(p)}(t)\equiv\hat{\Omega}^{(p)}(t)=\hat{\omega}^{(p)}(t)-\hat{R}^{(p)}(t), \quad p\geqslant 1,\ t\geqslant t_0, \quad (4.16)$$

where

$$\hat{R}_q^{(p)}(t)\equiv\hat{R}^{(p)}(t)=-\langle f_H(t)|[\hat{\omega}^{(0)}(t)\hat{I}-\hat{H}_0^R(t)]|\varkappa^{(p)}(t)\rangle\rangle, \quad p\geqslant 1. \quad (4.17)$$

Finally, combining Eq. (4.16) and inverse Fourier transform of Eq. (3.21), it is easy to derive a useful expression for the swarm particle flux:

$$\vec{\Gamma}(\vec{r},t)=\sum_{p=0}^{\infty}\hat{\Omega}^{(p+1)}(t)\odot_p\left(-\frac{\partial}{\partial\vec{r}}\right)^p n(\vec{r},t). \quad (4.18)$$

The physical interpretation afforded by Eqs. (4.14) and (4.15) permits an interpretation of the flux component and ''nonconservative'' corrections to the bulk transport coeffi-

cients given in Eq. (4.16). The flux drift velocity $\hat{\Omega}^{(1)}(t)$ represents the rate of change of the position of the center of mass due to external (gravitation) fields only. The presence of gravity results in a spatial variation in the energy throughout the swarm. Under such a condition, an energy-selective change of the number of swarm particles leads to a change in the position of the center of mass of the swarm. This effect on the bulk drift velocity $\vec{W}(t)$ is represented by $\hat{R}^{(1)}(t)$. Likewise the flux diffusion tensor $\hat{\Omega}^{(2)}(t)$ represents the rate of spreading of the swarm due to gravity and $\partial n/\partial \vec{r}$. An energy-selective change of the number of swarm particles may result in a variation of $\partial n/\partial \vec{r}$ throughout the swarm and a subsequent variation in the rate of change of the mean-squared width of the swarm. Such effects are expressed by the second-rank tensor $\hat{R}^{(2)}(t)$.

V. CONCLUDING REMARKS

In this paper we have analyzed the foundations of the transport theory of granular swarms. The Boltzmann-Lorentz equation was used to describe the motion of swarm particles in a dilute granular gas of inelastic hard spheres. The theory developed in this paper is valid under several assumptions. We have considered the special case when the background fluid is in the homogeneous cooling state, but the developed formalism is applicable to any nonstationary but *homogeneous* state of the background granular fluid. An infinite medium with no boundaries is an idealization, and proper analysis of real systems requires that boundaries and associated boundary conditions be taken into account. Furthermore, the homogeneous cooling state is unstable to long-wavelength perturbations that were not considered, so the results obtained apply only on time scales short compared to that for the growth of such perturbations.

We have applied the time-dependent perturbation method to study the evolution of the swarm from an arbitrary initial distribution. We have obtained Eq. (3.8) which is a nonhydrodynamic extension of the diffusion equation with transport coefficients that are time dependent and implicitly depend on the wave vector. The spatial dependence of the transport coefficients arises from their explicit dependence on the initial distribution. In other words, every Fourier component of the initial distribution has a corresponding set of transport coefficients $\{\hat{\omega}_{\vec{q}}^{(p)}(t)|p\geqslant 0\}$ which, according to Eq. (3.8), describes temporal evolution of the corresponding Fourier component of the number density $n(\vec{r},t)$. Our description of the *short-time* development of granular swarms is consistent with the generalized hydrodynamic description in which the diffusion coefficient depends on the wave vector [38]. Any transport coefficient can be represented as a function of solutions to the hierarchy of kinetic equations [Eqs. (3.11)–(3.17)]. This is similar to the Chapman-Enskog procedure. Namely, $\hat{\omega}^{(2)}$ corresponds to the Navier-Stokes hydrodynamics, and the transport coefficients $\hat{\omega}^{(3)}$ and $\hat{\omega}^{(4)}$ correspond to the Burnett and super-Burnett levels, respectively. Despite this similarity, our formalism is more general than the Chapman-Enskog theory, because it is valid for *all* times including the initial nonhydrodynamic regime. In fact, it contains the Chapman-Enskog theory as its long-time limit as demonstrated in Appendix A for the case of self-diffusion in a freely evolving granular gas. The fact that the transport coefficients appearing in Eq. (3.8) are tensors reflects the anisotropy of the granular swarm induced by the imposed external field.

For the special kind of initial state [Eq. (4.2)] whose one-particle distribution function is given by $f(\vec{r},\vec{v},t_0)=f_0(\vec{v})n(\vec{r},t_0)$, transport coefficients $\hat{\omega}^{(p)}(t)$, $p\geqslant 0$, are $\vec{q}$ independent at all times. In that case, they can be expressed as time derivatives of the spatial moments of the number density $n(\vec{r},t)$ [Eqs. (4.13)–(4.15)]. For more general initial conditions the *short-time* behavior of the number density cannot be characterized by singular distributions $\hat{\omega}^{(p)}(\vec{r},t)=\hat{\omega}^{(p)}(t)\delta(\vec{r})$, $p\geqslant 0$ [Eq. (4.6)]. Then, the temporal evolution of the spatial moment of the number density of sth order depends on the generalized transport coefficients $\hat{\omega}^{(p)}(\vec{r},t)$ of order $0\leqslant p\leqslant s$. Equations (4.9)–(4.11) describe the inertial regime in the dynamics of swarm particles subjected to an external field. This regime holds for times small enough compared to the characteristic relaxation times. Note that the results presented here are not restricted to small gradients in the density of swarm particles.

As a new aspect of granular swarm transport theory we have introduced the concept of non-particle-conserving collisions. If ''reactions'' are present, the collision operator J may be split into a particle conserving part J^{PC} and a ''reactive'' part J^R. Without going into details of the operator J^R, we have demonstrated the separation of the flux and ''reactive'' component of the transport coefficients [Eqs. (3.22) and (4.16)]. While the Chapman-Enskog method is valid only if the reaction term J^R in the kinetic equation can be treated as a small perturbation with respect to the collision term J^{PC} [39], our hierarchy of kinetic equations (3.1) and (3.2) is not restricted by this condition.

The problems that need further elaboration are problems involving boundaries and inhomogeneities of the background granular fluid. On the other hand, computer simulations have become an effective tool for gaining physical insight into various aspects of granular swarm behavior. It would be interesting to see them used to obtain accurate values of transport coefficients [Eqs. (4.13)–(4.15)].

APPENDIX A: DERIVATION OF EQ. (3.21)

Since the time derivative of $\hat{N}^{(p)}(\vec{q},t)$ is [see Eq. (3.4)]

$$\frac{\partial}{\partial t}\hat{N}^{(p)}(\vec{q},t)=\langle f_H(t)|\frac{\partial}{\partial t}\|\varkappa_q^{(p)}(t)\rangle\rangle, \quad p\geqslant 0, \tag{A1}$$

we obtain, from Eqs. (3.1) and (3.2),

$$\frac{\partial}{\partial t}\hat{N}^{(0)}(\vec{q},t)=\langle f_H(t)|\hat{H}_0(t)\|\varkappa_q^{(0)}(t)\rangle\rangle, \tag{A2}$$

$$\frac{\partial}{\partial t}\hat{N}^{(p)}(\vec{q},t)=\langle f_H(t)|\hat{H}_0(t)\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle + \langle f_H(t)|\hat{\vec{v}}\|\varkappa_{\vec{q}}^{(p-1)}(t)\rangle\rangle, \quad p\geqslant 1. \tag{A3}$$

From Eq. (3.20) and Eqs. (A2) and (A3), using definition (3.6) and Eq. (3.5), we arrive at

$$\vec{\Gamma}_{\vec{q}}(t)=\sum_{p=0}^{\infty}(-i\vec{q})^p\odot_p[\hat{\omega}_{\vec{q}}^{(p+1)}(t)n_{\vec{q}}(t) + \langle f_H(t)|[\hat{\omega}_{\vec{q}}^{(0)}(t)\hat{I}-\hat{H}_0(t)]\|\varkappa_{\vec{q}}^{(p+1)}(t)\rangle\rangle]. \tag{A4}$$

This leads immediately to Eq. (3.21).

APPENDIX B: DERIVATION OF EQ. (4.8)

Multiplying Eq. (4.6) by the function $\psi(\vec{r})$ and integrating over $\vec{r}$, we obtain

$$\frac{\partial}{\partial t}\int d\vec{r}\,\psi(\vec{r})n(\vec{r},t)-\sum_{p=0}^{\infty}\int d\vec{r}_1\,\hat{\omega}^{(p)}(\vec{r}_1,t) \odot_p\int d\vec{r}\,\psi(\vec{r})\left(-\frac{\partial}{\partial\vec{r}}\right)^p n(\vec{r}-\vec{r}_1,t)=0. \tag{B1}$$

We suppose that $n(\vec{r},t)$ together with its derivatives vanish at the boundaries of the domain of integration:

$$\left(\frac{\partial}{\partial\vec{r}}\right)^p n(\vec{r},t)\to 0, \quad |\vec{r}|\to\infty; \quad p\geqslant 0. \tag{B2}$$

By partial integration, we get immediately

$$\frac{\partial}{\partial t}\int d\vec{r}\,\psi(\vec{r})n(\vec{r},t)-\sum_{p=0}^{\infty}\int d\vec{r}_1\,\hat{\omega}^{(p)}(\vec{r}_1,t) \odot_p\int d\vec{r}\,n(\vec{r}-\vec{r}_1,t)\left(\frac{\partial}{\partial\vec{r}}\right)^p\psi(\vec{r})=0. \tag{B3}$$

Using, in addition, the equality

$$\frac{\partial}{\partial t}\int d\vec{r}\,\psi(\vec{r})n(\vec{r},t)=N(t)\frac{\partial}{\partial t}\langle\psi(\vec{r})\rangle_n+\frac{dN(t)}{dt}\langle\psi(\vec{r})\rangle_n, \tag{B4}$$

we readily obtain Eq. (4.8).

APPENDIX C: SELF-DIFFUSION

Here we apply our general formalism to the self-diffusion of swarm particles in the regime of homogeneous cooling and compare with results previously obtained by Chapman-Enskog expansion in the density gradient of the swarm particles [15]. We consider the system in the absence of an external field.

Self-diffusion is the simplest transport process when the swarm particles are mechanically equivalent to the fluid particles. The time evolution of the one-particle distribution function of the swarm particles is given by Eq. (1.1), where the collision operator has the form (1.7) with $\overline{\sigma}=\sigma$ and $m=m_0$, $\Delta=1$ in Eq. (1.3).

Under the assumption that the initial one-particle distribution function of swarm particles is given by $f(\vec{r},\vec{v},t_0)=f_0(\vec{v})n(\vec{r},t_0)$, the transport coefficients $\hat{\omega}_{\vec{q}}^{(p)}$, $p\geqslant 0$, are $\vec{q}$ independent at all times according to Eq. (4.4). Since we are interested in transport coefficients only, it can be supposed that the initial distribution for swarm particles is $n(\vec{r},t_0)=N_0\delta(\vec{r})$, where N_0 is the number of swarm particles. For this idealized initial condition, the $\vec{q}$ dependence of the tensors $\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle$, $p\geqslant 0$, in the hierarchy of kinetic equations (3.1) and (3.2) can be omitted. Indeed, from Eq. (4.3) and $n_{\vec{q}}(t_0)=N_0$, we obtain

$$\|\varkappa_{\vec{q}}^{(p)}(t)\rangle\rangle=\|\varkappa^{(p)}(t)\rangle\rangle N_0\to N_0\,\hat{f}^{(p)}(\vec{v},t), \quad p\geqslant 0, \quad t\geqslant t_0. \tag{C1}$$

Hence, from Eqs. (3.11)–(3.17) and Eq. (C1) we obtain the kinetic equations

$$\frac{\partial}{\partial t}f^{(0)}(\vec{v},t)=J[f^{(0)}](\vec{v},t), \quad f^{(0)}(\vec{v},t_0)=f_0(\vec{v}), \tag{C2}$$

$$\frac{\partial}{\partial t}\vec{f}^{(1)}(\vec{v},t)=J[\vec{f}^{(1)}](\vec{v},t)+\vec{v}f^{(0)}(\vec{v},t), \quad \vec{f}^{(1)}(\vec{v},t_0)=0, \tag{C3}$$

and expressions for the transport coefficients

$$\vec{W}(t)=\frac{1}{\int d\vec{v}\,f^{(0)}(\vec{v},t)}\int d\vec{v}\,\vec{v}f^{(0)}(\vec{v},t), \tag{C4}$$

$$\hat{D}(t)=\frac{1}{\int d\vec{v}\,f^{(0)}(\vec{v},t)}\left[\int d\vec{v}\,\vec{v}\otimes\vec{f}^{(1)}(\vec{v},t) - \vec{W}(t)\otimes\int d\vec{v}\,\vec{f}^{(1)}(\vec{v},t)\right]. \tag{C5}$$

Next, let us consider the long-time behavior of swarm particles. It is assumed that in the long-time limit there are solutions to the kinetic equations (C2) and (C3) of the form

$$f^{(0)}(\vec{v},t)=f_T^{(0)}(\vec{v}|T_H(t)), \quad \vec{f}^{(1)}(\vec{v},t)=\vec{f}_T^{(1)}(\vec{v}|T_H(t)). \tag{C6}$$

The notation $f_T(\vec{v}|T_H(t))$ means that f_T is a functional of the granular temperature $T_H(t)$ and that its time dependence occurs only through temperature $T_H(t)$. An evolution equation for the temperature has the well-known form

$$\frac{\partial T_H(t)}{\partial t} = -\zeta_H(T_H) T_H(t), \tag{C7}$$

where $\zeta(T_H)$ is the cooling rate [20]. Finally, the kinetic equations for this stage of evolution follow directly from Eqs. (C2),(C3) and Eq. (C7):

$$-\zeta_H(T_H) T_H(t) \frac{\partial}{\partial T_H} f_T^{(0)}(\vec{v}|T_H) = J[f_T^{(0)}](\vec{v}|T_H), \tag{C8}$$

$$-\zeta_H(T_H) T_H(t) \frac{\partial}{\partial T_H} \vec{f}_T^{(1)}(\vec{v}|T_H) = J[\vec{f}_T^{(1)}](\vec{v}|T_H) + \vec{v} f_T^{(0)}(\vec{v}|T_H). \tag{C9}$$

The solution of the zeroth-order kinetic equation (C8) must be proportional to $f_H(\vec{v},t)$; i.e., it has the form

$$f_T^{(0)}(\vec{v}|T_H) = \frac{1}{n_0} f_H(\vec{v}, T_H(t)) \tag{C10}$$

and, therefore, the drift velocity [Eq. (C4)] of swarm particles vanishes, i.e., $\vec{W}(t)=0$. Substitution of this into Eqs. (C9) and (C5) yields

$$-\zeta_H(T_H) T_H(t) \frac{\partial}{\partial T_H} \vec{f}_T^{(1)}(\vec{v}|T_H) = J[\vec{f}_T^{(1)}](\vec{v}|T_H) + \vec{v} \frac{1}{n_0} f_H(\vec{v}, T_H(t)), \tag{C11}$$

$$\hat{D}(T_H(t)) = \int d\vec{v}\, \vec{v} \otimes \vec{f}_T^{(1)}(\vec{v}|T_H(t)). \tag{C12}$$

Recently, the results (C11) and (C12) have been obtained from Chapman-Enskog solution to the Enskog-Lorentz equation [15] [Eqs. (24) and (26) therein], and the integral equation (C11) was approximately solved in a leading-order Sonine polynomial expansion.

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